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A Complexity-Efficient Sphere Decoder for MIMO Systems

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***Abstract*—It is well known that although the conventional sphere decoder (SD) achieves optimal maximum likelihood (ML) performance at a reduced complexity compared to the naive ML detector, the SD computational complexity varies with signal noise ratio (SNR) and is high in the low SNR region. This paper proposes a new idea to overcome these drawbacks that reduces the complexity significantly at a negligible performance loss. The main idea is to scale the search radius of the original SD by a factor that depends on the SNR. This factor tends to unity for high SNR, which means there is no performance loss for high SNRs. The resulting SD performs nearly-optimal ML detection over the whole range of SNRs, while keeping its complexity roughly constant. We give simulation results and theoretical analysis to confirm the advantages of the proposed SD. It is suitable practical implementation because of its effectively-reduced and almost-fixed complexity.**

I. Introduction

The optimal detector for spatial multiplexing in multiple-input multiple-output (MIMO) systems is the maximum likelihood (ML) detector. However, the complexity of the naive ML detector (i.e. exhaustive search) grows exponentially with the number of transmit antennas and with the order of the signal constellation [1]. As an alternative, the sphere decoder (SD) has been developed to attain low complexity with the ML performance, especially for higher signal noise ratio (SNR) [2]. The Fincke-Pohst SD (FP-SD) is an efficient strategy for enumerating all the lattice points within a hypersphere with a certain radius, while the more efficient alternative Schnorr-Euchner SD (SE-SD) is based on examining the lattice points inside the hypersphere in a different order and adjusting of the search radius whenever an admissible lattice point is reached [3][4].

There are two challenges when the SD based MIMO detection is implemented in very large scale integration (VLSI): (i) its high computational complexity in the low SNR region, and (ii) the variability of its complexity. To address the first issue, several SD variants have been developed, such as [5], which selects more reliable symbol candidates according to their conditional probabilities by exploiting the minimum mean square error (MMSE) criterion. However, the complexity is still high for near-ML performance or for a high order constellation. Probabilistic tree pruning approaches [6][7] sacrifice performance for complexity reduction. To overcome the second challenge, the K-best SD [8] and the fixed complexity SD (FSD) [9] are proposed, both of which can provide fixed computational complexity. The K-best SD traverses the tree breadth-first by only considering the $K > 1$ nodes in each level. However, its complexity is considerably higher than the complexity of the SD in order to approach ML performance [8]. The FSD ensures fixed complexity by combining a channel matrix ordering and a search through a small subset of the transmit constellation. It nevertheless has higher complexity than the SE-SD in the high SNR region [9].

In this paper, we propose a new complexity-efficient sphere decoder (CSD) with much lower complexity and almost-constant level of complexity. Moreover, it achieves almost optimal detection performance, as measured by symbol error rate (SER). Our main idea is to scale the search radius by a factor that depends on the SNR. This factor approaches one in high SNR, which resulting in the optimal performance. However, in low SNR, this factor results in pruning additional nodes, as compared to the conventional SD algorithms.

The rest of this paper is organized as follows. Section II and Section III describe the MIMO system model and the basic principle of SD algorithm, respectively. Section IV introduces our CSD algorithm and analyzes its complexity. Simulation results about performance and complexity are given in Section V. Finally, conclusions are drawn in Section VI.

II. System Model

Consider a spatial multiplexing MIMO system with N_T transmit antennas and N_R receive antennas. A rich scattering memoryless (flat fading) channel is assumed [1]. The received signal vector can be written as [1]

$$\tilde{\mathbf{y}} = \tilde{\mathbf{H}}\tilde{\mathbf{s}} + \tilde{\mathbf{n}} \tag{1}$$

