New Simple Approximations for Error Probability and Outage in Fading

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***Abstract*—A new class of approximations for the bit error rate (BER), the symbol error rate (SER), and the outage of wireless digital communication systems impaired by fading and noise is derived. As compared to conventional high signal-to-noise ratio (SNR) approximations [1], these new approximations are better by an order of magnitude in the high SNR regime and retain their accuracy for a wider range of SNRs. They require the first two terms of the Taylor expansion of the channel probability density function (PDF). The resulting approximations for two diversity combiners, multiple-antenna eigenmode transmission and several important modulation schemes are developed.**

***Index Terms*—Error probability, outage probability, finite signal-to-noise ratio (SNR), asymptotic SNR, fading.**

I. Introduction

PERFORMANCE analysis of wireless systems with various digital receiver techniques over fading channels involves averaging the performance metric $h(\gamma)$, where γ is the instantaneous signal-to-noise ratio (SNR) and $h(\cdot)$ represent measures such as outage, error rate, capacity and others. By representing $\gamma = \beta\bar{\gamma}$, where $\bar{\gamma}$ is the unfaded link SNR or the average SNR and β is a channel-dependent non-negative random variable, the average is performed over the probability density function (PDF) $f(\beta)$ [1]. Exact closed-form solutions may not always be possible or may be cumbersome and may not provide direct insight into important parameters that governs the system performance. Thus, simple approximations to develop insight and suitable for applications such as cross-layer optimal system design [2], is highly desirable. Several high-SNR approximations and bounds may be found in [1], [3]–[6].

Common $h(\gamma)$ includes $Q(\sqrt{\kappa\bar{\gamma}\beta})$ to represent the error probability of various modulation schemes, where $Q(\cdot)$ is the Gaussian Q-function. Typically, such $h(\gamma)$ decays exponentially at high SNR, i.e., $h(\gamma) = O(e^{-\bar{\gamma}\beta})$ as $\bar{\gamma} \to \infty$, then the high-SNR performance is dominated by the behavior of $f(\beta)$ at $\beta \to 0^+$. Wang and Giannakis [1] exploited this fact to suggest the approximation of $f(\beta)$ by the first term of its Taylor series expansion at $\beta = 0$, i.e., $f(\beta) \approx a\beta^t$ as $\beta \to 0^+$. In [1], the average error probability and outage probability were quantified in terms of coding gain G_c (also known as the SNR gain or combining gain) and diversity gain (diversity order) G_d as

$$\mathbb{E}[h(\gamma)] \approx (G_c\bar{\gamma})^{-G_d} \quad \text{as} \quad \bar{\gamma} \to \infty, \tag{1}$$

where G_c and G_d are expressed in terms of a and t. With this approach, they unified the analysis of many communication systems over a wide spectrum of fading channels and hence, this has been widely used in recent research ([1] has been cited over 400 times). However, eq. (1) is not accurate for the low SNR regime. For example, in some cases, eq. (1) is accurate only when the SNR exceeds 20 dB or more and error rates below 10^{-7}. However, current wireless systems operate at 3-20 dB with error rates as high as 10^{-2} [7]. Thus, new approximations more accurate than eq. (1) are desirable.

Although approximations and bounds for high to moderate SNRs are available, they are not general. For example, error probability bounds in [5] are applicable only to N-branch diversity systems employing two-dimensional signaling constellations. However, unified approach (e.g., [1]) accurate over a wide range of SNRs is still lacking and this paper fills this gap. The contributions are summarized as follows.

1) A unified analysis of digital communication systems impaired by fading and noise is presented by deriving a new class of approximations for the bit error rate (BER), symbol error rate (SER), and outage probability.
2) It is demonstrated that our new approximations are not only more accurate than the conventional approximation eq. (1) in the high SNR regime, but also retain accuracy for a wide range of SNRs.
3) The new approximations are computed by using only the first two terms of the Taylor series of $f(\beta)$ at $\beta = 0$. We show how to extract these two terms from the moment generating function (MGF).

