On the Application of Character Expansions for MIMO Capacity Analysis

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***Abstract*—To evaluate the unitary integrals, such as the well-known Harish–Chandra–Itzykson–Zuber integral, character expansions were developed by Balantekin, where the matrix integrand is a group member; i.e., a square matrix with a nonzero determinant. Recently, this method has been exploited to derive the joint eigenvalue distributions of the Wishart matrices; i.e., $\mathbf{HH}^*$ where $\mathbf{H}$ is the complex Gaussian random channel matrix of a multiple-input multiple-output (MIMO) system. The joint eigenvalue distributions are used to calculate the moment generating function of the mutual information (ergodic capacity) of a MIMO channel. In this paper, we show that the previous integration framework presented in the literature is not correct, and results in incorrect joint eigenvalue distributions for the Ricean and full-correlated Rayleigh MIMO channels. We develop a new framework to apply the character expansions for integrations over the unitary group, involving general rectangular complex matrices in the integrand. We derive the correct distribution functions and use them to obtain the capacity of the Ricean and correlated Rayleigh MIMO systems in a unified and straightforward approach. The integration technique proposed in this paper is general enough to be used for other unitary integrals in engineering, mathematics, and physics.**

***Index Terms*—Character expansion, Gaussian random matrix, group representation, MIMO capacity, unitary integration, Wishart matrix.**

I. Introduction

MULTIPLE-INPUT multiple-output (MIMO) systems, which deploy antenna arrays at both the transmitter and receiver, provide high-capacity and high-quality wireless communication links [1], [2]. MIMO systems have been investigated from a variety of aspects including the ergodic capacity [1] and the outage probability [3], by exact or asymptotic analysis [4], [5]. In the case of exact analysis, several results regarding the distribution of the channel matrix have been presented in the literature. It is shown that in an independent and identically distributed (i.i.d.) Rayleigh fading channel, the capacity of a MIMO system with N_t transmit antennas and N_r receive antennas scales almost linearly with the $\min(N_t, N_r)$ in the high signal-to-noise ratio (SNR) regime [1].

The capacity of MIMO systems is commonly analyzed by using the moment generating function (MGF) of the mutual information between the transmitter and receiver, for various assumptions about the statistics of the channel matrix. The first derivative of the MGF yields the ergodic capacity, and the probability of outage can be derived through a simple numerical integral [6]. The outage mutual information for Gaussian uncorrelated channels, obtained by using the MGF, is presented in [3], and the capacity of MIMO systems can be found in [7] when the channel is Ricean, and in [8] and [9] when the channel is semi-correlated Rayleigh; i.e., either the transmit antennas or the receive antennas are correlated. The case that the number of correlated antennas is less than or equal to the number of uncorrelated antennas is analyzed in [8], and the opposite case in [9]. All these works are based on the available results in the theory of Wishart random matrices ($\mathbf{HH}^*$ is a Wishart matrix when $\mathbf{H}$ is a complex Gaussian matrix). The joint probability density function (pdf) of the eigenvalues of the Wishart matrix, derived in [10] in the form of a hypergeometric function with matrix arguments, is typically used in the literature to obtain the MGF of the mutual information.

Recently, the character expansion method, introduced by Balantekin [11], has been used in [12] for the capacity analysis of the full-correlated[1] Rayleigh MIMO channel when $N_r = N_t$, and in [13] for the capacity analysis of the Ricean, and semi-correlated[2] and full-correlated Rayleigh MIMO channels with arbitrary numbers of transmit and receive antennas. These studies use character expansions to calculate the integrals over the unitary group, which are essential for obtaining the joint eigenvalue distributions of Wishart matrices [14]. In fact, unitary integrals have many applications in physics [15], [16]–[17], mathematics [18], and engineering [19]. In 1984, Balantekin presented a combinatorial formula for the character expansions of the $\mathcal{U}(N)$ group (the group of unitary matrices with dimension N) [20], and later he generalized those results in [11] and used the character expansions to simplify the integrations over the unitary group, in particular, to derive the well-known Harish–Chandra–Itzykson–Zuber integral [18], [21]. The integration steps presented by Balantekin [11] are as follows:

1) Expansion of the integrand by using the character expansion method.
2) Integration over unitary matrices by using the available results on the unitary group.
3) Re-summation of the expansion by using the Cauchy–Binet formula.

[1]Both the transmit and receive antennas are correlated.

[2]Only the transmit or the receive antennas are correlated.

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In Balantekin's work, the coefficient matrices appearing in the integrand are nonzero-determinant square matrices. However, when the channel matrix is non-square, integrals appear over unitary matrices with rectangular coefficient matrices ($M \times N, M \leqslant N$). To handle this problem, the following integration steps are proposed in [13]:

1) Assume $M = N$ so that the matrix integrand is a group member.
2) Apply Balantekin's three-step method to calculate the unitary integrals.
3) Find the limit of the final result when $N - M$ eigenvalues approach zero.

This integration method has been used in [13] to derive the joint pdf of the eigenvalues for the Ricean and semi-correlated Rayleigh channels when $M \leqslant N$, and the full-correlated Rayleigh channel when $M = N$. However, the joint eigenvalue distribution of the Ricean case is incorrect [7], [17]. In fact, it was observed in [17] (without providing any solution) that the above integration method produces incorrect results when applied to multiple unitary integrals with unequal dimensions (e.g., the Ricean and full-correlated Rayleigh MIMO channels when $M < N$).

In this paper, we first briefly introduce the MIMO system model, capacity formula, and character expansions of groups. By applying the character expansion method, we calculate four useful unitary integrals to develop our main tools for unitary integrations with rectangular complex coefficient matrices in the integrand. These results are the generalizations of the classical unitary integrals so that the coefficient matrices are not restricted to Hermitian, positive definite, diagonal and/or real matrices. We use the results of the unitary integrals to derive the joint eigenvalue distributions of the Ricean and full-correlated Rayleigh MIMO channels with arbitrary numbers of transmit and receive antennas. The joint eigenvalue distribution of the full-correlated non-square MIMO channel is a new result in random matrix theory [22]. By employing the derived eigenvalue distributions, we calculate the MGF of the mutual information and the capacity for the Ricean and full-correlated Rayleigh MIMO channels.

To make the paper self-content, we provide some examples of applying the integration method proposed in [13] to compare with the results in this paper, and justify the observation in [17]. In particular, we show that the MGF of the mutual information (and consequently the capacity) for the full-correlated Rayleigh MIMO channel is correctly obtained in [13] (Remark 3).

The integration steps along with the auxiliary propositions and lemmas presented in this paper, enabling the derivation of the joint eigenvalue distributions and MIMO capacity in a unified approach, are the main contributions of this paper. Furthermore, the proposed integration framework is a powerful and straightforward tool for evaluating other unitary integrals in communications, mathematics, and physics.

II. MIMO System Model and Capacity

Consider a narrow-band, flat-fading communication system with N_t transmit and N_r receive antennas ($\mathrm{MIMO}(N_t, N_r)$). The linear transformation between the transmit and receive antennas can be modeled as

$$\mathbf{y} = \sqrt{\rho}\,\mathbf{H}\mathbf{s} + \mathbf{n} \tag{1}$$

where $\mathbf{y} \in \mathcal{C}^{N_r}$ is the complex received vector, $\mathbf{s} \in \mathcal{C}^{N_t}$ is the transmitted vector, $\mathbf{n} \in \mathcal{C}^{N_r}$ is the additive noise, and $\mathbf{H} \in \mathcal{C}^{N_r \times N_t}$ is the channel matrix. To obtain the capacity, we assume the entries of both vectors $\mathbf{s}$ and $\mathbf{n}$ are i.i.d. complex Gaussian random variables with zero mean and unit variance, $\mathcal{CN}(0,1)$. Thus, $\mathrm{E}\{\mathbf{s}\mathbf{s}^*\} = \mathbf{I}$, where $\mathrm{E}\{\cdot\}$ and $(\cdot)^*$ denote the expectation and Hermitian (transpose conjugate), and $\mathbf{I}$ is the identity matrix. Consequently, ρ will be the average transmitted power at each signaling interval from each antenna.

Assuming that the channel matrix is known to the receiver only, the mutual information between the transmitter and receiver is obtained by

$$\mathcal{I} = \log\Big(\det\left[\mathbf{I} + \rho\mathbf{H}\mathbf{H}^*\right]\Big) \tag{2}$$

where $\log(\cdot)$ denotes the natural logarithm. By defining the MGF of $\mathcal{I}$ as

$$g(z) = \mathrm{E}\left\{\mathrm{e}^{z\mathcal{I}}\right\} = \mathrm{E}\Big\{\det\left[\mathbf{I} + \rho\mathbf{H}\mathbf{H}^*\right]^z\Big\} \tag{3}$$

and assuming that the channel matrix is updated for each transmission (fast fading), the ergodic capacity of the system is obtained by direct differentiation of $g(z)$:

$$C = \mathrm{E}\{\mathcal{I}\} = g'(0). \tag{4}$$

The generating function can be written in terms of the eigenvalues $\{\lambda_i\}$ of the matrix $\mathbf{H}\mathbf{H}^*$ as

$$\begin{aligned} g(z) &= \mathrm{E}\left\{\prod_{i=1}^{M}(1+\rho\lambda_i)^z\right\} \\ &= \prod_{i=1}^{M}\int_0^\infty d\lambda_i\,(1+\rho\lambda_i)^z\,P(\boldsymbol{\lambda}) \end{aligned} \tag{5}$$

where $M = \min\{N_t, N_r\}$, and $P(\boldsymbol{\lambda} = \{\lambda_1, \ldots, \lambda_M\})$ is the joint pdf of the M nonzero eigenvalues of $\mathbf{H}\mathbf{H}^*$. Assuming the singular value decomposition of $\mathbf{H}$ as $\mathbf{H} = \mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^*$, where $\mathbf{U} \in \mathcal{U}(N_r)$, $\mathbf{V} \in \mathcal{U}(N_t)$, and $\boldsymbol{\Sigma} = \mathrm{diag}(\{\sqrt{\lambda_i}\}) \in \mathcal{R}_+^{N_r \times N_t}$, it is shown that [14]

$$P(\boldsymbol{\lambda}) = \mathcal{K}_{M,N}\underset{M}{\Delta}(\boldsymbol{\lambda})^2\prod_{i=1}^{M}\lambda_i^{N-M}\int \mathrm{D}\mathbf{V}\int \mathrm{D}\mathbf{U}\,p(\mathbf{H}) \tag{6}$$

where $N = \max\{N_t, N_r\}$, the integrals are over all unitary matrices $\mathbf{U}$ and $\mathbf{V}$, and $\mathrm{D}\mathbf{U}$ denotes the standard Haar measure of $\mathcal{U}(N_r)$ [23]. In addition, $p(\mathbf{H})$ is the joint pdf of the elements of $\mathbf{H}$,

$$\mathcal{K}_{M,N}^{-1} = M!\prod_{j=1}^{M}(N-j)!\,(M-j)! \tag{7}$$

and

$$\underset{M}{\Delta}(\boldsymbol{\lambda}) = \Delta(\lambda_1, \ldots, \lambda_M) = \underset{M}{\det}\left[\lambda_i^{M-j}\right] = \prod_{i<j}(\lambda_i - \lambda_j) \quad (8)$$

is the Vandermonde determinant of vector $\boldsymbol{\lambda}$. ($\underset{M}{\det}[f(i,j)]$ denotes the determinant of a matrix with the (i,j)th element given by $f(i,j)$, and the subscript under the determinant is the dimension of the argument matrix.)