where $\tilde{\mathbf{s}} = (\tilde{s}_1, \tilde{s}_2, \ldots, \tilde{s}_{N_T})^T$ is the transmitted symbol vector, $\tilde{s}_i \in \mathcal{Q}$ (a complex constellation such as M^2-QAM), $\tilde{\mathbf{y}} = (\tilde{y}_1, \tilde{y}_2, \ldots, \tilde{y}_{N_R})^T$, $\tilde{y}_i$ is the signal received at the ith antenna ($i = 1, 2, \ldots, N_R$). $\tilde{\mathbf{H}}$ denotes the $N_R \times N_T$ Rayleigh fading channel matrix with independent identically distributed (i.i.d.) elements $\tilde{h}_{ij} \sim \mathcal{CN}(0, 1)$, where $\mathcal{CN}(0, 1)$ denotes complex Gaussian distribution with zero mean and unit variance. $\tilde{\mathbf{n}} = (\tilde{n}_1, \tilde{n}_2, \ldots, \tilde{n}_{N_R})^T$ is the vector of i.i.d. additive white Gaussian noise (AWGN) where $\tilde{n}_i \sim \mathcal{CN}(0, \sigma^2)$. In this paper, it is assumed that the channel matrix $\tilde{\mathbf{H}}$ is perfectly known by the receiver. For brevity, we also assume $N_T = N_R = N$ and uncoded MIMO system.

From [10], the complex channel matrix can be transformed to real matrix representation $\mathbf{y} = \mathbf{H}\mathbf{s} + \mathbf{n}$, where $\mathbf{y}, \mathbf{n} \in \mathcal{R}^n$, $\mathbf{H} \in \mathcal{R}^{n\times m}$ and $\mathbf{s} \in \mathcal{R}^m$ with $m = n = 2N$, that is,

$$\begin{bmatrix}\Re(\tilde{\mathbf{y}})\\ \Im(\tilde{\mathbf{y}})\end{bmatrix} = \begin{bmatrix}\Re(\tilde{\mathbf{H}}) & -\Im(\tilde{\mathbf{H}})\\ \Im(\tilde{\mathbf{H}}) & \Re(\tilde{\mathbf{H}})\end{bmatrix}\begin{bmatrix}\Re(\tilde{\mathbf{s}})\\ \Im(\tilde{\mathbf{s}})\end{bmatrix} + \begin{bmatrix}\Re(\tilde{\mathbf{n}})\\ \Im(\tilde{\mathbf{n}})\end{bmatrix} \tag{2}$$

where $\Re(\mathbf{y})$ and $\Im(\mathbf{y})$ are the real and the imaginary part of $\mathbf{y}$, respectively. Therefore, for real constellation, the optimal ML detection is given by the rule [10]

$$\hat{\mathbf{s}} = \arg\min_{\mathbf{s}\in\Omega^m} ||\mathbf{y} - \mathbf{H}\mathbf{s}||^2, \tag{3}$$

which means that $\hat{\mathbf{s}}$ is the element in $\mathbf{s} \in \Omega^m$ obtaining the minimum of $||\mathbf{y} - \mathbf{H}\mathbf{s}||^2$, and Ω^m is the set of constellation symbols in the m dimensional real constellation Ω. For example, M^2-QAM can be transformed to M-PAM, e.g., $\Omega = \{-3, -1, 1, 3\}$ in 16-QAM constellation. Because the computational complexity of ML detection increases exponentially with the number of transmit antennas N_T and the size of constellation, the SD algorithm [2] reduces the complexity of the ML detector.

III. Sphere Decoder Algorithm

The main idea of the SD is restricting the search space for detection from all the constellation points to a hypersphere with a certain radius d around the received signal. The FP-SD and SE-SD [3] [4] are two efficient methods to achieve this idea. Here, we briefly introduce the basic principle of SD algorithm. Based on equation (3) and the QR factorization of $\mathbf{H}$ ($\mathbf{H} = \mathbf{Q}\mathbf{R}$), where $\mathbf{R}$ is an upper-triangular matrix and $\mathbf{Q}$ is an unitary matrix, then let $\mathbf{z} = \mathbf{Q}^H\mathbf{y}$, so equation (3) can be equivalent to

$$\hat{\mathbf{s}} = \arg\min_{\mathbf{s}\in\Phi} ||\mathbf{z} - \mathbf{R}\mathbf{s}||^2. \tag{4}$$

Φ should be the set of all points which satisfy $||\mathbf{z} - \mathbf{R}\mathbf{s}||^2 \le d^2$. Further, it can be expanded to

$$\sum_{i=1}^{m}\left(z_i - \sum_{j=i}^{m} r_{i,j}s_j\right)^2 \le d^2. \tag{5}$$