II. Preliminaries

The Taylor series expansion of the PDF of β is given as

$$f(\beta) = a\beta^t + a_1\beta^{t+1} + O(\beta^{t+2}) \quad \text{as} \quad \beta \to 0^+, \tag{2}$$

where the real constants a, a_1 and t are assumed known. Since $f(\beta)$ at $\beta \to 0^+$ determines the high-SNR performance, we may simply replace $f(\beta)$ with $a\beta^t + a_1\beta^{t+1}$, the first two terms of the Taylor series. However, simply averaging the performance measure $h(\beta\bar{\gamma})$ over $a\beta^t + a_1\beta^{t+1}$ is not interesting and amounts to a rather trivial extension of [1]. However, by noting that $ax^t + a_1x^{t+1} = ax^te^{xa_1/a}$ as $x \to 0^+$, we propose the approximation

$$f(\beta) \approx a\beta^te^{-\alpha\beta} \tag{3}$$

as $\beta \to 0^+$, where $\alpha = -\frac{a_1}{a}$. Note that as we require eq. (3) to agree with the actual PDF only for suitably small β, it may not be a proper PDF. The intuition behind this approximation is that it matches the two-terms Taylor expansion of $f(\beta)$ as $\beta \to 0^+$ exactly; moreover its exponential form facilitates the derivation of handy closed-form solutions and offers significant improvement in the SNR range and accuracy.

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A. Use of MGF to obtain our new class of approximations

Our new class of approximations requires the Taylor expansion of $f(\beta)$ at $\beta = 0$. This expansion is readily computed if the explicit PDF is available. In many cases, however, it is not readily available, but the MGF is. Thus, it is natural to use the MGF to extract the required information.

Proposition 1. *When the MGF $M_\beta(s) = \mathbb{E}\left[e^{-s\beta}\right]$ can be expanded in an absolutely convergent series for $|s| > R$ of the form $M_\beta(s) = \frac{c_0}{s^\tau} + \frac{c_1}{s^{\tau+1}} + O\left(\frac{1}{s^{\tau+2}}\right)$ as $s \to \infty$, then $t = \tau - 1$, $a = \frac{c_0}{\Gamma(\tau)}$ and $a_1 = \frac{c_1}{\Gamma(\tau+1)}$, where $\Gamma(\cdot)$ is the Gamma function.*

Proof: Through term by term inverse laplace transformation of the asymptotic series of the MGF, the series expansion of the PDF can be expressed as $f(\beta) = \frac{c_0\beta^{\tau-1}}{\Gamma(\tau)} + \frac{c_1\beta^\tau}{\Gamma(\tau+1)} + O\left(\beta^{\tau+1}\right)$ as $\beta \to 0^+$ [8, Theorem 35.2]. ■

Example 1: Consider an N_r branch maximal ratio combining (MRC) system over independent and identically distributed (iid) Rayleigh fading. The MGF of the channel gain is $M_\beta(s) = \frac{1}{(1+s)^{N_r}}$, whose asymptotic expansion for $s \to \infty$ can be obtained as $M_\beta(s) = \frac{1}{s^{N_r}}\left(1 - \frac{N_r}{s}\right) + O(\frac{1}{s^{N_r+2}})$, and thus, a, a_1 and t are readily obtained.

III. Average Probability of Error

Here, we derive a new class of approximations for the BER or SER of several modulation schemes. Since the Q-function is a common representation of many BER or SER expressions, our main result is the following proposition.

Proposition 2. *For modulation formats with the conditional error rate of the form $Q(\sqrt{\kappa\overline{\gamma}\beta})$, the average is given by*

$$\mathbb{E}\left[Q\left(\sqrt{\kappa\overline{\gamma}\beta}\right)\right] = \frac{2^t a\sqrt{\kappa\overline{\gamma}}\,\Gamma(t+\frac{3}{2})}{\sqrt{\pi}(t+1)(2\alpha+\kappa\overline{\gamma})^{t+\frac{3}{2}}}$$
$$\,_2F_1\left(1, t+\frac{3}{2}, t+2, \frac{2\alpha}{2\alpha+\kappa\overline{\gamma}}\right) + O\left(\overline{\gamma}^{-(t+3)}\right), \quad (4)$$

where $\,_2F_1(p, q;\ r;\ z)$ is the Gauss hypergeometric function [9, Eq. 9.100].