Accordingly, for different channel statistics, the corresponding $p(\mathbf{H})$ should be inserted into (6), and the integrals over the unitary matrices should be calculated. The resulting $P(\boldsymbol{\lambda})$ can then be used to obtain the MGF of the mutual information, $g(z)$, and, consequently, the capacity.

III. Character Expansions of Groups

The group of unitary matrices $\mathcal{U}(N)$ is a subgroup of the group of complex invertible matrices, $\mathrm{GL}(N,\mathcal{C})$. A d–dimensional representation of the group $\mathrm{GL}(N,\mathcal{C})$ is a homomorphism from $\mathrm{GL}(N,\mathcal{C})$ into the $\mathrm{GL}(d,\mathcal{C})$. A d–dimensional representation of $\mathrm{GL}(N,\mathcal{C})$ is irreducible if it has no nontrivial invariant subspaces. The irreducible representations of $\mathrm{GL}(N,\mathcal{C})$ can be labeled by the N–dimensional ordered sets $\mathbf{r}_N = \{r_1, r_2, \ldots, r_N\}$, where $r_1 \geqslant r_2 \geqslant \cdots \geqslant r_N \geqslant 0$ are integers. The dimension $d_{\mathbf{r}_N}$ of the irreducible representation $\mathbf{r}_N$ is given by [24]

$$d_{\mathbf{r}_N} = \left[\prod_{i=1}^{N} \frac{(r_i + N - i)!}{(N-i)!}\right] \underset{N}{\det}\left[\frac{1}{(r_i - i + j)!}\right] \quad (9)$$

where the matrix elements inside the determinant with $r_i - i + j < 0$ are zero. Another useful formula for $d_{\mathbf{r}_N}$ is [13]

$$d_{\mathbf{r}_N} = \frac{\underset{N}{\Delta}(\mathbf{k})}{\prod_{i=1}^{N}(N-i)!} \quad (10)$$

where $k_i = r_i + N - i$ for $i = 1, \ldots, N$.

The character of a group element $\mathbf{X} \in \mathrm{GL}(N,\mathcal{C})$ in its representation $\mathbf{r}_N$ is defined by Weyl's character formula as [25]

$$\chi_{\mathbf{r}_N}(\mathbf{X}) = \mathrm{tr}\left\{\mathbf{X}^{(\mathbf{r}_N)}\right\} = \frac{\underset{N}{\det}\left[x_i^{r_j+N-j}\right]}{\underset{N}{\Delta}(\mathbf{x})} \quad (11)$$

where $\mathbf{X}^{(\mathbf{r}_N)}$ denotes the representation matrix of $\mathbf{X}$, and $\mathbf{x} = (x_1, \ldots, x_N)^T$ are the eigenvalues of $\mathbf{X}$. In this case, the following equation holds for $\mathbf{X}$ [11]:

$$\mathrm{etr}\{\mathbf{X}\} = \sum_{\mathbf{r}_N} \alpha_{\mathbf{r}_N} \chi_{\mathbf{r}_N}(\mathbf{X}) \quad (12)$$

where $\mathrm{etr}\{\mathbf{X}\} = \exp\left(\mathrm{tr}\{\mathbf{X}\}\right)$, the summation is over all irreducible representations of $\mathrm{GL}(N,\mathcal{C})$, and the expansion coefficient $\alpha_{\mathbf{r}_N}$ is defined as

$$\alpha_{\mathbf{r}_N} = \underset{N}{\det}\left[\frac{1}{(r_i - i + j)!}\right] = \left[\prod_{i=1}^{N} \frac{(N-i)!}{(r_i + N - i)!}\right] d_{\mathbf{r}_N}. \quad (13)$$

Lemma 1: The orthogonality relation between the unitary group matrix elements implies that [23]

$$\int \mathrm{D}\mathbf{U}\, U_{ij}^{(\mathbf{r}_N)} U_{kl}^{(\mathbf{r}'_N)*} = \frac{1}{d_{\mathbf{r}_N}} \delta_{\mathbf{r}_N \mathbf{r}'_N} \delta_{ik}\, \delta_{jl} \quad (14)$$

where $U_{ij}^{(\mathbf{r}_N)}$ denotes the (i,j)th element of the representation matrix of $\mathbf{U}$, and $d_{\mathbf{r}_N}$ is the dimension of the representation. ▲

Proposition 1: Assume $\mathbf{A}, \mathbf{B} \in \mathrm{GL}(N,\mathcal{C})$, $\mathbf{U} \in \mathcal{U}(N)$, and $\mathbf{r}_N$ and $\mathbf{r}'_N$ are two representations of $\mathrm{GL}(N,\mathcal{C})$. Then, [26]

$$\int \mathrm{D}\mathbf{U}\, \chi_{\mathbf{r}_N}(\mathbf{U}\mathbf{A})\, \chi_{\mathbf{r}'_N}(\mathbf{U}^*\mathbf{B}) = \frac{1}{d_{\mathbf{r}_N}} \chi_{\mathbf{r}_N}(\mathbf{A}\mathbf{B})\, \delta_{\mathbf{r}_N \mathbf{r}'_N}. \quad ▲$$

Proposition 2: Assume $\mathbf{A}, \mathbf{B} \in \mathrm{GL}(N,\mathcal{C})$, $\mathbf{U} \in \mathcal{U}(N)$, and $\mathbf{r}_N$ is a representation of $\mathrm{GL}(N,\mathcal{C})$. Then, [26]

$$\int \mathrm{D}\mathbf{U}\, \chi_{\mathbf{r}_N}(\mathbf{U}\mathbf{A}\mathbf{U}^*\mathbf{B}) = \frac{1}{d_{\mathbf{r}_N}} \chi_{\mathbf{r}_N}(\mathbf{A})\, \chi_{\mathbf{r}_N}(\mathbf{B}). \quad ▲$$

IV. Unitary Integration by Character Expansions

To develop our main tools for unitary integration by character expansions, we calculate the following four unitary integrals in this section:

1) The unitary integral

$$\mathrm{I}_1^{M,N} = \int \mathrm{D}\mathbf{U}\, \mathrm{etr}\{\beta \mathbf{U}\mathbf{A}\mathbf{C}\mathbf{U}^*\mathbf{B}\} \quad (15)$$

where $\mathbf{U} \in \mathcal{U}(N)$, $\mathbf{B} \in \mathrm{GL}(N,\mathcal{C})$, $\mathbf{A} \in \mathcal{C}^{N\times M}$, and $\mathbf{C} \in \mathcal{C}^{M\times N}$ are complex matrices with rank M ($M \leqslant N$), and β is a complex scalar, is equal to

$$\mathrm{I}_1^{M,N} = \frac{\beta^{M\left(\frac{M+1}{2}-N\right)} \prod_{i=1}^{M}(N-i)!}{\underset{M}{\Delta}(\mathbf{a})\underset{N}{\Delta}(\mathbf{b})\underset{M}{\det}[\mathbf{C}\mathbf{A}]^{N-M}} \times \underset{N}{\det}\begin{bmatrix} \exp(\beta a_i b_j) \,\Big|_{M}^{i=1} \\ b_j^{N-i} \,\Big|_{N}^{i=M+1} \end{bmatrix} \quad (16)$$

where $\mathbf{a} \in \mathcal{C}^M$ and $\mathbf{b} \in \mathcal{C}^N$ represent the eigenvalues of the matrices $\mathbf{C}\mathbf{A}$ and $\mathbf{B}$, respectively.

2) The unitary integral

$$\mathrm{I}_2^{M,N} = \int \mathrm{D}\mathbf{U}\, \mathrm{etr}\{\beta\,(\mathbf{U}\mathbf{A}\mathbf{C} + \mathbf{U}^*\mathbf{B}\mathbf{D})\} \quad (17)$$

where $\mathbf{U} \in \mathcal{U}(N)$, $\mathbf{A} \in \mathcal{C}^{N\times P}$ and $\mathbf{C} \in \mathcal{C}^{P\times N}$ are complex matrices with rank P ($P \leqslant N$), $\mathbf{B} \in \mathcal{C}^{N\times Q}$ and $\mathbf{D} \in \mathcal{C}^{Q\times N}$ are complex matrices with rank Q ($Q \leqslant N$), and β is a complex scalar, is equal to

$$\mathrm{I}_2^{M,N} = \frac{\beta^{M\left(\frac{M+1}{2}-N\right)} \prod_{i=1}^{M}(N-i)!}{\underset{M}{\Delta}(\mathbf{a}^2)\prod_{i=1}^{M} a_i^{2(N-M)}} \times \underset{M}{\det}\left[a_i^{N-j}\, I_{N-j}(2\beta a_i)\right] \quad (18)$$

where $M = \min(P,Q)$, and $\mathbf{a}^2 \in \mathcal{C}^M$ represents the nonzero eigenvalues of the matrix $\mathbf{A}\mathbf{C}\mathbf{B}\mathbf{D}$.

3) The double unitary integral

$$\mathrm{I}_3^{M,N} = \int \mathrm{D}\mathbf{V} \int \mathrm{D}\mathbf{U}\, \mathrm{etr}\{\beta\,(\mathbf{UAV}^*\mathbf{B} + \mathbf{U}^*\mathbf{CVD})\} \quad (19)$$

where $\mathbf{U} \in \mathcal{U}(N)$, $\mathbf{V} \in \mathcal{U}(M)$, $\mathbf{A}, \mathbf{C} \in \mathcal{C}^{N\times M}$ and $\mathbf{B}, \mathbf{D} \in \mathcal{C}^{M\times N}$ are complex matrices with rank M ($M \leqslant N$), and β is a complex scalar, is equal to

$$\mathrm{I}_3^{M,N} = \frac{\beta^{M(1-N)} \prod_{i=1}^{M}(N-i)!(M-i)!}{\underset{M}{\Delta}(\mathbf{a}^2)\underset{M}{\Delta}(\mathbf{b}^2)\underset{M}{\det}\,[\mathbf{DABC}]^{\frac{N-M}{2}}} \times \underset{M}{\det}\,[I_{N-M}(2\beta a_i b_j)] \quad (20)$$

where $\mathbf{a}^2, \mathbf{b}^2 \in \mathcal{C}^M$ represent the eigenvalues of the matrices $\mathbf{DA}$ and $\mathbf{BC}$, respectively.

To our best knowledge, this integral was previously known only for the case that $\mathbf{D} = \mathbf{A}^*$ and $\mathbf{B} = \mathbf{C}^*$ [10], [17], [27].