From the above inequality, it can be solved by considering the above conditions in the order from m to 1. For example, given the symbols $s_{i+1}, \ldots, s_m$, the element s_i can be chosen from the range of $LB_i \le s_i \le UB_i$ where

$$LB_i = \left\lceil \frac{1}{r_{i,i}}\left(z_i - \sum_{j=i+1}^{m} r_{i,j}s_j - d_i\right)\right\rceil \tag{6}$$

$$UB_i = \left\lfloor \frac{1}{r_{i,i}}\left(z_i - \sum_{j=i+1}^{m} r_{i,j}s_j + d_i\right)\right\rfloor \tag{7}$$

and $d_i^2 = d^2 - \sum_{k=i+1}^{m}\left(z_k - \sum_{j=k}^{m} r_{k,j}s_j\right)^2$, where $\lceil\cdot\rceil$ denotes the smallest integer greater than or equal to its argument and $\lfloor\cdot\rfloor$ denotes the largest integer less than or equal to its argument. For the FP-SD, [11] suggests a way to choose the initial radius d. The method utilizes the noise variance per antenna $\sigma_r^2 = \frac{\sigma^2}{2}$, where σ_r^2 denotes the noise variance per real dimension. Accordingly, the initial radius can be set to

$$d^2 = \alpha n \sigma_r^2. \tag{8}$$

Here, α is the parameter chosen to ensure that at least one signal vector is found inside the hypersphere with high probability. If none found, the radius increases and the SD runs again. Note that for the actual transmitted signal vector $\mathbf{s}$, the distance metric

$$||\mathbf{y} - \mathbf{H}\mathbf{s}||^2 = ||\mathbf{n}||^2 \tag{9}$$

is a χ_n^2 distributed variable with n degrees of freedom [11]. So we can choose α to satisfy

$$\int_0^{\frac{\alpha n}{2}} \frac{\lambda^{\frac{n}{2}-1}}{\Gamma(\frac{n}{2})} e^{-\lambda} d\lambda = 1 - \varepsilon \tag{10}$$

where $1-\varepsilon$ is set to the value very close to 1, such as $1-\varepsilon = 0.999$. If no point is found, the probability $1-\varepsilon$ should be increased and the search should be run again.

However, in the SE-SD, the initial radius can be set to a very large value such as $d = \infty$ to make sure that at least one point is included in the hypersphere. The main idea of the SE-SD is to compute their connecting branch weights and then explore them in an increasing order of these weights and update the radius to be $||\mathbf{y} - \mathbf{H}\hat{\mathbf{s}}_{new}||^2$, where $\hat{\mathbf{s}}_{new}$ is a found candidate satisfying (5). This progress continues until all the points in the hypersphere are found.

The SE-SD is a much more attractive method for simulation and implementation by contrast to the FP-SD. Our new CSD can be exploited to improve either of these SD methods. It reduces the complexity significantly and achieves almost fixed complexity at a negligible performance loss.

IV. Complexity-Efficient Sphere Decoder

As discussed in [4], although the FP-SD and SE-SD achieve optimal performance with much lower complexity compared to the naive ML detector, the computational complexity is variable and still high within the low SNR region. We propose the CSD based on SNR, which efficiently reduces the complexity and obtains approximately fixed complexity.

A. Algorithm Theory

Our main idea is scaling the search radius, which is defined as

$$d_{CSD}^2 = \frac{\rho}{\rho + C} \times d^2, \tag{11}$$

where d_{CSD} is the radius in the CSD, ρ is the SNR of the MIMO system, d is the radius used in basic FP-SD or SE-SD, and C is a certain constant to guarantee that more nodes are pruned in the low SNR region and fewer points are pruned in the high SNR region. Because of

$$\lim_{\rho\to\infty} \frac{\rho}{\rho + C} = 1, \tag{12}$$

the performance of the proposed SD reverts to that of the original SD when the SNR is sufficiently high.

Corresponding to the FP-SD, we apply our idea in this algorithm, which is called the C-FP-SD. So base on (8), the initial radius should be

$$\begin{aligned} d_{CSD}^2 &= \frac{\rho}{\rho+C} \times d^2 \\ &= \frac{\rho}{\rho+C} \times \alpha n \sigma_r^2 \\ &= \frac{\alpha n m}{4(\rho+C)}. \end{aligned} \tag{13}$$

The last step for the above equation is base on that the SNR in complex MIMO system is given by $\rho = \frac{NE(|s|^2)}{\sigma^2}$, and we know $m = 2N$ and assume the average energy of each symbol $E(|s|^2) = 1$.

