Improved K-Best Sphere Detection for Uncoded and Coded MIMO Systems

Shuangshuang Han, Tao Cui, and Chintha Tellambura, *Fellow, IEEE*

***Abstract*—The conventional K-best sphere decoder (KSD) keeps the best K nodes at each level of the search tree. In addition to retaining the best K nodes, we also consider all the nodes whose costs are within a certain margin of the cost of the Kth best node. The resulting algorithm is called improved K-best sphere decoder (IKSD). Three IKSD variants are considered in this letter, which are fixed threshold, normalized threshold and adaptive threshold IKSD. The proposed IKSD requires a smaller K (indicating lower complexity) while still achieving a better and near optimal performance compared to the conventional KSD. These gains are confirmed by the simulation results. For example, for the fixed threshold IKSD in a 4×4 16-QAM multiple-input multiple-output (MIMO) system, with $K = 2$, it achieves the same performance as the conventional KSD ($K = 16$), yielding about 80% complexity savings. For coded MIMO systems, the IKSD is also extended as a list sphere decoder for joint iterative detection and decoding.**

***Index Terms*—MIMO, ML, sphere decoder, tree search, wireless communications.**

I. Introduction

The K-best sphere decoder (KSD) [1] for spatial multiplexing multiple-input multiple-output (MIMO) detection has received significant attention recently because of its fixed throughput and parallel implementation. In contrast, the conventional SD uses the depth-first tree search resulting in non-constant throughput, which limits the decoding efficiency. The KSD is also known as the M-algorithm or as beam search in the Artificial Intelligence literature. Instead of a depth-first tree traversal, the KSD performs a breadth-first search and retains only K best nodes at each layer.

Although it has a fixed detection complexity, the KSD does not guarantee the Maximum Likelihood (ML) performance [1]. To do so, the KSD typically requires very large values of K, which results in a higher complexity than that of the conventional SD. Nevertheless, due to advantages of the KSD, several variants have been proposed to further reduce its complexity or/and improve its performance, e.g., [2]–[7].

Since the performance loss of KSD may be due to the likelihood of early discarding the ML solution, in this letter, we propose an improved KSD (IKSD) by replacing the strict value K in the conventional KSD with a hypersphere radius determined by the cost of Kth best node and a threshold Δ. The IKSD achieves the quasi-ML performance with a much lower complexity than the conventional KSD.

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S. Han and C. Tellambura are with the Department of Electrical and Computer Engineering, University of Alberta, Edmonton, AB, Canada T6G 2V4 (e-mail: {shan7, chintha}@ece.ualberta.ca).

T. Cui is with the Department of Electrical Engineering, California Institute of Technology, Pasadena, CA 91125, USA (e-mail: taocui@caltech.edu).

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Main Contributions:

1) An IKSD is proposed, which expands the fixed K nodes at each layer in the conventional KSD to a slightly bigger list, which includes all the nodes with a partial cost of f equal to or less than the Kth node cost f_K plus a small value Δ ($f \leq f_K + \Delta$). This Δ could be derived by off-line computation. The likelihood of discarding the ML solution is thus smaller than the conventional KSD.
2) Three specific IKSD are proposed in this letter with different choices of the threshold Δ (fixed threshold, normalized threshold and adaptive threshold IKSD). The parameter Δ controls the extra number of nodes visited by the IKSD. Furthermore, the closed-form expression of Δ is obtained for the normalized threshold.
3) By leveraging the IKSD, the soft extension of the IKSD for coded MIMO systems is also derived in this letter. This method increases the possibility of the candidate list including the ML point, and reduces the complexity with close performance to the conventional soft KSD detection [1].

The rest of this letter is organized as follows. Section II presents the new IKSD, introduces three specific IKSD, and discusses how to derive the threshold. The soft IKSD detection in coded MIMO systems is also proposed in this Section. Simulation results and discussions for both the performance and the complexity are given in Section III. Finally, conclusions are drawn in Section IV.

II. Improved K-best Sphere Decoder

A. Improved K-best SD

The proposed IKSD consists of a search through a small subset of the complete transmit constellation. The most important part of the algorithm is to determine the subset of the complete transmit constellation that needs to be searched. The IKSD is described in Algorithm 1.

When the initial sphere radius d is sufficiently large, the algorithm achieves its maximal complexity. When it is smaller, the complexity is reduced with the degradation in performance due to the lost lattice points outside the radius. In our simulation, we choose $d^2 = \gamma m \sigma_n^2$ [1], where $m = 2N$ (N is the number of transmit antennas), σ_n^2 is the noise variance, and $\gamma \geq 1$ is chosen to guarantee the lattice point can be captured.