4) The double unitary integral

$$\mathrm{I}_4^{M,N} = \int \mathrm{D}\mathbf{V} \int \mathrm{D}\mathbf{U}\, \mathrm{etr}\{\beta \mathbf{UAV}^*\mathbf{BVCU}^*\mathbf{D}\} \quad (21)$$

where $\mathbf{U} \in \mathcal{U}(N)$, $\mathbf{V} \in \mathcal{U}(M)$, $\mathbf{B} \in \mathrm{GL}(M, \mathcal{C})$, $\mathbf{D} \in \mathrm{GL}(N, \mathcal{C})$, $\mathbf{A} \in \mathcal{C}^{N\times M}$ and $\mathbf{C} \in \mathcal{C}^{M\times N}$ are complex matrices with rank M ($M \leqslant N$), and β is a complex scalar, is equal to

$$\mathrm{I}_4^{M,N} = \frac{\prod_{i=1}^{M}(N-i)!(M-i)!}{\underset{M}{\Delta}(\mathbf{a})\underset{M}{\Delta}(\mathbf{b})\underset{N}{\Delta}(\mathbf{c})\beta^{\frac{M(M-1)}{2}}} \times \sum_{\mathbf{k}_M} \frac{\underset{M}{\det}\left[a_i^{k_j}\right]\underset{M}{\det}\left[(\beta b_i)^{k_j}\right]}{\underset{M}{\Delta}(\mathbf{k})\prod_{j=1}^{M}(k_j+N-M)!} \times \underset{N}{\det}\left[c_i^{k_j+N-M}\Big|_{j=1}^{M}, c_i^{N-j}\Big|_{j=M+1}^{N}\right] \quad (22)$$

where $\mathbf{k}_M$ represents all irreducible representations of $\mathrm{GL}(M, \mathcal{C})$, and $\mathbf{a}, \mathbf{b} \in \mathcal{C}^M$ and $\mathbf{c} \in \mathcal{C}^N$ represent the eigenvalues of the matrices $\mathbf{CA}$, $\mathbf{B}$, and $\mathbf{D}$, respectively. To our best knowledge, this integral has been solved in the literature only when $N = M$, $\mathbf{A} = \mathbf{C}^*$, and $\mathbf{B}$ and $\mathbf{D}$ are positive definite Hermitian matrices [12], [13].

A. Calculation of I_1

We start with the Harish–Chandra–Itzykson–Zuber integral [18], [21], defined as

$$\mathrm{I}_1^N = \int \mathrm{D}\mathbf{U}\, \mathrm{etr}\{\beta \mathbf{UAU}^*\mathbf{B}\} \quad (23)$$

where $\mathbf{U} \in \mathcal{U}(N)$, $\mathbf{A}, \mathbf{B} \in \mathrm{GL}(N, \mathcal{C})$, and $\beta \in \mathcal{C}$ is a scalar. By absorbing β into $\mathbf{B}$ and applying the character expansion formula (12), we obtain

$$\mathrm{J}_1^N = \int \mathrm{D}\mathbf{U}\, \mathrm{etr}\{\mathbf{UAU}^*\mathbf{B}\} \quad (24)$$

$$= \sum_{\mathbf{r}_N} \alpha_{\mathbf{r}_N} \int \mathrm{D}\mathbf{U}\, \chi_{\mathbf{r}_N}(\mathbf{UAU}^*\mathbf{B})$$

$$= \sum_{\mathbf{r}_N} \frac{\alpha_{\mathbf{r}_N}}{d_{\mathbf{r}_N}} \chi_{\mathbf{r}_N}(\mathbf{A})\, \chi_{\mathbf{r}_N}(\mathbf{B}) \quad (25)$$

$$= \sum_{\mathbf{r}_N} \left[\prod_{i=1}^{N} \frac{(N-i)!}{(r_i+N-i)!}\right] \frac{\underset{N}{\det}\left[a_i^{r_j+N-j}\right]}{\underset{N}{\Delta}(\mathbf{a})} \frac{\underset{N}{\det}\left[b_i^{r_j+N-j}\right]}{\underset{N}{\Delta}(\mathbf{b})} \quad (26)$$

where (25) is achieved by applying Proposition 2, and (26) is the result of substituting $\alpha_{\mathbf{r}_N}$ from (13) and the characters from (11). The vectors $\mathbf{a} = (a_1, a_2, \ldots, a_N)^T$ and $\mathbf{b} = (b_1, b_2, \ldots, b_N)^T$ represent the eigenvalues of the matrices $\mathbf{A}$ and $\mathbf{B}$, respectively. Thus,

$$\mathrm{J}_1^N = \frac{\prod_{i=1}^{N}(N-i)!}{\underset{N}{\Delta}(\mathbf{a})\underset{N}{\Delta}(\mathbf{b})} \sum_{\mathbf{k}_N} \underset{N}{\det}\left[a_i^{k_j}\right]\underset{N}{\det}\left[b_i^{k_j}\right] \prod_{j=1}^{N} \frac{1}{k_j!} \quad (27)$$

where $k_j = r_j + N - j$. Considering the fact that

$$\exp(z) = \sum_{k=0}^{\infty} \frac{1}{k!} z^k$$

and by applying the Cauchy–Binet formula (Lemma 3 in Appendix A), we have

$$\mathrm{J}_1^N = \frac{\prod_{i=1}^{N}(N-i)!}{\underset{N}{\Delta}(\mathbf{a})\underset{N}{\Delta}(\mathbf{b})} \underset{N}{\det}[\exp(a_i b_j)]. \quad (28)$$

By replacing $\mathbf{B}$ by $\beta\mathbf{B}$ or, equivalently, replacing $\mathbf{b}$ by $\beta\mathbf{b}$ in (28), we conclude that

$$\mathrm{I}_1^N = \frac{\beta^{-\frac{N(N-1)}{2}} \prod_{i=1}^{N}(N-i)!}{\underset{N}{\Delta}(\mathbf{a})\underset{N}{\Delta}(\mathbf{b})} \underset{N}{\det}[\exp(\beta a_i b_j)]. \quad (29)$$

Now, assume $N - M$ eigenvalues of matrix $\mathbf{A}$ in (24) approach zero. To find

$$\mathrm{J}_1^{M,N} = \lim_{\{a_{M+1},\ldots,a_N\}\longrightarrow 0} \mathrm{J}_1^N$$

one can directly apply Lemma 5 (Appendix B) to (28) to obtain (35) [13], [17]. On the other hand, from (25), we have

$$\mathrm{J}_1^{M,N} = \sum_{\mathbf{r}_N} \frac{\alpha_{\mathbf{r}_N}}{d_{\mathbf{r}_N}} \chi_{\mathbf{r}_N}(\mathbf{B}) \lim_{\{a_{M+1},\ldots,a_N\}\longrightarrow 0} \chi_{\mathbf{r}_N}(\mathbf{A}). \quad (30)$$

Calculation of (30) requires the following results:

Proposition 3: Assume $\mathbf{A} \in \mathrm{GL}(N, \mathcal{C})$ with eigenvalues $\{a_i\}$, $i = 1, \ldots, N$, and $\mathbf{r}_N$ is a representation of $\mathrm{GL}(N, \mathcal{C})$. Then,

$$\lim_{\{a_{M+1},\ldots,a_N\}\to 0} \chi_{\mathbf{r}_N}(\mathbf{A}) = \begin{cases} \chi_{\mathbf{r}_M}\left(\widehat{\mathbf{A}}\right), & \text{if } r_{M+1} = \cdots = r_N = 0; \\ 0, & \text{otherwise;} \end{cases}$$

where $\hat{\mathbf{A}} \in \mathrm{GL}(M,\mathcal{C})$ is the matrix with eigenvalues $\{a_i\}$, $i = 1,\dots,M$, and $\mathbf{r}_M$ is a representation of $\mathrm{GL}(M,\mathcal{C})$. (See Appendix C for proof.) ▲

According to Proposition 3, the summation terms in (30) are nonzero only if $\mathbf{r}_N = \mathbf{r}_{M,N}$, where $\mathbf{r}_{M,N} \triangleq \{\mathbf{r}_M, 0, 0, \dots, 0\}$.

Proposition 4: Assume $\mathbf{r}_M$ and $\mathbf{r}_{M,N} = \{\mathbf{r}_M, 0, 0, \dots, 0\}$ are irreducible representations of $\mathrm{GL}(M,\mathcal{C})$ and $\mathrm{GL}(N,\mathcal{C})$, respectively. Then,

$$\alpha_{\mathbf{r}_{M,N}} = \alpha_{\mathbf{r}_M} \tag{31}$$

$$d_{\mathbf{r}_{M,N}} = d_{\mathbf{r}_M} \prod_{i=1}^{M} \left[\frac{(M-i)!}{(N-i)!} \times \frac{(r_i+N-i)!}{(r_i+M-i)!}\right] \tag{32}$$

$$\frac{\alpha_{\mathbf{r}_{M,N}}}{d_{\mathbf{r}_{M,N}}} = \prod_{i=1}^{M} \frac{(N-i)!}{(r_i+N-i)!} \tag{33}$$

where $\alpha_{\mathbf{r}_{M,N}}$ and $\alpha_{\mathbf{r}_M}$ are the corresponding expansion coefficients defined in (13), and $d_{\mathbf{r}_{M,N}}$ and $d_{\mathbf{r}_M}$ are the corresponding dimensions of the representations defined in (9). (See Appendix D for proof.) ▲

Applying Propositions 3 and 4 to (30) gives

$$\begin{aligned} \mathrm{J}_1^{M,N} &= \sum_{\mathbf{r}_M} \frac{\alpha_{\mathbf{r}_M}}{d_{\mathbf{r}_{M,N}}} \chi_{\mathbf{r}_M}\left(\hat{\mathbf{A}}\right) \chi_{\mathbf{r}_{M,N}}(\mathbf{B}) \qquad (34)\\ &= \frac{\prod_{i=1}^{M}(N-i)!}{\underset{M}{\Delta}(\mathbf{a})\underset{N}{\Delta}(\mathbf{b})} \\ &\quad \times \sum_{\mathbf{k}_M} \det_M\left[a_i^{k_j}\right] \det_N\left[b_i^{k_j+N-M}\Big|_{j=1}^{M}, b_i^{N-j}\Big|_{j=M+1}^{N}\right] \\ &\quad \times \prod_{j=1}^{M} \frac{1}{(k_j+N-M)!} \end{aligned}$$

where $k_j = r_j + M - j$. By considering the power series expansion of $\exp(z)$, and applying the generalized Cauchy–Binet formula (Lemma 4 in Appendix A), we obtain

$$\mathrm{J}_1^{M,N} = \frac{\prod_{i=1}^{M}(N-i)!}{\underset{M}{\Delta}(\mathbf{a})\underset{N}{\Delta}(\mathbf{b})\prod_{i=1}^{M} a_i^{N-M}} \det_N\begin{bmatrix} \exp(a_i b_j) \Big|_{M}^{i=1} \\ b_j^{N-i} \Big|_{N}^{i=M+1}\end{bmatrix}. \tag{35}$$

Replacing $\mathbf{B}$ by $\beta\mathbf{B}$ or, equivalently, replacing $\mathbf{b}$ by $\beta\mathbf{b}$ in (35) gives the result for $\mathrm{I}_1^{M,N}$ in (16).