Because the radius is not updated in the C-FP-SD just like in the FP-SD, so the following process for this algorithm is just same to we have mentioned in Section III.

By comparison to the FP-SD, the SE-SD further reduces the complexity. We also can improve the SE-SD according to our new method called the C-SE-SD, which is described as follows.

The C-SE-SD function:

1) Initial the radius $d_{CSE} = \infty$;
2) Generate all the children denoted by the set $\mathcal{T}$ in the k-th level which satisfy

$$\left(z_k - \sum_{j=k}^{m} r_{k,j} s_j\right)^2 \leq d_k^2, \tag{14}$$

where $d_k^2 = d_{CSE}^2 - \sum_{i=k+1}^{m} \left(z_i - \sum_{j=i}^{m} r_{i,j} s_j\right)^2$ and $k \in \{m, m-1, \ldots, 1\}$;
3) Sort the components in $\mathcal{T}$ by the increasing order of the branch cost c_i in this level,

$$c_i = \left(z_i - r_{i,i} s_i - \sum_{j=i+1}^{m} r_{i,j} s_j\right)^2 \tag{15}$$

and $s_i \in \mathcal{T}$, with $i \in \{1, 2, \ldots, N'\}$ and N' is the number of elements in $\mathcal{T}$;
4) For $i = 1$ to N'

    $\hat{s}_k = s_i$;
    if $c_i < d_k^2$ then
        if $k = 1$ then
            $\hat{\mathbf{s}}_{new} = \hat{\mathbf{s}}$;
            $d_{CSE}^2 = \frac{\rho}{\rho+C} \, ||\mathbf{z} - \mathbf{R}\hat{\mathbf{s}}_{new}||^2$;
        else
            $k = k - 1$;
            Go to 2);
        end
    end
5) end

The C-SE-SD is the variant according to the SE-SD [3], while it achieves critical improvement in computational complexity of the SD and almost equivalent SER to the optimal detection.

B. Complexity Analysis

The SD has received enormous attraction because of the reduced complexity over the exhaustive search; hence, it is very important to evaluate its complexity for the implementation. Reference [11] has analyzed the complexity of the FP-SD, which is easier than the complexity evaluation of the SE-SD, where the update of radius and the ordering in each level make it much more difficult to analyze the complexity theoretically. Therefore, in this paper, we only analyze the C-FP-SD theoretically, while resorting to simulation to evaluate the complexity of the C-SE-SD.

The complexity of the SD is proportional to the average number of nodes visited by each symbol detection in the searching tree. From [11], the complexity is related to the number of antennas, the initial radius and the noise variance. In this paper, we only consider the number of nodes visited by all the levels, so the expected complexity of the SD is given by

$$C(m, \sigma_r^2, d^2) = \sum_{k=1}^{m} (\text{The number of nodes visited in } k \text{ level within the hypersphere of radius } d), \tag{16}$$

where d^2 is provided by (8).

Furthermore, we only show the theoretical complexity for 16-QAM systems, which is equivalent to two real 4-PAM constellations $\left(\{-\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{3}{2}\}\right)$. Other constellations may be analyzed similarly, but are omitted for brevity. For consistency with the results of [11], we do not set the average energy of transmitted signals to 1. Therefore, according to [11, Theorem 2], the complexity of the FP-SD becomes

$$C_{FP}(m, \sigma_r^2, d^2) = \sum_{k=1}^{m} \sum_{q} \frac{1}{2^k} \sum_{l=0}^{k} \binom{k}{l} \times g_{kl}(q) \gamma \left(\frac{d^2}{2(\sigma_r^2 + q)}, \frac{n-m+k}{2}\right), \tag{17}$$

where $g_{kl}(q)$ is the coefficient of x^q in the polynomial

$$(1 + x + x^4 + x^9)^l (1 + 2x + x^4)^{k-l}.$$

Similarly, due to $\sigma_r^2 = \frac{m}{\rho} \times \frac{L^2-1}{12}$ (for 4-PAM, $L = 4$) and (13), the expected complexity of the C-FP-SD is given as

$$C_{C-FP}(m, \rho, d^2) = \sum_{k=1}^{m} \sum_{q} \frac{1}{2^k} \sum_{l=0}^{k} \binom{k}{l} \times g_{kl}(q) \gamma \left(\frac{\alpha n \rho}{2(\rho+C)(1+\frac{12\rho q}{m(L^2-1)})}, \frac{n-m+k}{2}\right). \tag{18}$$