Proof: By averaging $Q(\sqrt{\kappa\overline{\gamma}\beta})$ over eq. (3) with the help of [9, Eq. 6.286.1], the first term of the right hand side of eq. (4) is obtained. Since the error term of eq. (3) is $O(\beta^{t+2})$, by considering a small neighborhood of $\beta = 0$, we can show that the approximation error is $O\left(\overline{\gamma}^{-(t+3)}\right)$. ■

Remarks:

1) Proposition 2 covers many coherent modulation schemes. For example, it gives average BER of binary phase shift keying (BPSK) ($\kappa = 2$) and coherently detected binary frequency shift keying (FSK) with orthogonal signaling ($\kappa = 1$) or minimum correlation ($\kappa = 1.217$), and average SER of M-ary PAM ($\kappa = 6/(M^2 - 1)$ with eq. (4) multiplied by a constant $2(M-1)/M$).
2) If t is an integer, $\,_2F_1(\cdots)$ in eq. (4) can be replaced with a finite polynomial type expression (e.g., [10, Sec. 14.4]). However, the former is valid for all real values of t, which may be useful in cases such as Nakagami-m fading with non-integer m, and is easy to compute. Moreover, since $2\alpha/(2\alpha + \kappa\overline{\gamma}) < 1$, the series is absolutely convergent.
3) Since $\,_2F_1(p, q;\ r;\ z) = 1$ as $z \to 0$, eq. (4) clearly indicates that the diversity order is $G_d = 1 + t$, and the coding gain is $G_c = \left\{\frac{2^t a\sqrt{\kappa}\,\Gamma(t+\frac{3}{2})}{\sqrt{\pi}(t+1)}\right\}^{-\frac{1}{1+t}}$. These values are exactly the same as found by [1].
4) The error term of eq. (4) is $O(\overline{\gamma}^{-(G_d+2)})$ – see Fig. 2.
5) Last but not least, for this method to work, a_1 must be negative. This is certainly true for many cases.

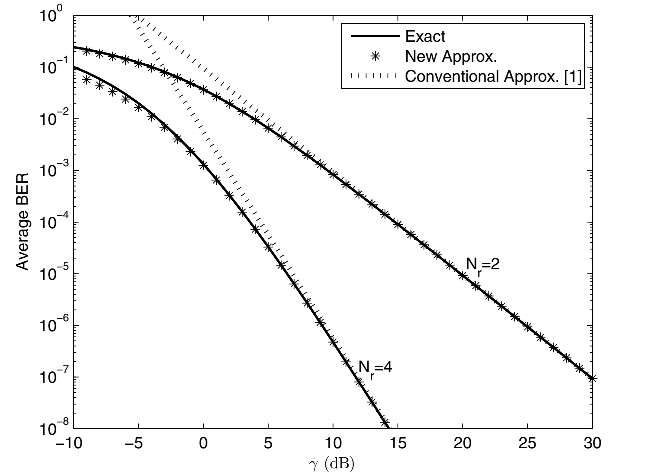

Fig. 1. BER of BPSK for N_r branch MRC with independent channel gains $\beta_i \sim G(1, i), i = 1, 2, \ldots, N_r$.

We next present two applications and generalizations of Proposition 2.

A. Performance in N_r branch MRC in independent fading

Consider extending Example 1 to the more general case with $\beta = \sum \beta_i$ where β_i, $i = 1, \ldots, N_r$, are the independently distributed channel gains of the N_r branches. Suppose for each branch, we have the following information:

$$M_{\beta_i}(s)\Big|_{s\to\infty} = \frac{c_i}{s^{\mu_i}} + \frac{d_i}{s^{\mu_i+1}} + O(\frac{1}{s^{\mu_i+2}}), \quad i = 1, \ldots, N_r.$$

Since the MGF of β is the product of the MGF's of the summands, we find

$$M_\beta(s)\Big|_{s\to\infty} = \frac{\prod_i c_i}{s^{\sum \mu_i}} + \frac{\sum_j d_j \prod_{i\neq j} c_i}{s^{1+\sum \mu_i}} + O\left(\frac{1}{s^{2+\sum \mu_i+2}}\right), \quad i = 1, \ldots, N_r.$$

Thus, a, a_1 and t are immediately found via Proposition 1.

Fig. 1. shows the BER of MRC with BPSK over independently faded N_r branches with Gamma distributed channel gains, i.e. $\beta_i \sim G(k_i, \theta_i)$, $i = 1, 2, \ldots, N_r$, where the parameters of Gamma distribution are chosen to be $k_i = 1$ and $\theta_i = i$. Clearly, for each branch, $c_i = \frac{1}{i}$, $\mu_i = 1$ and $d_i = -\frac{1}{i^2}$ and a, a_1, and t are readily obtained. The exact result, the new approximation, eq. (4) and conventional one, eq. (1) are compared. Note that while the conventional one is accurate only at high SNR, say above 10 dB, eq. (4) works for medium-to-high SNRs and even at very low SNRs, -10 dB.