Remark 1: According to Propositions 1 and 2, the result of a unitary integral over $\mathcal{U}(N)$ is determined by the dimension of its corresponding representation ($d_{\mathbf{r}_N}$). On the other hand, Propositions 3 and 4 reveal that performing the limit on the zero eigenvalues shrinks $\mathbf{r}_N$ to $\mathbf{r}_M$, $\chi_{\mathbf{r}_N}$ to $\chi_{\mathbf{r}_M}$, and $\alpha_{\mathbf{r}_N}$ to $\alpha_{\mathbf{r}_M}$. However, it **does not** shrink $d_{\mathbf{r}_N}$ to $d_{\mathbf{r}_M}$. Therefore, any assumption on the dimensions of the matrices, that changes the dimension of the unitary matrix, produces an incorrect result for the original unitary integral even after applying the limit. ◇

Example 1: Consider the case in which $M > N$ in (15). Consequently, the matrix $\mathbf{AC}$ has a full rank of N. Therefore,

$$\mathrm{I}_1^{M,N} = \mathrm{I}_1^{N,N} = \mathrm{I}_1^{N} \text{ when } M > N.$$

On the other hand, one may follow the approach in [13] and set N to be equal to M. Thus, the matrices $\mathbf{AC}$ and $\mathbf{B}$ have a full rank of M, which results in

$$\mathrm{I}_1^{M} = \frac{\beta^{-\frac{M(M-1)}{2}} \prod_{i=1}^{M}(M-i)!}{\underset{M}{\Delta}(\mathbf{a})\underset{M}{\Delta}(\mathbf{b})} \det_M[\exp(\beta a_i b_j)]. \tag{36}$$

The next step is to find the limit of (36) when $M-N$ eigenvalues of both matrices $\mathbf{AC}$ and $\mathbf{B}$ approach zero. Applying Lemma 5 (Appendix B) to (36) gives

$$\begin{aligned} \lim_{\left\{\substack{a_{N+1},\dots,a_M \\ b_{N+1},\dots,b_M}\right\}\to 0} \mathrm{I}_1^{M} &= \frac{\beta^{N\left(\frac{N+1}{2}-M\right)} \prod_{i=1}^{N}(M-i)!}{\underset{N}{\Delta}(\mathbf{a})\underset{N}{\Delta}(\mathbf{b})\prod_{i=1}^{N}(a_i b_i)^{M-N}} \\ &\quad \times \det_N[\exp(\beta a_i b_j)] \\ &\neq \mathrm{I}_1^{N}. \end{aligned}$$ △

Interested readers may refer to Appendix F for more examples.

B. *Calculation of* I_2

The second unitary integral examined in this paper is

$$\mathrm{I}_2^{N} = \int \mathrm{D}\mathbf{U}\, \mathrm{etr}\{\beta(\mathbf{UA} + \mathbf{U}^*\mathbf{B})\} \tag{37}$$

where $\mathbf{U} \in \mathcal{U}(N)$, $\mathbf{A}, \mathbf{B} \in \mathrm{GL}(N,\mathcal{C})$, and β is a complex scalar. Absorbing β into $\mathbf{A}$ and $\mathbf{B}$, and applying the character expansion formula (12) yields

$$\begin{aligned} \mathrm{J}_2^{N} &= \int \mathrm{D}\mathbf{U}\, \mathrm{etr}\{\mathbf{UA} + \mathbf{U}^*\mathbf{B}\} \qquad (38)\\ &= \sum_{\mathbf{r}_N}\sum_{\mathbf{r}'_N} \alpha_{\mathbf{r}_N}\alpha_{\mathbf{r}'_N} \int \mathrm{D}\mathbf{U}\, \chi_{\mathbf{r}_N}(\mathbf{UA})\, \chi_{\mathbf{r}'_N}(\mathbf{U}^*\mathbf{B}) \\ &= \sum_{\mathbf{r}_N} \frac{\alpha_{\mathbf{r}_N}^2}{d_{\mathbf{r}_N}} \chi_{\mathbf{r}_N}(\mathbf{AB}) \qquad (39)\\ &= \frac{\prod_{i=1}^{N}(N-i)!}{\underset{N}{\Delta}(\mathbf{a}^2)} \sum_{\mathbf{r}_N} \frac{1}{\prod_{j=1}^{N}(r_j+N-j)!} \\ &\quad \times \det_N\left[\frac{1}{(r_j-j+i)!}\right] \det_N\left[a_i^{2(r_j+N-j)}\right] \qquad (40) \end{aligned}$$

where (39) is obtained by applying Proposition 1, and (40) is the result of substituting $\alpha_{\mathbf{r}_N}$ and $d_{\mathbf{r}_N}$ from (13), and the character function from (11). The vector $\mathbf{a}^2 = \left(a_1^2, a_2^2, \dots, a_N^2\right)^T$ represents the eigenvalues of matrix $\mathbf{AB}$. Thus,

$$\mathrm{J}_2^{N} = \frac{\prod_{i=1}^{N}(N-i)!}{\underset{N}{\Delta}(\mathbf{a}^2)} \sum_{\mathbf{k}_N} \det_N\left[\frac{1}{k_j!\,(k_j-N+i)!}\right] \det_N\left[a_i^{2k_j}\right]$$

where $k_j = r_j + N - j$. Considering the power series expansion of the modified Bessel function as

$$\frac{I_n(2z)}{z^n} = \sum_{k=0}^{\infty} \frac{1}{k!\,(k+n)!}\, z^{2k} \tag{41}$$

and applying the Cauchy–Binet formula (Lemma 2 in Appendix A) gives

$$\mathrm{J}_2^N = \frac{\prod_{i=1}^{N}(N-i)!}{\underset{N}{\Delta}(\mathbf{a}^2)} \det_N \left[\frac{I_{-N+i}(2a_j)}{a_j^{-N+i}}\right].$$

Since $I_{-n}(x) = I_n(x)$, and by replacing $\mathbf{a}^2$ by $\beta^2\mathbf{a}^2$, we have

$$\mathrm{I}_2^N = \frac{\prod_{i=1}^{N}(N-i)!}{\beta^{\frac{N(N-1)}{2}}\underset{N}{\Delta}(\mathbf{a}^2)} \det_N \left[a_i^{N-j}\, I_{N-j}(2\beta a_i)\right]. \tag{42}$$

Now, assume $N-M$ eigenvalues of matrix $\mathbf{AB}$ in (38) approach zero. To find

$$\mathrm{J}_2^{M,N} = \lim_{\{a_{M+1},\ldots,a_N\}\longrightarrow 0} \mathrm{J}_2^N$$

one can directly apply Lemma 5 (Appendix B) to obtain (45). On the other hand, applying Propositions 3 and 4 to (39) gives

$$\mathrm{J}_2^{M,N} = \sum_{\mathbf{r}_M} \frac{\alpha_{\mathbf{r}_M}^2}{d_{\mathbf{r}_{M,N}}}\, \chi_{\mathbf{r}_M}\left(\widehat{\mathbf{AB}}\right) \tag{43}$$

$$= \frac{\prod_{i=1}^{M}(N-i)!}{\underset{M}{\Delta}(\mathbf{a}^2)} \times \sum_{\mathbf{k}_M} \det_M \left[\frac{1}{k_j!\,(k_j-N+i)!}\right] \det_M \left[a_i^{2(k_j-N+M)}\right] \tag{44}$$

where $k_j = r_j + N - j$. Considering the power series expansion of $I_n(z)$(41), and by applying the Cauchy–Binet formula (Lemma 2 in Appendix A), we obtain

$$\mathrm{J}_2^{M,N} = \frac{\prod_{i=1}^{M}(N-i)!}{\underset{M}{\Delta}(\mathbf{a}^2)\prod_{i=1}^{M} a_i^{2(N-M)}} \det_M \left[a_i^{N-j}\, I_{N-j}(2a_i)\right]. \tag{45}$$

Replacing $\mathbf{A}$ and $\mathbf{B}$ by $\beta\mathbf{AC}$ and $\beta\mathbf{BD}$, respectively, or, equivalently, replacing $\mathbf{a}^2$ by $\beta^2\mathbf{a}^2$ in (45) gives the result for $\mathrm{I}_2^{M,N}$ in (18).

C. Calculation of I_3

By absorbing β into $\mathbf{B}$ and $\mathbf{C}$ in (19), the integral over $\mathbf{U}$ in $\mathrm{I}_3^{M,N}$ is equivalent to J_2. Since the N–dimensional matrices $\mathbf{AV}^*\mathbf{B}$ and $\mathbf{CVD}$ have rank M, we use the result of $\mathrm{J}_2^{M,N}$ in (43) to obtain

$$\mathrm{J}_3^{M,N} = \sum_{\mathbf{r}_M} \frac{\alpha_{\mathbf{r}_M}^2}{d_{\mathbf{r}_{M,N}}} \int D\mathbf{V}\, \chi_{\mathbf{r}_M}(\mathbf{VDAV}^*\mathbf{BC})$$

$$= \sum_{\mathbf{r}_M} \frac{\alpha_{\mathbf{r}_M}^2}{d_{\mathbf{r}_{M,N}}} \frac{1}{d_{\mathbf{r}_M}}\, \chi_{\mathbf{r}_M}(\mathbf{DA})\, \chi_{\mathbf{r}_M}(\mathbf{BC}) \tag{46}$$

where the last equality is achieved by applying Proposition 2. Thus, substituting from (11), (13), and (33) gives

$$\mathrm{J}_3^{M,N} = \frac{\prod_{i=1}^{M}(N-i)!(M-i)!}{\underset{M}{\Delta}(\mathbf{a}^2)\underset{M}{\Delta}(\mathbf{b}^2)} \times \sum_{\mathbf{k}_M} \det_M\left[a_i^{2k_j}\right]\det_M\left[b_i^{2k_j}\right] \prod_{j=1}^{M} \frac{1}{k_j!\,(k_j+N-M)!}$$

where $k_j = r_j + M - j$, and $\mathbf{a}^2, \mathbf{b}^2 \in \mathcal{C}^M$ are the eigenvalues of the matrices $\mathbf{DA}$ and $\mathbf{BC}$, respectively. Considering the power series expansion of $I_n(z)$ (41), and applying the Cauchy–Binet formula (Lemma 3 in Appendix A) yields

$$\mathrm{J}_3^{M,N} = \frac{\prod_{i=1}^{M}(N-i)!(M-i)!}{\underset{M}{\Delta}(\mathbf{a}^2)\underset{M}{\Delta}(\mathbf{b}^2)\prod_{i=1}^{M}(a_ib_i)^{N-M}} \det_M\left[I_{N-M}(2a_ib_j)\right]. \tag{47}$$

Replacing $\mathbf{b}^2$ by $\beta^2\mathbf{b}^2$ in (47) gives the result for $\mathrm{I}_3^{M,N}$ in (20).

D. Calculation of I_4

By absorbing β into $\mathbf{B}$ in (21), the integral over $\mathbf{U}$ in $\mathrm{I}_4^{M,N}$ is equivalent to J_1. Since the N–dimensional matrix $\mathbf{AV}^*\mathbf{BVC}$ has rank M, we use the result of $\mathrm{J}_1^{M,N}$ in (34) to obtain

$$\mathrm{J}_4^{M,N} = \sum_{\mathbf{r}_M} \frac{\alpha_{\mathbf{r}_M}}{d_{\mathbf{r}_{M,N}}}\, \chi_{\mathbf{r}_{M,N}}(\mathbf{D}) \int D\mathbf{V}\, \chi_{\mathbf{r}_M}(\mathbf{VCAV}^*\mathbf{B})$$

$$= \sum_{\mathbf{r}_M} \frac{\alpha_{\mathbf{r}_M}}{d_{\mathbf{r}_{M,N}}} \frac{1}{d_{\mathbf{r}_M}} \chi_{\mathbf{r}_M}(\mathbf{CA})\chi_{\mathbf{r}_M}(\mathbf{B})\, \chi_{\mathbf{r}_{M,N}}(\mathbf{D}) \tag{48}$$

where the last equality is achieved by applying Proposition 2. Thus, substituting from (10), (11), and (33) gives

$$\mathrm{J}_4^{M,N} = \frac{\prod_{i=1}^{M}(N-i)!(M-i)!}{\underset{M}{\Delta}(\mathbf{a})\underset{M}{\Delta}(\mathbf{b})\underset{N}{\Delta}(\mathbf{c})} \times \sum_{\mathbf{k}_M} \frac{\det_M\left[a_i^{k_j}\right]\det_M\left[b_i^{k_j}\right]}{\underset{M}{\Delta}(\mathbf{k})\prod_{j=1}^{M}(k_j+N-M)!} \times \det_N\left[c_i^{k_j+N-M}\Big|_{j=1}^{M},\, c_i^{N-j}\Big|_{j=M+1}^{N}\right] \tag{49}$$

where $k_j = r_j + M - j$, and $\mathbf{a}, \mathbf{b} \in \mathcal{C}^M$ and $\mathbf{c} \in \mathcal{C}^N$ represent the eigenvalues of the matrices $\mathbf{CA}$, $\mathbf{B}$, and $\mathbf{D}$, respectively. Although it is possible to generalize the Cauchy–Binet formula (Appendix A) to the cases of three or more determinants in the summation, since $\underset{M}{\Delta}(\mathbf{k})$ in (49) cannot be represented as a multiplication of any function of k_j's, it seems (49) cannot be further simplified.