In this paper, we define a measure of complexity for the range of SNR as

$$\eta = \frac{E(C - \bar{C})^2}{\bar{C}^2}, \tag{19}$$

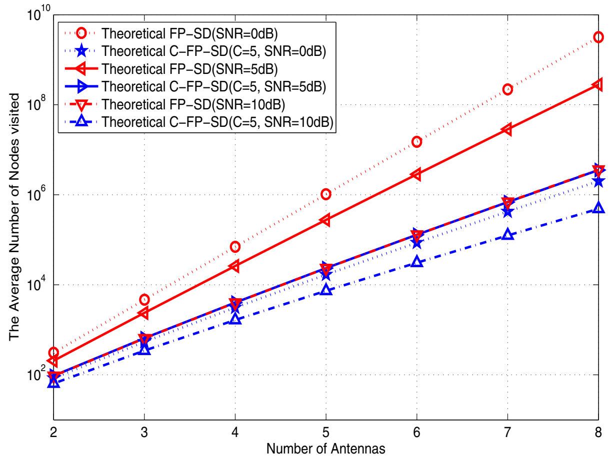

Fig. 1. Theoretical complexity comparison of the C-FP-SD and FP-SD with 16-QAM and different number of antennas.

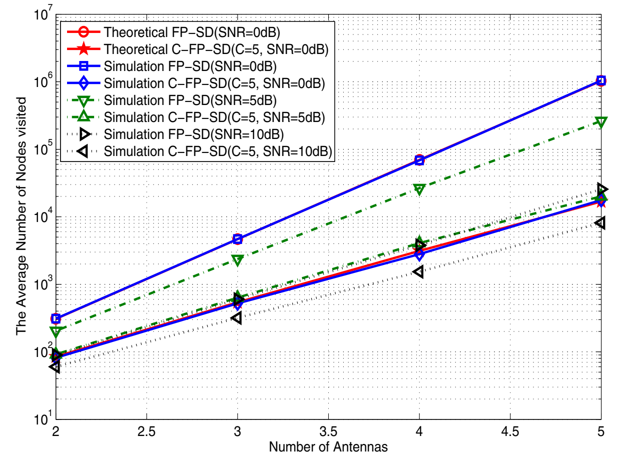

Fig. 2. Theoretical and simulation complexity comparison of the C-FP-SD and FP-SD with 16-QAM and different number of antennas.

where $\bar{C}$ and $E(C)$ denote the mean and the expectation of C, respectively. So with η smaller, the complexity is more fixed. From the above theoretical analysis, $\eta = 1.78$ for the FP-SD, while $\eta = 0.69$ for the C-FP-SD which means our new CSD efficiently reduces the variability of the complexity.

Fig. 1 shows the theoretical complexity (18) as a function of the number of antennas for several SNR values, the constant $C = 5$ as an example. The complexity of the C-FP-SD is much lower than the original FP-SD. Furthermore, the gap between the proposed algorithm and the FP-SD increases with the increasing number of antennas. For example, it can reduce the complexity 5×10^3 times by compared to the FP-SD for an 8×8 MIMO system when $\text{SNR} = 0\,\text{dB}$.

The complexity gap between these two algorithms depends on the SNR. The C-FP-SD provides more improvement in the low SNR region, while the improvement decreases with SNR increasing (Fig. 1).

Remarks:

1) The new CSD further reduces the computational complexity of the basic SD with near-ML performance, especially in the lower SNR region, while maintains the low complexity and near-optimal SER performance of the basic SD in the high SNR region. Another significant advantage is that the CSD effectively reduces the variability of the complexity of the basic SD.
2) This method can also be applied to other types of tree search algorithms for the detection of MIMO systems, such as many SD variants [5][7] according to channel ordering or different stopping criterion and so on.
3) In this paper, we do not discuss in detail the effects of the constant C in (11), which should not be very large. That is because that when the SNR is large enough, such as $20\,\text{dB}$, the complexity of the SE-SD is already low that is not needed to improve any further.