However, to further ascertain the relative accuracy levels of the new approximation eq. (4) and the conventional one

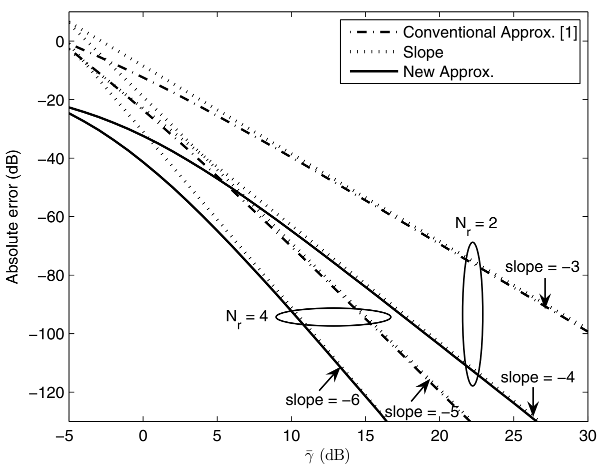

Fig. 2. Accuracy of new and conventional approximations for N_r branch MRC in independent fading with $\beta_i \sim G(1,i), i=1,2$.

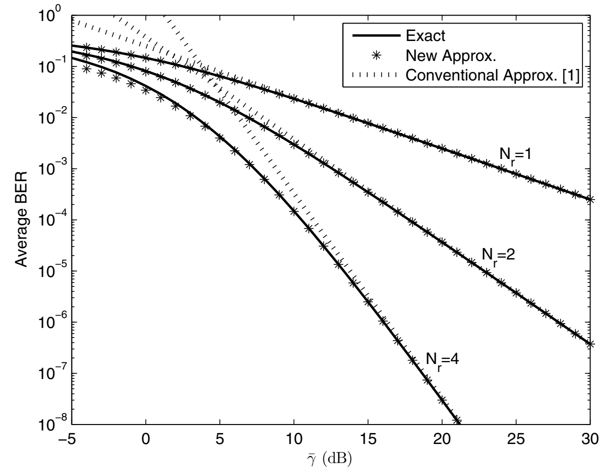

Fig. 3. BER of BPSK for N_r branch SC with iid Rayleigh fading.

eq. (1), we present Fig. 2, where the absolute errors of both of these approximations, i.e., the absolute value of the difference $(exact - approximate)$ are plotted in log-log scale. The slope lines are also drawn to indicate the orders of the errors and hence verify that as $\overline{\gamma} \to \infty$, the error term of eq. (1) has a negative slope of G_d+1, whereas that of eq. (4) is G_d+2. This advantage translate into the improved accuracy of the new approximation over eq. (1) in the whole SNR range.

B. Performance in N_r branch SC in iid Rayleigh fading

In selection combining (SC), the branch with the strongest SNR is selected. For N_r branch SC in iid Rayleigh fading, the PDF of β can be written as

$$f(\beta) = N_r e^{-\beta}(1-e^{-\beta})^{N_r-1}, \quad \beta \geq 0. \tag{5}$$

We need the first two non-zero terms of the Taylor expansion of $f(\beta)$. By differentiating $f(\beta)$, we obtain

$$\phi(l) = \frac{\partial^l f(\beta)}{\partial \beta^l}\Big|_{\beta=0} = N_r \sum_{k=0}^{N_r-1} \binom{N_r-1}{k}(-1)^{k+l}(k+1)^l, \tag{6}$$

where $l = 0,1,\ldots$ We find that $\phi(l)=0$ for $l < N_r-1$, thus $t = N_r - 1$. Then, $a = \frac{\phi(N_r-1)}{(N_r-1)!}$ and $a_1 = \frac{\phi(N_r)}{N_r!}$. Proposition 2 can now readily be applied.

The average BER of the system with BPSK is plotted against $\overline{\gamma}$ in Fig. 3. For each value of N_r, exact result, conventional approximation eq. (1) and new approximation eq. (4) are compared. For $N_r=1$, eq. (4) coincides with the exact result. For larger values of N_r, it is highly accurate from high to medium range while being fairly accurate at low SNR values. Its improved accuracy over eq. (1) is again evident.