Replacing $\mathbf{B}$ by $\beta\mathbf{B}$ or, equivalently, replacing $\mathbf{b}$ by $\beta\mathbf{b}$ gives the result for $\mathrm{I}_4^{M,N}$ in (22).

V. MIMO Capacity Analysis

In this section, we use the results of the unitary integrals, derived in Section IV, to obtain the capacity for the Ricean and full-correlated Rayleigh MIMO channels. Without loss of generality, we assume $N_t \leqslant N_r$ or, equivalently, $M = \min(N_t, N_r) = N_t$ and $N = \max(N_t, N_r) = N_r$.

Accordingly, the results in this section can be used for $N_t > N_r$ by replacing the corresponding parameters.

A. Ricean MIMO Channel

In this scenario, the channel matrix can be modeled as $\mathbf{H} = \mathbf{G} + \mathbf{G}_0$, where $\mathbf{G}_0$ denotes the mean matrix, and $\mathbf{G}$ is the standard Rayleigh matrix; i.e., all elements of $\mathbf{G}$ are i.i.d. $\mathcal{CN}(0,1)$ random variables. In this case,

$$\begin{aligned} p(\mathbf{H}) &= \mathrm{etr}\{-(\mathbf{H}-\mathbf{G}_0)(\mathbf{H}-\mathbf{G}_0)^*\} \\ &= \exp\left(-\sum_{i=1}^{M}(\lambda_i+\gamma_i)\right)\mathrm{etr}\{\mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^*\mathbf{G}_0^* + \mathbf{G}_0\mathbf{V}\boldsymbol{\Sigma}^*\mathbf{U}^*\} \end{aligned}$$

where $\boldsymbol{\gamma} = (\gamma_1, \ldots, \gamma_M)^T$ are the M nonzero eigenvalues of the matrix $\mathbf{G}_0\mathbf{G}_0^*$, and the second equality comes from $\mathbf{H} = \mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^*$. Thus, from (6), we have

$$\begin{aligned} P(\boldsymbol{\lambda}) &= \mathcal{K}_{M,N} \underset{M}{\Delta}(\boldsymbol{\lambda})^2 \prod_{i=1}^{M} \lambda_i^{N-M} \mathrm{e}^{-(\lambda_i+\gamma_i)} \\ &\quad \times \int \mathrm{D}\mathbf{V} \int \mathrm{D}\mathbf{U}\ \mathrm{etr}\{\mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^*\mathbf{G}_0^* + \mathbf{U}^*\mathbf{G}_0\mathbf{V}\boldsymbol{\Sigma}^*\}. \end{aligned} \tag{50}$$

The double unitary integral in (50) is similar to I_3. Therefore, by employing $\mathrm{I}_3^{M,N}$ in (20), the joint eigenvalue distribution of the Ricean MIMO channel is obtained as

$$P(\boldsymbol{\lambda}) = \frac{\mathcal{G}}{M!} \underset{M}{\Delta}(\boldsymbol{\lambda}) \underset{M}{\det}\left[f_j(\lambda_i)\right] \tag{51}$$

where

$$\mathcal{G} = \frac{1}{\underset{M}{\Delta}(\boldsymbol{\gamma})} \prod_{i=1}^{M} \mathrm{e}^{-\gamma_i}\, \gamma_i^{-\frac{N-M}{2}}$$

and

$$f_j(\lambda_i) = \mathrm{e}^{-\lambda_i}\, \lambda_i^{\frac{N-M}{2}}\, I_{N-M}\left(2\sqrt{\lambda_i\gamma_j}\right).$$

Substituting (51) into (5) gives

$$\begin{aligned} g(z) &= \frac{\mathcal{G}}{M!} \prod_{i=1}^{M} \int_0^\infty d\lambda_i\, (1+\rho\lambda_i)^z \underset{M}{\Delta}(\boldsymbol{\lambda}) \underset{M}{\det}\left[f_j(\lambda_i)\right] \\ &= \frac{\mathcal{G}}{M!} \prod_{i=1}^{M} \int_0^\infty d\lambda_i\, (1+\rho\lambda_i)^z \\ &\qquad \times \sum_{\mathbf{a}} \mathrm{S}(\mathbf{a}) \prod_{i=1}^{M} \lambda_i^{M-a_i} \sum_{\mathbf{b}} \mathrm{S}(\mathbf{b}) \prod_{i=1}^{M} f_{b_i}(\lambda_i) \\ &= \frac{\mathcal{G}}{M!} \sum_{\mathbf{a}} \sum_{\mathbf{b}} \mathrm{S}(\mathbf{a})\mathrm{S}(\mathbf{b}) \prod_{i=1}^{M} \int_0^\infty d\lambda (1+\rho\lambda)^z \lambda^{M-a_i} f_{b_i}(\lambda) \\ &= \mathcal{G} \underset{M}{\det}\left[\int_0^\infty d\lambda\, (1+\rho\lambda)^z \lambda^{M-i} f_j(\lambda)\right] \end{aligned} \tag{52}$$

where the second equality comes from (64) (Appendix E), and the last equality comes from (65). Therefore,

$$g(z) = \mathcal{G} \underset{M}{\det}\left[\int_0^\infty d\lambda (1+\rho\lambda)^z \mathrm{e}^{-\lambda} \lambda^{\frac{N+M}{2}-i} I_{N-M}\left(2\sqrt{\lambda\gamma_j}\right)\right]. \tag{53}$$

Differentiating (53) with respect to z (Appendix E) yields

$$C = g'(0) = \mathcal{G} \sum_{m=1}^{M} \underset{M}{\det}[L_{m,ij}] \tag{54}$$

where

$$L_{m,ij} = \int_0^\infty d\lambda \log(1+\rho\lambda)\, \mathrm{e}^{-\lambda}\, \lambda^{\frac{N+M}{2}-i}\, I_{N-M}\left(2\sqrt{\lambda\gamma_j}\right)$$

if $i = m$; and

$$L_{m,ij} = \frac{(N-i)!}{(N-M)!}\, \gamma_j^{\frac{N-M}{2}}\, \Phi\Big(N-i+1, N-M+1; \gamma_j\Big)$$

if $i \neq m$; and $\Phi\big(a,b;x\big)$ is the confluent hypergeometric function [28].

Remark 2: In the case that some of M nonzero eigenvalues of the matrix $\mathbf{G}_0\mathbf{G}_0^*$ are equal, one can use Lemma 5 in Appendix B to obtain the appropriate joint eigenvalue distribution. One such example occurs when the MIMO channel is an i.i.d. Ricean fading channel, where the elements of the mean matrix $\mathbf{G}_0$ are equal or, equivalently, the elements of the channel matrix $\mathbf{H}$ are i.i.d. $\mathcal{CN}(g,1)$ random variables (g is a complex constant). Therefore, $\gamma = MN|g|^2$ is the *only* nonzero eigenvalue of the matrix $\mathbf{G}_0\mathbf{G}_0^*$. By applying Lemma 5 to (51), the joint pdf of the eigenvalues for the i.i.d. Ricean MIMO channel is obtained as

$$P(\boldsymbol{\lambda}) = \frac{\mathcal{G}_\gamma}{M!} \underset{M}{\Delta}(\boldsymbol{\lambda}) \underset{M}{\det}\left[f_j(\lambda_i)\right]$$

where

$$\mathcal{G}_\gamma = \mathcal{K}_{M-1,N-1}\, \mathrm{e}^{-\gamma}\, \gamma^{-\frac{N+M}{2}+1}$$

and

$$f_j(\lambda_i) = \begin{cases} \mathrm{e}^{-\lambda_i}\, \lambda_i^{\frac{N-M}{2}}\, I_{N-M}\left(2\sqrt{\gamma\lambda_i}\right) & \text{, if } j = 1; \\ \mathrm{e}^{-\lambda_i}\, \lambda_i^{N-j} & \text{, if } j > 1. \end{cases}$$

Following the same procedure as in (52) gives

$$g(z) = \mathcal{G}_\gamma \underset{M}{\det}\left[\int_0^\infty d\lambda\, (1+\rho\lambda)^z \lambda^{M-i} f_j(\lambda)\right]. \tag{55}$$

Differentiating (55) with respect to z (Appendix E) results in

$$C = g'(0) = \mathcal{G}_\gamma \sum_{m=1}^{M} \underset{M}{\det}[L_{m,ij}]$$

where

$$L_{m,ij} = \int_0^\infty d\lambda \log(1+\rho\lambda)\, \mathrm{e}^{-\lambda}\, \lambda^{\frac{N+M}{2}-i}\, I_{N-M}\left(2\sqrt{\lambda\gamma}\right)$$

if $i = m, j = 1$;

$$L_{m,ij} = \int_0^\infty d\lambda \log(1+\rho\lambda)\, e^{-\lambda}\, \lambda^{N+M-i-j}$$

if $i = m, j > 1$;

$$L_{m,ij} = \frac{(N-i)!}{(N-M)!}\, \gamma^{\frac{N-M}{2}}\, \Phi\Big(N-i+1, N-M+1; \gamma\Big)$$

if $i \neq m, j = 1$; and

$$L_{m,ij} = (N+M-i-j)!$$

if $i \neq m, j > 1$. ◇

B. Full-Correlated Rayleigh MIMO Channel

In this scenario, the channel matrix is correlated at both sides of the communication link. Thus, it can be modeled as $\mathbf{H} = \mathbf{R}^{\frac{1}{2}}\mathbf{G}\mathbf{T}^{\frac{1}{2}}$, where $\mathbf{R}$ and $\mathbf{T}$ denote the receiver and transmitter correlation matrices, respectively, and $\mathbf{G}$ is the standard Rayleigh matrix; i.e., all elements of $\mathbf{G}$ are i.i.d. $\mathcal{CN}(0,1)$ random variables. In this case,

$$\begin{aligned} p(\mathbf{H}) &= \mathcal{N}_{\mathbf{R},\mathbf{T}}\, \mathrm{etr}\{-\mathbf{H}\mathbf{T}^{-1}\mathbf{H}^*\mathbf{R}^{-1}\} \\ &= \mathcal{N}_{\mathbf{R},\mathbf{T}}\, \mathrm{etr}\{-\mathbf{U}\mathbf{\Sigma}\mathbf{V}^*\mathbf{T}^{-1}\mathbf{V}\mathbf{\Sigma}^*\mathbf{U}^*\mathbf{R}^{-1}\} \end{aligned}$$

where $\mathcal{N}_{\mathbf{R},\mathbf{T}}^{-1} = \det[\mathbf{R}]^M \det[\mathbf{T}]^N$, and the second equality comes from $\mathbf{H} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^*$. Thus, from (6), we have

$$\begin{aligned} P(\boldsymbol{\lambda}) &= \mathcal{K}_{M,N}\, \mathcal{N}_{\mathbf{R},\mathbf{T}}\, \underset{M}{\Delta}(\boldsymbol{\lambda})^2 \prod_{i=1}^{M} \lambda_i^{N-M} \\ &\quad\times \int \mathrm{D}\mathbf{V} \int \mathrm{D}\mathbf{U}\ \mathrm{etr}\left\{-\mathbf{U}\mathbf{\Sigma}\mathbf{V}^*\mathbf{T}^{-1}\mathbf{V}\mathbf{\Sigma}^*\mathbf{U}^*\mathbf{R}^{-1}\right\}. \quad (56) \end{aligned}$$