V. Simulation Results

In this section, the CSD is simulated for uncoded MIMO systems with different number of antennas over a flat Rayleigh fading channel. Due to the negligible difference between their performance in terms of SER, the FP-SD, C-FP-SD, SE-SD and C-SE-SD are compared only in terms of complexity, measured by the average total number of nodes visited by each algorithm. After that, the performance and complexity are simulated and compared for different SD strategies in 4×4 MIMO systems, in which 16-QAM modulation is exploited. In our simulations, we choose $C = 5$ for all the cases.

In Fig. 2, we compare the theoretical and simulation complexity for the C-FP-SD and FP-SD when $\text{SNR} = 0\,\text{dB}$. We set $1 - \varepsilon = 0.9999$. For different number of antennas, the simulation results of complexity are agree with the theoretically derived complexity. Furthermore, the simulation complexity for different SNRs is also given. Our C-FP-SD achieves significant complexity reduction in the low SNR region and for relatively large number of antennas (Section IV-B).

Fig. 3 shows the simulation complexity comparison among the FP-SD, C-FP-SD, SE-SD and C-SE-SD. The C-SE-SD gets the lowest complexity in these four algorithms, for example, in 8×8 MIMO system ($\text{SNR} = 0\,\text{dB}$), the complexity of the C-SE-SD is around 10^3 while it is 10^5 for the SE-SD and more than 10^9 for the FP-SD. Therefore, our new CSD efficiently reduces the complexity and the C-SE-SD is suitable for practical implementation.

Finally, the SER performance and the average number of nodes visited for different decoders in 4×4 16-QAM MIMO system are shown in Fig. 4 and Fig. 5. In our simulations, we compare the CSD with the K-best SD with $K = 4$ (K is the number of detected nodes in each level) [8], the FSD with $p = 1$ (p is the number of level where the maximum number of nodes are considered) [9], the FP-SD and the SE-SD.

Fig. 4 gives the SER performance comparison among these

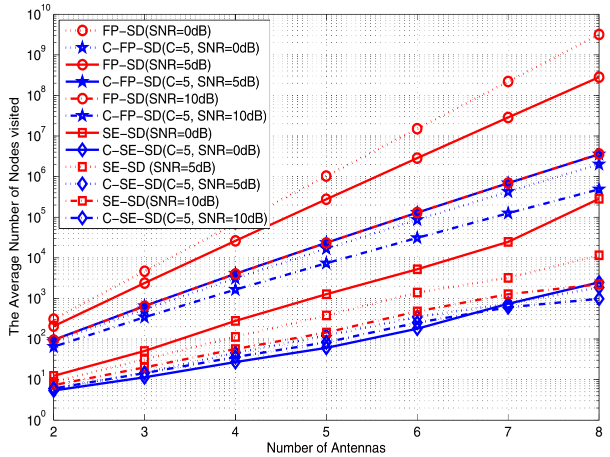

Fig. 3. Theoretical and simulation complexity comparison of different decoders with 16-QAM and different number of antennas.

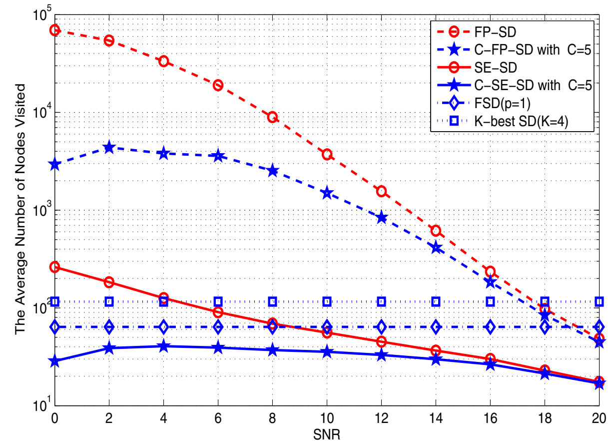

Fig. 5. Complexity comparison of different decoders for an 4×4 MIMO system with 16-QAM.