In this letter, only the standard QR matrix decomposition is applied. The channel matrix ordering (e.g. [5]) is not included; however, it can improve the performance of the proposed IKSD.

For the tree search process, the conventional KSD sorts all the child nodes based on their partial costs, and selects the K

Algorithm 1: The IKSD Algorithm

Input : Δ, K, $\mathbf{z}$, $\mathbf{H}$, d
Output: $\hat{\mathbf{s}}$

1 Initial the sphere radius d and the partial cost $f_{best} = 0$, and take the root s_0 (level $k = m$) as the start node. ;
2 **for** $p \leftarrow 1$ **to** $length(f_{best})$ **do**
3   Expand the pth node, generate all its successors $\forall s \in \Omega$, and calculate the partial costs: $f_t = f_{best} + f_{k,t}$, where $f_{k,t} = (z_{k,p} - r_{k,k}s)^2$;
4 **end**
5 Sort all the components of f in an ascending order;
6 **if** *The number of the elements is less than* K **then**
7   Keep all the candidates with $f \leq d^2$ to obtain $\mathcal{T}$;
8 **else**
9   Only keep the elements whose cost indexes satisfy $f \leq f_K + \Delta$ in $\mathcal{T}$;
10 **end**
11 Replace the f_{best} to be the adjusted f;
12 **if** $k \neq 1$ **then** Calculate $\mathbf{z}_t = \mathbf{z}_t - R_{:,k}s_t$ $(\forall s_t \in \mathcal{T})$, $k = k - 1$ and go to step 2;
13 **else** Return the first element in $\mathcal{T}$ as the estimated $\hat{\mathbf{s}}$;

best paths. In the proposed IKSD, instead of choosing exactly K nodes, we keep the additional nodes whose costs are close to the cost of the Kth node, f_K. For example, at the ith level (where $i = 1, 2, \ldots, m$, $m = 2N$), supposing that the nodes are also sorted, if the cost difference between the Kth node and the $(K+r)$th node $(r = 1, 2 \ldots)$ is less than Δ, all $K+r$ nodes are retained.

B. Threshold Rules

The choice of Δ is the main challenge of the IKSD, If Δ is too large, then more nodes are visited and the complexity increases; while if Δ is too small, the performance improvement is limited compared to the conventional KSD. Depending on the parameterization of Δ, a flexible performance-complexity trade-off could be achieved. Based on different choices of the threshold Δ, three types of IKSD are proposed next.

1) ***Fixed Threshold IKSD:*** Intuitively, Δ could be a predefined constant, resulting in the fixed threshold IKSD. This choice is motivated by the fact that it is important to prune less aggressively in the early stage. A fixed Δ can perfectly serve this purpose. The value of Δ can be determined off-line through calculation, e.g., by the analysis in the Section II-C. For example, the proper value for the 4×4 16-QAM MIMO system with noise variance σ_n^2, Δ could be set to be $0.25\sigma_n^2$, which is obtained by both theoretical and numerical analysis.

2) ***Normalized Threshold IKSD:*** The threshold could be defined to be dependent on the cost of the Kth node at each level. From the theoretical analysis in the Section II-C, we will see that this will correspond to reducing the probability of pruning the true solution by a constant ratio compared to the KSD. Thus, the threshold could be given as

$$\Delta = \tau f_K. \tag{1}$$

This is called normalized threshold IKSD, which adaptively updates Δ in the searching process. The closed-form of Δ and τ will be derived in the Section II-C.

3) ***Adaptive Threshold IKSD:*** If the signal-to-noise ratio (SNR) is known or could be estimated, SNR-dependent Δ may be defined as

$$\Delta = \frac{\sigma_n^2}{\ln \rho + 1}, \tag{2}$$

where σ_n^2 is the noise variance and ρ is the SNR in the MIMO system. With this adaptive threshold IKSD, Δ decreases with increasing SNR. The motivation of this threshold choice lies in the fact that the cumulative costs are larger in the low SNR region while they are smaller in the high SNR region. Therefore, a large Δ should be chosen in the former case while a small value for the latter case.

Other choices of the threshold may be possible. However, all the proposed threshold rules reduce the probability of early dropping the ML solution when traversing the search tree, resulting in performance gains compared to the conventional KSD with the same value of K. Furthermore, the proposed IKSD with K outperforms the KSD with a larger K, while the former also obtains lower complexity than the latter, which will be shown in Section III.