The double unitary integral in (56) is similar to I_4. Therefore, by employing $\mathrm{I}_4^{M,N}$ in (22), the joint eigenvalue distribution of the full-correlated Rayleigh MIMO channel is obtained as

$$P(\boldsymbol{\lambda}) = \sum_{\mathbf{k}_M} \frac{(-1)^{\frac{M(M-1)}{2}}\mathcal{A}}{M!\, \underset{M}{\Delta}(\mathbf{k})}\, \underset{M}{\Delta}(\boldsymbol{\lambda}) \underset{M}{\det}\left[\lambda_i^{k_j+N-M}\right] \quad (57)$$

where

$$\mathcal{A} = \frac{\prod_{i=1}^{M} x_i^N \prod_{j=1}^{N} y_j^M}{\underset{M}{\Delta}(\mathbf{x})\underset{N}{\Delta}(\mathbf{y})} \times \frac{\underset{M}{\det}\left[(-x_i)^{k_j}\right] \underset{N}{\det}\left[y_i^{k_j+N-M}\Big|_{j=1}^{M}, y_i^{N-j}\Big|_{j=M+1}^{N}\right]}{\prod_{j=1}^{M}(k_j+N-M)!} \quad (58)$$

is independent of $\boldsymbol{\lambda}$, $M = \min(N_t, N_r)$, $N = \max(N_t, N_r)$, $\mathbf{x} \in \mathcal{R}^M$ represents the eigenvalues of $\mathbf{R}^{-1}$ or $\mathbf{T}^{-1}$, whichever has the dimension M, and $\mathbf{y} \in \mathcal{R}^N$ represents the eigenvalues of $\mathbf{R}^{-1}$ or $\mathbf{T}^{-1}$, whichever has the dimension N.

Since $\mathbf{H} = \mathbf{R}^{\frac{1}{2}}\mathbf{G}\mathbf{T}^{\frac{1}{2}}$ in the correlated scenario, we absorb the transmit power factor $\sqrt{\rho}$ into $\mathbf{R}^{\frac{1}{2}}$ to simplify the calculation of the MGF. Considering the fact that

$$\underset{M}{\Delta}(\boldsymbol{\lambda}) = \underset{M}{\det}\left[\left(\frac{\lambda_i}{1+\lambda_i}\right)^{M-j}\right] \prod_{i=1}^{M} (1+\lambda_i)^{M-1}$$

and by applying Leibniz formula (64) (Appendix E) to expand the determinants in (57), we can write

$$\begin{aligned} g(z) &= \sum_{\mathbf{k}_M} \frac{(-1)^{\frac{M(M-1)}{2}}\mathcal{A}}{M!\, \underset{M}{\Delta}(\mathbf{k})} \prod_{i=1}^{M} \int_0^\infty d\lambda_i\, (1+\lambda_i)^{z+M-1} \\ &\quad \times \sum_{\mathbf{a}} \mathrm{S}(\mathbf{a}) \prod_{i=1}^{M} \left(\frac{\lambda_i}{1+\lambda_i}\right)^{M-a_i} \sum_{\mathbf{b}} \mathrm{S}(\mathbf{b}) \prod_{i=1}^{M} \lambda_i^{k_{b_i}+N-M} \\ &= \sum_{\mathbf{k}_M} \frac{(-1)^{\frac{M(M-1)}{2}}\mathcal{A}}{\underset{M}{\Delta}(\mathbf{k})} \frac{1}{M!} \sum_{\mathbf{a}} \sum_{\mathbf{b}} \mathrm{S}(\mathbf{a})\mathrm{S}(\mathbf{b}) \\ &\quad \times \prod_{i=1}^{M} \int_0^\infty d\lambda\, (1+\lambda)^{z+a_i-1} \lambda^{k_{b_i}+N-a_i} \\ &= \sum_{\mathbf{k}_M} \frac{(-1)^{\frac{M(M-1)}{2}}\mathcal{A}}{\underset{M}{\Delta}(\mathbf{k})} \underset{M}{\det}\left[\int_0^\infty d\lambda (1+\lambda)^{z+i-1} \lambda^{k_j+N-i}\right] \end{aligned} \quad (59)$$

where the last equality comes from (65). By following the same approach as in [13, Section V-A], and taking the integrals in (59) by parts, $(M-i)$ times, we have [29]

$$\begin{aligned} g(z) &= \sum_{\mathbf{k}_M} \frac{(-1)^{\frac{M(M-1)}{2}}\mathcal{A}}{\underset{M}{\Delta}(\mathbf{k})} \times (-1)^{\frac{M(M-1)}{2}} \underset{M}{\Delta}(\mathbf{k}) \\ &\quad \times \prod_{j=1}^{M} \frac{\int_0^\infty d\lambda\, (1+\lambda)^{z+M-1}\, \lambda^{k_j+N-M}}{(z+j-1)^{j-1}} \quad (60) \\ &= \frac{\prod_{i=1}^{M} x_i^N \prod_{j=1}^{N} y_j^M}{\underset{M}{\Delta}(\mathbf{x})\underset{N}{\Delta}(\mathbf{y}) \prod_{i=1}^{M-1}(z+i)^i} \\ &\quad \times \sum_{\mathbf{k}_M} \underset{M}{\det}\left[(-x_i)^{k_j}\right] \underset{N}{\det}\left[y_i^{k_j+N-M}\Big|_{j=1}^{M}, y_i^{N-j}\Big|_{j=M+1}^{N}\right] \\ &\quad \times \prod_{i=1}^{M} \int_0^\infty d\lambda\, (1+\lambda)^{z+M-1} \frac{\lambda^{k_i+N-M}}{(k_i+N-M)!} \quad (61) \end{aligned}$$

where the last equality is obtained by substituting $\mathcal{A}$ from (58).

By replacing $\mathbf{x}$ by $\rho^{-1}\mathbf{x}$, and applying the generalized Cauchy–Binet formula (Lemma 4 in Appendix A), according to the fact that

$$\int_0^\infty d\lambda (1+\lambda)^{z+M-1} e^{\lambda x} = \sum_{k=0}^{\infty} \left[\int_0^\infty d\lambda (1+\lambda)^{z+M-1} \frac{\lambda^k}{k!}\right] x^k$$

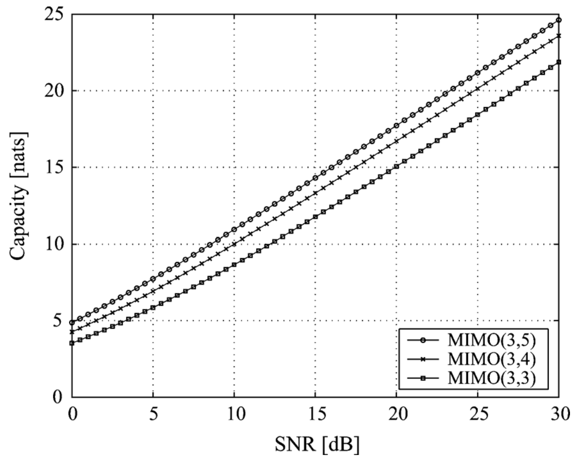

Fig. 1. The capacity of Ricean MIMO channels with $\boldsymbol{\gamma} = (0.1, 0.3, 0.7)^T$.

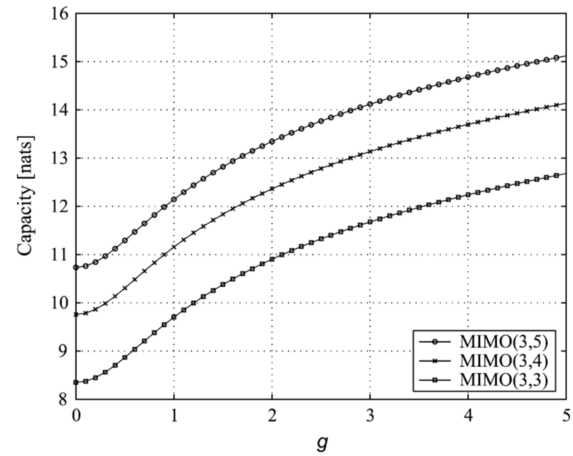

Fig. 2. The capacity of i.i.d. Ricean MIMO channels at SNR=10 dB with $\mathcal{CN}(g,1)$ distribution.

we conclude that

$$g(z) = \frac{(-1)^{M(N-M)} \prod_{i=1}^{M} x_i^M \prod_{j=1}^{N} y_j^M}{\rho^{\frac{M(M-1)}{2}} \underset{M}{\Delta}(\mathbf{x}) \underset{N}{\Delta}(\mathbf{y}) \prod_{i=1}^{M-1} (z+i)^i} \times \det_N \left[\begin{array}{ll} \int_0^\infty d\lambda \, (1+\rho\lambda)^{z+M-1} \, \mathrm{e}^{-\lambda x_i y_j} & \Big|_{M}^{i=1} \\ y_j^{N-i} & \Big|_{N}^{i=M+1} \end{array} \right] = \frac{\mathcal{B}}{\prod_{i=1}^{M-1}(z+i)^i} \det_N \left[\begin{array}{ll} F_z\left(\rho^{-1} x_i y_j, \, M\right) & \Big|_{M}^{i=1} \\ y_j^{N+M-i} & \Big|_{N}^{i=M+1} \end{array} \right] \tag{62}$$

where

$$F_z\big(x, \, M\big) = x^M \int_0^\infty d\lambda \, (1+\lambda)^{z+M-1} \, \mathrm{e}^{-\lambda x} = x^{-z} \, \mathrm{e}^{x} \, \Gamma\big(z+M, x\big)$$

and

$$\mathcal{B} = \frac{(-1)^{M(N-M)} \, \rho^{\frac{M(M-1)}{2}}}{\underset{M}{\Delta}(\mathbf{x}) \underset{N}{\Delta}(\mathbf{y})}$$

and $\Gamma\big(\alpha, x\big)$ is the upper incomplete Gamma function [28]. Note that for integer m, we have

$$F_m\big(x, \, M\big) = (M+m-1)! \sum_{k=0}^{M+m-1} \frac{x^{k-m}}{k!}.$$

Differentiating (62) with respect to z (Appendix E) yields

$$C = g'(0) = 1 - M + \frac{\mathcal{B}}{\prod_{i=1}^{M-1} i^i} \sum_{m=1}^{M} \det_N \left[L_{m,ij} \right] \tag{63}$$

where

$$L_{m,ij} = \rho\big(\rho^{-1} x_i y_j\big)^M \int_0^\infty d\lambda \log(1+\rho\lambda) \, (1+\rho\lambda)^{M-1} \, \mathrm{e}^{-\lambda x_i y_j}$$

if $i = m \leqslant M$; and

$$L_{m,ij} = \begin{cases} F_0\Big(\rho^{-1} x_i y_j, \, M\Big) & , \text{if } i \neq m \leqslant M; \\ y_j^{N+M-i} & , \text{if } i > M. \end{cases}$$

Remark 3: In this section, we used the joint eigenvalue distribution of the full-correlated MIMO channel, derived in (57), to calculate $g(z)$. As a result, the term $\underset{M}{\Delta}(\mathbf{k})$, which is the result of the unitary integral over $\mathbf{V}$ (48, 49), is omitted in (60), and the summation in (61) takes the form of the generalized Cauchy–Binet formula (Lemma 4 in Appendix A).