C. Generalization to other modulation schemes

Proposition 2 covers only coherent modulation formats whose conditional error rates lend to a Q-function representation. However, this representation does not hold for noncoherent and differential modulations. Two examples of these modulations are considered below by deriving new closed-form approximations. The error term $O\left(\frac{1}{\overline{\gamma}^{t+3}}\right)$ is not shown for brevity.

1) M-ary differential phase shift keying (MDPSK): For MDPSK, averaging conditional SER expression [11, Eq. (19)] over $f(\beta) = a\beta^t e^{-\alpha\beta}$ as $\beta \to 0^+$, the SER can be obtained as

$$P_s(\overline{\gamma}) = \frac{a\Gamma(t+1)}{\pi\alpha^{t+1}} I_V\left(0, \pi - \frac{\pi}{M}, \cos\frac{\pi}{M}, \frac{\overline{\gamma}}{\alpha}\sin^2\frac{\pi}{M}, t+1\right), \tag{7}$$

where $I_V(\theta_l, \theta_u, r, p, q)$ is the integral of the form $\int_{\theta_l}^{\theta_u}\left[(1+r\cos\theta)/(1+p+c\cos\theta)\right]^q d\theta$ with closed-form solution available in [11, Eq. (11)].

2) Noncoherent correlated binary signaling: The average BER of noncoherently detected equal energy, equiprobable binary signals can be obtained as follows by averaging [11, Eq. (21)] over $f(\beta) = a\beta^t e^{-\alpha\beta}$ as $\beta \to 0^+$

$$P_b(\overline{\gamma}) = \frac{a\Gamma(t+1)}{2\pi\alpha^{t+1}} I_V\left(0, \pi, \frac{-2uv}{u^2+v^2}, \frac{(u^2-v^2)^2\overline{\gamma}}{2\alpha(u^2+v^2)}, t+1\right), \tag{8}$$

where $u = \sqrt{(1-\sqrt{1-\rho^2})/2}$, $v = \sqrt{(1+\sqrt{1-\rho^2})/2}$, and $0 \leq \rho \leq 1$ is the absolute value of correlation coefficient. $\rho = 0$ corresponds to orthogonal binary FSK.

IV. Outage Probability

Outage probability, an important quality-of-service measure for fading channel communication, is the probability that the instantaneous SNR falls below a predefined threshold γ_T.

Proposition 3. *As per the approximation eq.* (3), $f(\beta) \approx a\beta^t e^{-\alpha\beta}$ *as* $\beta \to 0^+$, *the outage* $\Pr[\overline{\gamma}\beta \leq \gamma_T]$ *is*

$$P_{out}(\gamma_T, \overline{\gamma}) = \frac{a}{\alpha^{t+1}}\gamma(t+1, \alpha\gamma_T/\overline{\gamma}) + O\left(\frac{1}{\overline{\gamma}^{t+3}}\right), \tag{9}$$

where $\gamma(a,x)$ *is the incomplete Gamma function [9, Eq. 8.350.1].*

Proof: immediately follows from the definition of $\gamma(a,x)$. ■

Conventionally, analogous to eq. (1), high-SNR outage is approximated as $P_{out}(\gamma_T, \overline{\gamma}) \approx (O_c\overline{\gamma})^{-O_d}$ [1] where O_d, and O_c are outage diversity and coding gain. Note that since $\gamma(n,x) = x^n/n$ as $x \to 0$, eq. (9) indicates that $O_d = t+1$ and $O_c = \frac{1}{\gamma_T}\left(\frac{a}{t+1}\right)^{-\frac{1}{t+1}}$, which is consistent with [1].

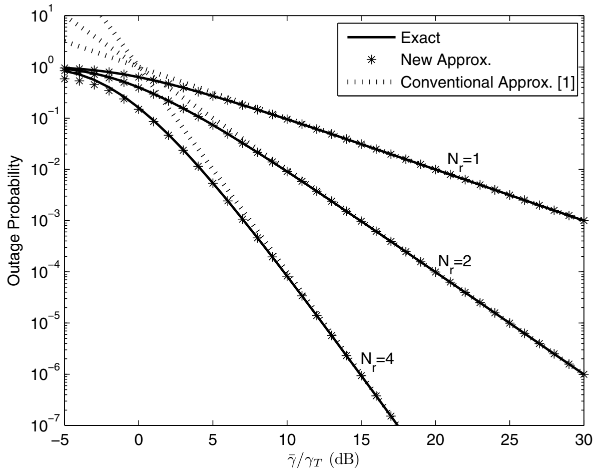

Fig. 4. Outage probability of N_r branch SC over iid Rayleigh fading.