C. Theoretical Analysis

Since the elements $n_1, \ldots, n_m$ in the noise vector $\mathbf{n}$ are values from independent identical distributed Gaussian, $\sum_{i=k}^{m} n_i^2$ becomes the chi-square random variable with $m-k+1$ degrees of freedom. Because $f_t = \sum_{i=k}^{m}\left(z_i - \sum_{j=i}^{m} r_{i,j}s_j\right)^2 = \sum_{i=k}^{m} n_i^2$, the probability of the new cost of nodes greater than the Kth node cost is

$$P_K = Pr(f_t > f_K) = 1 - Pr(f_t \leq f_K) = 1 - F(f_K; m-k+1), \tag{3}$$

where $F(f_K; m-k+1) = \gamma(\frac{m-k+1}{2}, \frac{f_K}{2})\Gamma(\frac{m-k+1}{2})$ is the cumulative distribution function (CDF) of f_t, and $\gamma(k, x)$ and $\Gamma(k)$ are incomplete Gamma function and Gamma function, respectively.

In order to reduce the probability of discarding the ML solution, we can decrease the probability in (3) by a predefined ratio λ $(0 < \lambda < 1)$, which is given as

$$P_\Delta = Pr(f_t > f_K + \Delta) = \lambda P_K. \tag{4}$$

Where λ could be set to be a number close to 1 in order to constrain the incremental complexity, such as $\lambda = 0.9$.

Therefore, the probability of $f_t \leq f_K + \Delta$ is $1 - \lambda P_K$. Thus, Δ can be defined as

$$\Delta = F^{-1}(1 - \lambda P_K; m-k+1) - f_K. \tag{5}$$

For the fixed threshold IKSD, Δ could be predefined to be a deterministic value according to the above equation. By calculating the values of Δ, we found an interesting result. For example in a 4×4 MIMO system, when $\lambda = 0.9$, Δ is always between 0.2 to 0.3 for all $1, \ldots, m$ degrees of freedom, calculated by Eq. (5). Thus, it is appropriate to choose $\Delta = 0.25$ for a 4×4 MIMO system. Similarly, a proper fixed threshold could also be derived by this simple off-line calculation for other MIMO systems.

For the normalized threshold IKSD, based on (1) and (5), τ is shown as

$$\tau = \frac{F^{-1}(1-\lambda P_K; m-k+1) - f_K}{f_K}. \tag{6}$$

When SNR is sufficiently high, P_K can be approximated to be

$$\lim_{\sigma_n^2 \to 0} P_K = \lim_{\sigma_n^2 \to 0} 1 - F(f_K; m-k+1). \tag{7}$$

When $x \to 0$, the probability density function of the chi-squared distribution is

$$f(x;k) = \frac{1}{2^{\frac{k}{2}}\Gamma(\frac{k}{2})} x^{\frac{k}{2}-1}\exp(-x/2) \approx \frac{1}{2^{\frac{k}{2}}\Gamma(\frac{k}{2})} x^{\frac{k}{2}-1}. \tag{8}$$

Then, the CDF $F(x;k)$ is

$$F(x;k) = \int_0^x \frac{1}{2^{k/2}\Gamma(\frac{k}{2})} x^{k/2-1} dx = \frac{x^{k/2}}{(k/2)2^{k/2}\Gamma(\frac{k}{2})} \tag{9}$$

and $F^{-1}(P;k) = \left(\frac{k}{2}2^{\frac{k}{2}}\Gamma(\frac{k}{2})P\right)^{\frac{2}{k}}$. Therefore, in the high SNR region, (6) could be derived by the closed-form in (10).

D. Soft Extension of the IKSD

For coded MIMO systems, the conventional KSD supports soft outputs [1], where the best K nodes left at last iteration form the candidate list used by the iterative detection and decoding. However, the conventional KSD in coded MIMO systems results in an increasing complexity in order to achieve the near optimal performance by a sufficient large K. Therefore, we propose the list IKSD by extending the proposed IKSD as a list sphere decoder for coded MIMO systems.

The list IKSD generates a list $\mathcal{L}$ of $N_{\mathcal{I}}$ candidates when searching the tree. This list includes $N_{\mathcal{I}} = K + N_{\Delta}$ estimates, and the size of the list satisfies $1 \leq N_{\mathcal{I}} < 2^{N_c \cdot N}$, where $N_c = \log_2(|\mathcal{Q}|)$ is the number of bits per modulated symbol and N_{Δ} is the number of extra nodes visited by the list IKSD compared to the list KSD. The coded spatial multiplexing MIMO system model and the detail of the MIMO detector and the channel decoder are referred to [8].