On the other hand, Lemma 4 is derived [29] by taking the limit of both sides of the Cauchy–Binet formula (Lemma 3 in Appendix A), when $N - M$ elements of the vector $\mathbf{x}$ approach zero. Therefore, to calculate a summation with the form of Lemma 4, one can directly apply Lemma 4, or apply Lemma 3 by assuming $M = N$, and then, find the limit of the result when $N - M$ elements of vector $\mathbf{x}$ approach zero.

In [13], the authors derive $P(\boldsymbol{\lambda})$ for the full-correlated case by assuming $M = N$, and use it to obtain $g(z)$ by employing Lemma 3. In the end, they apply the limit on the $N - M$ zero eigenvalues to obtain $g(z)$ for $M \leqslant N$. Hence, the MGF of the mutual information (and the capacity) for the full-correlated Rayleigh MIMO channel is correctly derived in [13]. ◇

VI. Simulation Results

To verify the analytical expressions of the capacity with the simulation results, we include four figures in this section, each one demonstrating the results for MIMO systems with $N_t = 3$ transmitter antennas and $N_r = 3$, 4, and 5 receiver antennas. In all figures, the solid curves are from analytic expressions, and the symbols are obtained by computer simulations.

- Fig. 1 shows the capacity of MIMO systems versus SNR (ρ) when the channel is Ricean fading. Here, the eigenvalues of the matrix $\mathbf{G}_0^*\mathbf{G}_0$ are $\boldsymbol{\gamma} = (0.1, 0.3, 0.7)^T$.
- Fig. 2 shows the capacity of MIMO systems at $\rho = 10$ dB when the channel is i.i.d. Ricean fading, and all elements of the channel matrix have $\mathcal{CN}(g,1)$ distribution.

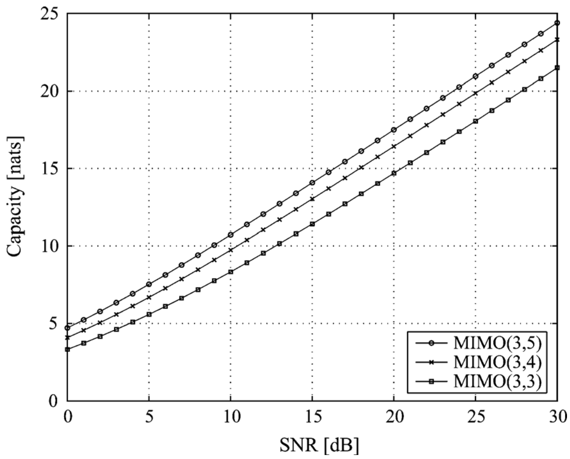

Fig. 3. The capacity of full-correlated Rayleigh MIMO channels with $d_\lambda = 2$ and $\delta = 10°$.

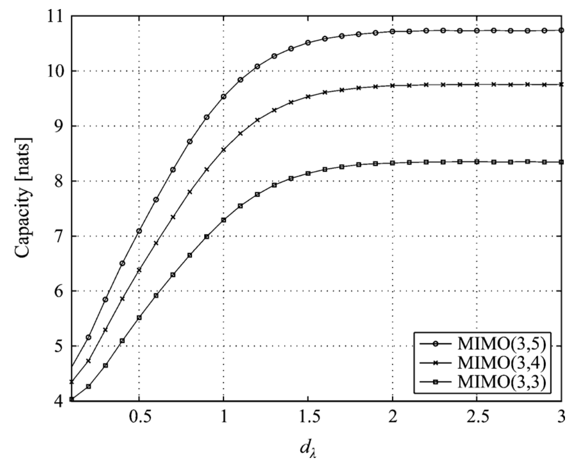

Fig. 4. The capacity of full-correlated Rayleigh MIMO channels at SNR=10 dB with $\delta = 10°$.

- Fig. 3 shows the capacity of MIMO systems versus SNR when the channel is full-correlated Rayleigh fading. The elements of the correlation matrices $\mathbf{R}$ and $\mathbf{T}$ are generated from the following expression [13]:

$$T_{ab} = \int_{-180}^{180} \frac{\mathrm{d}\phi}{\sqrt{2\pi\delta^2}} \exp\left(2\pi i(a-b)d_\lambda \sin\left(\frac{\phi\pi}{180}\right) - \frac{\phi^2}{2\delta^2}\right)$$

  where δ in degrees is the angle spread, measured from the vertical to the linear antenna array, and $d_\lambda = d_{\min}/\lambda_s$ is the normalized minimum distance between antennas (λ_s is the signal wavelength). The results in Fig. 3 are obtained by assuming $d_\lambda = 2$ and $\delta = 10°$.
- Fig. 4 shows the capacity of MIMO systems at $\rho = 10$ dB when the channel is full-correlated Rayleigh fading. Here, the results are obtained by assuming $\delta = 10°$.

As observed, the results from analytic expressions are consistent with the results from simulations, which verifies our analysis.

VII. Conclusions

Unitary integrals appear in several fields of science and engineering. In this paper, we showed that changing the dimension of the unitary matrix produces incorrect result for the original unitary integral even after applying the limit. We developed a precise framework to use the character expansions for integrations over the unitary group, where the coefficient matrices appearing in the integrand can be general rectangular complex matrices. We solved some of the *well-known but not solved in general form* unitary integrals to obtain the joint eigenvalue distributions (and the capacity) for the Ricean and correlated Rayleigh MIMO channels. Although some of the results of this paper have been derived before in the literature (using considerably more complicated methods), this paper demonstrates the power and neatness of the character expansion method to obtain those results in their general forms. The approach presented in this paper can be used to solve other unitary integrals accordingly.

Appendix A
Generalized Cauchy–Binet Formula

The following two lemmas have been proved in [30] as the Cauchy–Binet formula:

Lemma 2: Given vector $\mathbf{x}$ with dimension N, and power series expansions $f_i(z) = \sum_{k=0}^{\infty} a_k^{(i)} z^k$ convergent for $|z| < \xi$, then if $|x_i| < \xi$ for all $1 \leqslant i \leqslant N$, one can write

$$\sum_{\mathbf{k}_N} \det_N \left[a_{k_j}^{(i)}\right] \det_N \left[x_i^{k_j}\right] = \det_N \left[f_i(x_j)\right]$$

where $\mathbf{k}_N$ represents all irreducible representations of $\mathrm{GL}(N, \mathcal{C})$. ▲

Lemma 3: Given vectors $\mathbf{x}$ and $\mathbf{y}$ with dimension N, and a power series expansion $f(z) = \sum_{k=0}^{\infty} a_k z^k$ convergent for $|z| < \xi$, then if $|x_i y_j| < \xi$ for all $1 \leqslant i, j \leqslant N$, one can write

$$\sum_{\mathbf{k}_N} \det_N \left[x_i^{k_j}\right] \det_N \left[y_i^{k_j}\right] \prod_{j=1}^{N} a_{k_j} = \det_N \left[f(x_i y_j)\right]$$

where $\mathbf{k}_N$ represents all irreducible representations of $\mathrm{GL}(N, \mathcal{C})$. ▲

In the case of unequal dimension vectors $\mathbf{x}$ and $\mathbf{y}$, we have Lemma 4 [26].

Lemma 4 (Generalized Cauchy–Binet Formula): Given vectors $\mathbf{x}$ and $\mathbf{y}$ with dimensions M and N, respectively ($M \leqslant N$), and a power series expansion $f(z) = \sum_{k=0}^{\infty} a_k z^k$ convergent for $|z| < \xi$, then if $|x_i y_j| < \xi$ for all $1 \leqslant i \leqslant M$ and $1 \leqslant j \leqslant N$, one can write

$$\sum_{\mathbf{k}_M} \det_M \left[x_i^{k_j}\right] \det_N \left[y_i^{k_j+N-M}\Big|_{j=1}^{M}, y_i^{N-j}\Big|_{j=M+1}^{N} \right] \prod_{j=1}^{M} a_{k_j+N-M}$$
$$= \frac{1}{\prod_{i=1}^{M} x_i^{N-M}} \det_N \begin{bmatrix} f(x_i y_j) & \Big|_{i=1}^{M} \\ y_j^{N-i} & \Big|_{i=M+1}^{N} \end{bmatrix}$$

where $\mathbf{k}_M$ represents all irreducible representations of $\mathrm{GL}(M, \mathcal{C})$. ▲

Appendix B
Generalized l'Hôpital Rule

Lemma 5: If we define

$$R(x_1, \ldots, x_N) = \frac{\det_N [f_i(x_j)]}{\Delta(x_1, \ldots, x_N)}$$

then, [30]

$$\lim_{\{x_{M+1}, \ldots, x_N\} \longrightarrow x_0} R(x_1, \ldots, x_N) = \frac{\det_N [\mathbf{F}]}{\Delta(x_1, \ldots, x_M) \prod_{i=1}^{M} (x_i - x_0)^{N-M} \prod_{j=1}^{N-M-1} j!}$$

where

$$\mathbf{F} = \left(f_i(x_j) \Big|_{j=1}^{M}, \; f_i^{(N-j)}(x_0) \Big|_{j=M+1}^{N} \right)$$

and $f^{(k)}$ denotes the kth derivative of the function f. ▲

Appendix C
Proof of Proposition 3

Proof: Since $\det_N \left[a_i^{r_j+N-j} \right] = \det_N \left[a_j^{r_i+N-i} \right]$, we define $f_i(a_j) = a_j^{r_i+N-i}$ and apply Lemma 5 in Appendix B to the character function (11) to obtain

$$\lim_{\{a_{M+1}, \ldots, a_N\} \longrightarrow a_0} \chi_{\mathbf{r}_N}(\mathbf{A}) = \lim_{\{a_{M+1}, \ldots, a_N\} \longrightarrow a_0} \frac{\det_N \left[a_j^{r_i+N-i} \right]}{\Delta(a_1, \ldots, a_N)}$$
$$= \frac{\det_N [\mathbf{F}]}{\Delta(a_1, \ldots, a_M) \prod_{i=1}^{M} (a_i - a_0)^{N-M} \prod_{j=1}^{N-M-1} j!}$$

where

$$\mathbf{F} = \left(a_j^{r_i+N-i} \Big|_{j=1}^{M}, \frac{(r_i+N-i)!}{(r_i-i+j)!} \, a_0^{r_i-i+j} \Big|_{j=M+1}^{N} \right).$$

Note that all entries of $\mathbf{F}$ with $r_i - i + j < 0$ are zero.