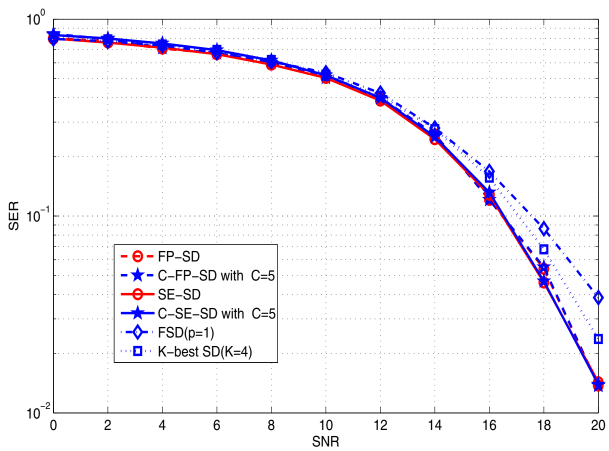

Fig. 4. Performance comparison of different decoders for an 4×4 MIMO system with 16-QAM.

decoders. It is clear that our new CSD including the C-FP-SD and C-SE-SD performs very close to the optimal decoder (SE-SD), and the performance loss in the low SNR region can be negligible. Moreover, it is apparently better than K-best SD and FSD in the higher SNR region.

We compare the average number of nodes visited for all the decoders in Fig. 5. In the low SNR region, the complexity of the C-FP-SD gets lower complexity than the FP-SD, while the C-SE-SD obtains complexity reduction than the SE-SD. More importantly, the complexity of our proposed C-SE-SD is even less than K-best SD and FSD in all SNR region with better performance than them as shown in Fig. 4. Furthermore, we get $\eta = 0.0545$ for the C-SE-SD according to (19) which means approximately fixed complexity.

VI. Conclusions

In this paper, we proposed an improved variant of the SD, which provides significant complexity improvement at a negligible performance loss. The idea is to tighten the hypersphere by reducing the radius according to SNR. Our new CSD achieves almost equivalent SER performance to the optimum ML detection at an almost fixed low complexity. Furthermore, it outperforms the K-best SD and the FSD not only in SER performance but also in computational complexity. Because of the reduction in both computational complexity and the variability of complexity, the proposed CSD is suitable for hardware implementation in practice.

References

[1] E. Biglieri, R. Calderbank, A. Constantinides, A. Goldsmith, A. Paulraj, and H. V. Poor, *MIMO Wireless Communications*. New York, NY, USA: Cambridge University Press, 2007.

[2] E. Viterbo and J. Boutros, "A universal lattice code decoder for fading channels," *IEEE Trans. Inf. Theory*, vol. 45, no. 5, pp. 1639–1642, Jul. 1999.

[3] E. Agrell, T. Eriksson, A. Vardy, and K. Zeger, "Closest point search in lattices," *IEEE Trans. Inf. Theory*, vol. 48, no. 8, pp. 2201–2214, Aug. 2002.

[4] M. Damen, H. El Gamal, and G. Caire, "On maximum-likelihood detection and the search for the closest lattice point," *IEEE Trans. Inf. Theory*, vol. 49, no. 10, pp. 2389–2402, Oct. 2003.

[5] J.-S. Kim, S.-H. Moon, and I. Lee, "A new reduced complexity ML detection scheme for MIMO systems," *IEEE Trans. Commun.*, vol. 58, no. 4, pp. 1302–1310, Apr. 2010.

[6] T. Cui, T. Ho, and C. Tellambura, "Statistical pruning for near maximum likelihood detection of MIMO systems," in *Proc. IEEE Int. Conf. on Commun.(ICC)*, Jun. 2007, pp. 5462 –5467.

[7] B. Shim and I. Kang, "On further reduction of complexity in tree pruning based sphere search," *IEEE Trans. Commun.*, vol. 58, no. 2, pp. 417–422, Feb. 2010.

[8] Z. Guo and P. Nilsson, "Algorithm and implementation of the K-best sphere decoding for MIMO detection," *IEEE J. Sel. Areas Commun.*, vol. 24, no. 3, pp. 491–503, Mar. 2006.

[9] L. Barbero and J. Thompson, "Fixing the complexity of the sphere decoder for MIMO detection," *IEEE Trans. Wireless Commun.*, vol. 7, no. 6, pp. 2131–2142, Jun. 2008.

[10] O. Damen, A. Chkeif, and J.-C. Belfiore, "Lattice code decoder for space-time codes," *IEEE Commun. Lett.*, vol. 4, no. 5, pp. 161–163, May 2000.

[11] B. Hassibi and H. Vikalo, "On the sphere-decoding algorithm I. expected complexity," *IEEE Trans. Signal Process.*, vol. 53, no. 8, pp. 2806–2818, Aug. 2005.