III. Simulation Results and Discussions

A. MIMO detection

In this section, the performance and complexity of the IKSD (Algorithm 1) are assessed. Both the symbol error rate (SER) and the average number of nodes (complexity) visited by the new IKSD are compared with those of the conventional KSD [1]. Although the three versions of the IKSD outperform the conventional KSD, in this letter, only the fixed threshold IKSD and the normalized threshold IKSD are shown due to the space limitation. The ML curve is from the conventional SD. In order to compare with the KSD fairly, the initial radius for both the proposed IKSD and KSD is chosen to be the same ($\gamma = 10$). Furthermore, in order to highlight the advantage of the proposed IKSD, the channel detection ordering is not included for all the algorithms.

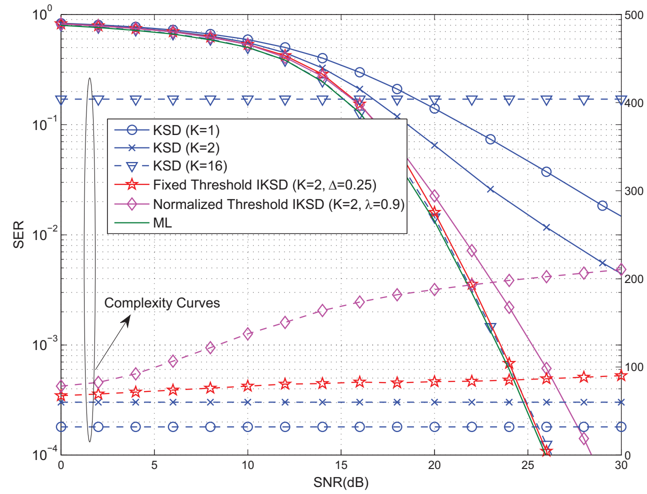

Fig. 1. Comparison of the IKSD and KSD for an uncoded 4×4 MIMO 16-QAM system.

Fig. 1 (the left axis) firstly shows the impact of the SER performance of the proposed IKSD. An uncoded 4×4 MIMO system with 16-QAM is simulated over a flat Rayleigh fading channel ($\sigma_n^2 = 1$). Note that the performance of the IKSD by the fixed threshold ($K = 2, \Delta = 0.25$) is so close to the ML curve; while the conventional KSD needs to set $K = 16$ for achieving the similar SER. Furthermore, the fixed threshold IKSD outperforms the normalized threshold ($K = 2, \lambda = 0.9$).

We also provide a complexity comparison between the IKSD and the KSD in the right axis. The complexity of the proposed fixed threshold IKSD is lower than that of the KSD when achieving the quasi-ML performance. For example, the conventional KSD ($K = 16$) searches about 4×10^2 nodes, while the fixed threshold IKSD only needs 80 nodes visited on average – an 80% complexity saving. Moreover, for $K = 2$, with 30% increase in complexity, the fixed threshold IKSD provides $7\,\mathrm{dB}$ gain (at an SER of 10^{-2}) over the KSD. Note that as expected the complexity curves for the conventional KSD are flat as a function of SNR; similarly, the fixed threshold IKSD has a virtually flat complexity curve. To quantify such flatness, a complexity variability index has been introduced in [9]. This index is 7×10^{-3}, affirming that the fixed threshold IKSD has a virtually constant complexity.

$$\begin{aligned} \tau &= \frac{F^{-1}\left[1-\lambda\left(1-\frac{(f_K)^{(m-k+1)/2}}{((m-k+1)/2)2^{(m-k+1)/2}\Gamma((m-k+1)/2)}\right); m-k+1\right] - f_K}{f_K} \\ &= \frac{\left[\frac{(m-k+1)}{2}2^{(m-k+1)/2}\Gamma((m-k+1)/2)\left(1-\lambda\left(1-\frac{(f_K)^{(m-k+1)/2}}{((m-k+1)/2)2^{(m-k+1)/2}\Gamma((m-k+1)/2)}\right)\right)\right]^{\frac{2}{k}} - f_K}{f_K}. \end{aligned} \tag{10}$$

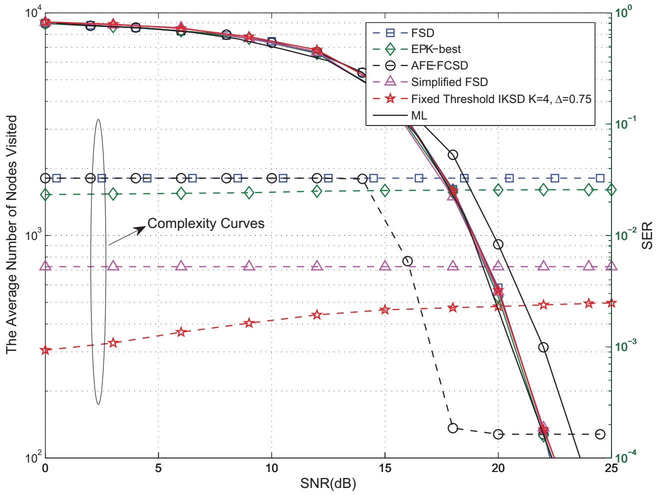

Fig. 2. Comparison of different SD algorithms for an uncoded 8×8 MIMO 16-QAM system.