As observed, all diagonal entries of $\mathbf{F}$ from $i = j = M+1, \ldots, N$ are in the form of $\frac{(r_i+N-i)!}{r_i!} \, a_0^{r_i}$. Therefore, if $a_0 = 0$, then $\det_N [\mathbf{F}] = 0$, unless $r_{M+1} = r_{M+2} = \cdots = r_N = 0$. In this case, by defining $\mathbf{r}_{M,N} = \{\mathbf{r}_M, 0, 0, \ldots, 0\}$, we have

$$\lim_{\{a_{M+1}, \ldots, a_N\} \longrightarrow 0} \chi_{\mathbf{r}_{M,N}}(\mathbf{A}) = \frac{\det_N \begin{bmatrix} \mathbf{Q}_{M \times M} & \mathbf{0}_{M \times (N-M)} \\ \mathbf{W}_{(N-M) \times M} & \mathbf{P}_{(N-M) \times (N-M)} \end{bmatrix}}{\Delta(a_1, \ldots, a_M) \prod_{i=1}^{M} a_i^{N-M} \prod_{j=1}^{N-M-1} j!}$$

where

$$\mathbf{Q} = \begin{pmatrix} a_1^{r_1+N-1} & \cdots & a_M^{r_1+N-1} \\ \vdots & \ddots & \vdots \\ a_1^{r_M+N-M} & \cdots & a_M^{r_M+N-M} \end{pmatrix}$$

and

$$\mathbf{P} = \begin{pmatrix} (N-M-1)! & \cdots & 0 & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \cdots & 2! & 0 & 0 \\ 0 & \cdots & 0 & 1! & 0 \\ 0 & \cdots & 0 & 0 & 0! \end{pmatrix}.$$

By column factoring of $\mathbf{Q}$, we obtain

$$\det_N \begin{bmatrix} \mathbf{Q} & \mathbf{0} \\ \mathbf{W} & \mathbf{P} \end{bmatrix} = \det_M [\mathbf{Q}] \det_{N-M} [\mathbf{P}]$$
$$= \left[\det_M \left[a_j^{r_i+M-i} \right] \prod_{i=1}^{M} a_i^{N-M} \right] \prod_{j=1}^{N-M-1} j!$$

independent of $\mathbf{W}$ [31]. Therefore,

$$\lim_{\{a_{M+1}, \ldots, a_N\} \longrightarrow 0} \chi_{\mathbf{r}_{M,N}}(\mathbf{A}) = \frac{\det_M \left[a_i^{r_j+M-j} \right]}{\Delta(a_1, \ldots, a_M)}$$
$$= \chi_{\mathbf{r}_M} \left(\widehat{\mathbf{A}} \right)$$

where $\widehat{\mathbf{A}} \in \mathrm{GL}(M, \mathcal{C})$ is the matrix with eigenvalues $(a_1, a_2, \ldots, a_M)^T$. ■

Appendix D
Proof of Proposition 4

Proof: From the definition of $\alpha_{\mathbf{r}_N}$ in (13), and noting that the matrix elements inside the determinant with $r_i - i + j < 0$ are zero, we have

$$\alpha_{\mathbf{r}_{M,N}} = \det_N \begin{bmatrix} \mathbf{Q}_{M \times M} & \mathbf{T}_{M \times (N-M)} \\ \mathbf{0}_{(N-M) \times M} & \mathbf{R}_{(N-M) \times (N-M)} \end{bmatrix}$$
$$= \det_M [\mathbf{Q}] \det_{N-M} [\mathbf{R}]$$

where $Q_{ij}^{-1} = (r_i - i + j)!$ for $i, j = 1, \ldots, M$, and

$$\mathbf{R} = \begin{pmatrix} \frac{1}{0!} & \frac{1}{1!} & \frac{1}{2!} & \cdots & \frac{1}{(N-M-2)!} & \frac{1}{(N-M-1)!} \\ 0 & \frac{1}{0!} & \frac{1}{1!} & \cdots & \frac{1}{(N-M-3)!} & \frac{1}{(N-M-2)!} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{1}{0!} & \frac{1}{1!} \\ 0 & 0 & 0 & \cdots & 0 & \frac{1}{0!} \end{pmatrix}$$

so that $\det_{N-M} [\mathbf{R}] = 1$. Thus,

$$\alpha_{\mathbf{r}_{M,N}} = \det_M [\mathbf{Q}] = \alpha_{\mathbf{r}_M}.$$

From (13), we have

$$d_{\mathbf{r}_{M,N}} = \alpha_{\mathbf{r}_{M,N}} \frac{\prod_{i=1}^{M} (r_i+N-i)! \prod_{i=M+1}^{N} (N-i)!}{\prod_{i=1}^{N} (N-i)!}$$
$$= \alpha_{\mathbf{r}_M} \frac{\prod_{i=1}^{M} (r_i+N-i)!}{\prod_{i=1}^{M} (N-i)!}.$$

Thus,

$$\frac{\alpha_{\mathbf{r}_M}}{d_{\mathbf{r}_{M,N}}} = \prod_{i=1}^{M} \frac{(N-i)!}{(r_i+N-i)!}$$

and, by using (13) once more,

$$d_{\mathbf{r}_{M,N}} = d_{\mathbf{r}_M} \prod_{i=1}^{M} \left[\frac{(M-i)!}{(N-i)!} \times \frac{(r_i+N-i)!}{(r_i+M-i)!} \right]. \quad \blacksquare$$

APPENDIX E
LEIBNIZ FORMULA FOR DETERMINANTS

The Leibniz formula for the determinant expansion [31] is as follows:

$$\det_M [X_{ij}] = \sum_{\mathbf{a}} \mathrm{S}(\mathbf{a}) \prod_{i=1}^{M} X_{ia_i} \tag{64}$$

$$= \frac{1}{M!} \sum_{\mathbf{a}} \sum_{\mathbf{b}} \mathrm{S}(\mathbf{a})\mathrm{S}(\mathbf{b}) \prod_{i=1}^{M} X_{a_i b_i} \tag{65}$$

where the vector $\mathbf{a} = (a_1, a_2, \ldots, a_M)^T$ is a permutation of integers $(1, 2, \ldots, M)$, $\mathrm{S}(\mathbf{a}) = +1$ if the permutation is even, $\mathrm{S}(\mathbf{a}) = -1$ if the permutation is odd, and the summation is over all possible permutations.

In addition, from (64), we have

$$\begin{aligned} \frac{\partial}{\partial z} \det_M [X_{ij}] &= \sum_{\mathbf{a}} \mathrm{S}(\mathbf{a}) \frac{\partial}{\partial z} \prod_{i=1}^{M} X_{ia_i} \\ &= \sum_{\mathbf{a}} \mathrm{S}(\mathbf{a}) \sum_{m=1}^{M} \frac{\partial}{\partial z} X_{ma_m} \prod_{i=1, i\neq m}^{M} X_{ia_i} \\ &= \sum_{m=1}^{M} \sum_{\mathbf{a}} \mathrm{S}(\mathbf{a}) \frac{\partial}{\partial z} X_{ma_m} \prod_{i=1, i\neq m}^{M} X_{ia_i} \\ &= \sum_{m=1}^{M} \det_M [X_{m,ij}] \end{aligned}$$

where

$$X_{m,ij} = \begin{cases} \frac{\partial}{\partial z} X_{ij} & , \text{if } i = m; \\ X_{ij} & , \text{otherwise.} \end{cases}$$

APPENDIX F

Example 2: By following the approach in [13] and setting $M = N$, from (46), we have

$$\mathrm{J}_3^{N,N} = \sum_{\mathbf{r}_N} \frac{\alpha_{\mathbf{r}_N}^2}{d_{\mathbf{r}_N}^2} \chi_{\mathbf{r}_N}(\mathbf{DA})\, \chi_{\mathbf{r}_N}(\mathbf{BC}) \tag{66}$$

$$= \frac{\prod_{i=1}^{N} [(N-i)!]^2}{\underset{N}{\Delta}(\mathbf{a}^2)\underset{N}{\Delta}(\mathbf{b}^2)} \det_N [I_0(2a_i b_j)]. \tag{67}$$

The next step is to find the limit of $\mathrm{J}_3^{N,N}$ when $N - M$ eigenvalues of both matrices $\mathbf{DA}$ and $\mathbf{BC}$ approach zero. Applying Propositions 3 and 4 to (66) gives

$$\lim_{\left\{ \substack{a_{M+1},\ldots,a_N \\ b_{M+1},\ldots,b_N} \right\} \to 0} \mathrm{J}_3^{N,N} = \sum_{\mathbf{r}_M} \frac{\alpha_{\mathbf{r}_M}^2}{d_{\mathbf{r}_{M,N}}^2} \chi_{\mathbf{r}_M}(\mathbf{DA})\, \chi_{\mathbf{r}_M}(\mathbf{BC}) \tag{68}$$

$$= \frac{\prod_{i=1}^{M} [(N-i)!]^2}{\underset{M}{\Delta}(\mathbf{a}^2)\underset{M}{\Delta}(\mathbf{b}^2) \prod_{i=1}^{M} (a_i b_i)^{2(N-M)}} \det_M [I_0(2a_i b_j)] \tag{69}$$

$$\neq \mathrm{J}_3^{M,N}$$

where $\mathrm{J}_3^{M,N}$ is derived in (46) and (47), corresponding to (68) and (69), respectively.

As explained in Remark 1, increasing the dimension of the integral over $\mathbf{V}$ from M to N generates the incorrect results in (68) and (69). Consequently, the joint eigenvalue distribution of the Ricean MIMO channel in [13, Eq. (52)], obtained based on (69), is incorrect [7], [10]. Interested readers may calculate [13, Eq. (52)] for a MIMO system with $N_t = 1$ and $N_r = 2$, which results in a non-pdf function. $\triangle$

Example 3: By setting $M = N$, from (48), we have

$$\mathrm{J}_4^{N,N} = \sum_{\mathbf{r}_N} \frac{\alpha_{\mathbf{r}_N}}{d_{\mathbf{r}_N}^2} \chi_{\mathbf{r}_N}(\mathbf{CA})\, \chi_{\mathbf{r}_N}(\mathbf{B})\, \chi_{\mathbf{r}_N}(\mathbf{D}). \tag{70}$$

The next step is to find the limit of $\mathrm{J}_4^{N,N}$ when $N - M$ eigenvalues of both matrices $\mathbf{CA}$ and $\mathbf{B}$ approach zero. Applying Propositions 3 and 4 to (70) gives

$$\lim_{\left\{ \substack{a_{M+1},\ldots,a_N \\ b_{M+1},\ldots,b_N} \right\} \to 0} \mathrm{J}_4^{N,N} = \sum_{\mathbf{r}_M} \frac{\alpha_{\mathbf{r}_M}}{d_{\mathbf{r}_{M,N}}^2} \chi_{\mathbf{r}_M}(\mathbf{CA}) \chi_{\mathbf{r}_M}(\mathbf{B}) \chi_{\mathbf{r}_{M,N}}(\mathbf{D})$$

$$\neq \mathrm{J}_4^{M,N} \tag{71}$$

where $\mathrm{J}_4^{M,N}$ is derived in (48). Similar to Example 2, increasing the dimension of the integral over $\mathbf{V}$ from M to N generates the incorrect result in (71). $\triangle$

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