The outage probability, eq. (9), of N_r branch SC in iid Rayleigh fading is plotted against $\overline{\gamma}/\gamma_T$ in Fig. 4, along with the exact value and the conventional approximation. As with the BER case, the latter is accurate only at high SNR values. However, with eq. (9), the accuracy holds from high SNRs to medium SNRs (even low SNRs in some cases). For example, with $N_r = 4$, eq. (9) is accurate from 0 dB and above; whereas the conventional approximation, from 15 dB and above.

V. Other Applications

Our new approximations facilitate rapid analysis of error and outage performance of wireless systems. In addition to the two classical diversity combining examples, we next show a multiple input multiple output (MIMO) example.

Example 2: In MIMO, if the transmission is only along the strongest eigenmode so that the receiver output SNR is maximized, this is known as beamforming. The distribution of this eigenmode is given in [12, Eq. (9)]. The extraction of a, a_1 and t is illustrated next.

Consider a $\mathbf{H} \in \mathbb{C}^{N_r \times N_t}$ MIMO channel whose columns are independent zero mean complex Gaussian random vectors. Let $n = \max(N_t, N_r)$ and $m = \min(N_t, N_r)$. The cumulative distribution function (CDF) of the instantaneous SNR for the strongest eigenmode is given by [12]

$$F_\Lambda(x) = K_{m,n}^{-1}|\Psi(x)| = K_{m,n}^{-1} \sum_{\forall\sigma} \mathrm{sgn}(\sigma) \prod_{i=1}^m \psi_{\sigma(i),i}(x),$$

where $K_{m,n} = \prod_{k=1}^m \Gamma(n-k+1)\Gamma(m-k+1)$ and $|\cdot|$ denotes the determinant. The second equality is obtained by applying Leibniz formula for the determinant of $m \times m$ matrix $\Psi(x) = [\psi_{i,j}(x)]_{m\times m}$ where $\psi_{i,j}(x) = \gamma(n-m+i+j-1, x)$ and $\gamma(\cdot,\cdot)$ is the incomplete gamma function. The summation is over all permutations σ of $\{1\ldots, m\}$ and $\mathrm{sgn}(\cdot)$ is $+1$ or -1 depending on whether the permutation σ is even or odd. Since $\gamma(a,z) \approx z^a\left(\frac{1}{a} - \frac{z}{(a+1)}\right)$ as $z \to 0^+$, we find

$$F_\Lambda(x) = K_{m,n}^{-1} x^{mn} \sum_{\forall\sigma} \mathrm{sgn}(\sigma) \left(\prod_{i=1}^m \frac{1}{(u-1+\sigma(i)+i)} - x\sum_{k=1}^m \frac{1}{(u+\sigma(k)+k)} \prod_{\substack{i=1\\ i\neq k}}^m \frac{1}{(u-1+\sigma(i)+i)}\right) + O(x^{mn+2}),$$

where $u = n - m$. We thus obtain $t = mn - 1$ and a and a_1 (details omitted for brevity). Having derived the required parameters, the error and outage performance can be instantly obtained using our derived results.

VI. Conclusion

New simple approximations for the average error probability and the outage of wireless systems impaired by fading and noise were derived. Illustrative examples of MRC, SC and MIMO beamforming were presented. The advantages of the new approximations are two fold:

1) They are much more accurate: on a log-log scale, the approximation error decays at a rate of $(G_d + 2)$ compared to that of $(G_d + 1)$ for eq. (1)).
2) They are accurate over a wide range of SNRs.

The price for these advantages is that these approximations require the first two terms of the Taylor expansion of $f(\beta)$ at $\beta = 0$, whereas the conventional approximation requires only the first term. For a large class of problems, when the explicit PDF is unavailable, those two terms can be extracted from the MGF (Proposition 1). With the potential to facilitate rapid performance analysis of a myriad of wireless systems, the new approximations may be used in many applications hitherto severed by eq. (1).

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