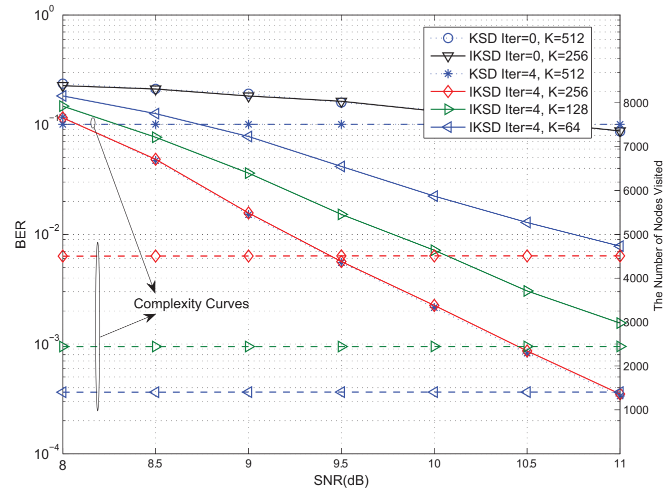

Fig. 3. Comparison of the IKSD and conventional KSD for a coded 4×4 MIMO 16-QAM system.

In order to show the advantages of the proposed IKSD, the comparison with other algorithms (FSD [5], EP K-best [4], AFE-FCSD [7], and simplified FSD [6]) is shown in Fig. 2. Achieving the near-optimal SER performance, the complexity of our proposed IKSD is only 23%, 27% and 59.5% of that of the FSD, EP K-best and simplified FSD, respectively. Although the AFE-FCSD obtains lower complexity than the proposed IKSD when $\mathrm{SNR} \geq 18\,\mathrm{dB}$, the latter gains $1.5\,\mathrm{dB}$ than the former at an SER of 10^{-4}. Above all, the proposed IKSD achieves the best trade-off between performance and complexity among all these algorithms.

B. Detection for Coded MIMO systems

We next assess the advantages of the IKSD in a 4×4 coded MIMO system. The performance measured by the bit error rate (BER), and the complexity of generating the candidate list are investigated. The systematic recursive convolutional code with rate $R = 1/2$ is exploited to encode the transmitted bits sequence with the frame length 8192, where the feed-forward and feedback-generating polynomials are $G_1(D) = 1 + D^2$ and $G_2(D) = 1 + D + D^2$ with memory length 2 [8], respectively. A random interleaver is exploited here.

In order to show the effects of K, the performance and complexity for different K are investigated in Fig. 3. By increasing K, more nodes are visited in the searching process, resulting in an increasing complexity of the iterative detection and decoding. However, the BER performance improves when K is larger. As shown in the left axis, by using 4 maximum iterations, the proposed list IKSD with $K = 256$ achieves the performance of the conventional KSD with $K = 512$.

As shown in the right axis of Fig. 3, when K decreases, the complexity degrades more. For example, the average number of nodes visited is around 4.5×10^3 with $K = 256$, approximately 2.4×10^3 with $K = 128$ and about 1.4×10^3 with $K = 64$, respectively. However, the conventional list KSD visits about 7.5×10^3 nodes with $K = 512$. Considering the performance and the complexity, the proposed list IKSD gains 40% complexity savings with the same performance.

IV. CONCLUSIONS

This letter proposed an improved K-best sphere decoder (IKSD), which achieves the quasi-ML performance at a reduced and roughly fixed complexity. Unlike the conventional KSD which retains a fixed number of K nodes per level, our main idea expands this number to all the nodes whose cost is less than $f_K + \Delta$. The conventional KSD is thus a special case when $\Delta = 0$. The motivation of keeping additional nodes is to reduce the likelihood of the conventional KSD to discard the ML solution early. For coded MIMO systems, a soft extension of the IKSD was developed as the list IKSD. It uses the IKSD to generate the candidate list for joint iterative detection and decoding, resulting in complexity savings over the conventional list KSD.

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