Asymptotically-Exact Performance Bounds of AF Multi-Hop Relaying over Nakagami Fading

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***Abstract*—A new class of upper bounds on the end-to-end signal-to-noise ratio (SNR) of channel-assisted amplify-and-forward (AF) multi-hop ($N \geq 2$) relay networks is presented. It is the half-harmonic mean of the minimum of the first $P \geq 0$ hop SNRs and the minimum of the remaining $N - P$ hop SNRs. The parameter P varies between 0 to N and may be chosen to provide the tightest bound. The closed-form cumulative distribution function and moment generating function are derived for independent and non-identically distributed Rayleigh fading and for independent and identically distributed Nakagami-m fading, where m is an integer. The resulting outage probability and the average symbol error rate bounds are asymptotically-exact. The asymptotic-exactness holds for any $0 \leq P \leq N$. As applications, two cases of multi-hop multi-branch relay networks (i) the best branch selection and (ii) maximal ratio combining reception are treated. Numerical results are provided to verify the comparative performance against the existing bounds.**

***Index Terms*—Multi-hop relay networks, amplify-and-forward, Nakagami-m fading, average symbol error rate, outage probability.**

I. Introduction

MULTI-HOP relay networks achieve broader coverage and enhanced throughput due to shorter hops and can also provide network connectivity to locations where traditional single-hop networks may not reach [1]. As well, the battery life of the terminals may be prolonged due to lower power requirements [1]. Moreover, such networks also achieve spatial diversity gains to enhance the system performance. Due to these reasons, their performance has been widely researched [2]–[7].

Prior related research: For a multi-hop network with non-regenerative relays, exact closed-form analytical performance results for a number of hops $N \geq 3$ appear to be intractable; even for $N = 2$ case, the exact analytical results are rather complicated. Thus, previous performance analyses provide bounds on the end-to-end[1] signal-to-noise ratio (SNR) [2], [5], [8], [9] or asymptotic approximations and numerical methods [3], [4], [6], [10], [11]. For example, in [2], the multi-hop SNR is upper bounded by the geometric mean of hop SNRs. The moment generating function (MGF), the probability density function (PDF), and the cumulative distribution function (CDF) of the upper bound are then derived. Closed-form lower bounds on the outage probability and the average bit error rate (BER) of the coherent binary modulation are also derived. In [8], the results of [2] is used to study the performance of multi-hop semi-blind relays over generalized fading channels. In [5], the bound of [8] is further employed for performance analysis of certain multi-hop relay networks. Reference [9] proposes an SNR upper bound for a multi-hop channel-assisted amplify-and-forward (CA-AF) relay network by using the minimum SNR of all hops [9, Eq. (11)]. The average BER of several modulation schemes over fading channels is also computed. Reference [12, Ch. 3, pp. 31-38] analyzes the performance of a multi-hop CA-AF relay network over Weibull fading by using the upper bound of [9].

Examples for approximations and/or numerical methods are [3], [4], [6], [10], and [11]. In [3], the outage probability of a multi-hop CA-AF relay network over Nakagami-m fading is evaluated. The MGF of the reciprocal of the SNR is derived in closed-form, and the outage probability is computed via numerical Laplace-transform inversion. A comprehensive performance analysis of a multiple-hop and multiple-branch cooperative network is proposed in [6]. The main idea is to relate the MGF of X to the MGF of $1/X$, which requires numerical integration in some cases. Reference [10] provides an asymptotic analysis of the error rates of multi-hop multi-branch relay networks. Moreover, the performance of multi-hop AF relays over independent and non identically distributed (i.n.i.d.) Rayleigh fading channels is studied in [4]. In [11], the asymptotic BER of multi-hop AF relaying over Nakagami-m fading is analyzed.

Motivation and our contribution: Although the performance bounds of [2], [5] and [8] are tight in low SNRs, they weaken for high SNRs and for severe fading environments such as Rayleigh fading. These bounds may thus not provide an accurate assessment of system performance. Moreover, while the performance analyses of multi-hop CA-AF relay networks [3] and that of multi-hop multi-branch relay network [6] are available, the performance metrics are not in closed-forms. Thus, these gaps in the performance analysis of multi-hop relay networks, arising mainly due to the intractability of the problem, motivated us to develop new asymptotically-exact performance bounds.

In this work, a class of new upper bounds is derived for the SNR of a N-hop $\{N \geq 2\}$ CA-AF relay network. The key idea is to bound the SNR by the half-harmonic mean of the minimum of the first P hop SNRs and the minimum of the next $N - P$ hop SNRs, where $0 \leq P \leq N$, and N is the number of hops in the system. Here, P is a free parameter used

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[1]Throughout this letter, SNR directly refers to the end-to-end SNR, unless otherwise stated.

to provide flexibility and generality. For example, the special cases $P = 0$ or $P = N$ result in the bounds of Hasna [9]. It may also be viewed as a tunable parameter to get the tightest bound. The CDF and the MGF are derived in closed-form. For the sake of brevity, only the cases of i.n.i.d. Rayleigh fading channels, and independent and identically distributed (i.i.d.) Nakagami-m fading, where m is a positive integer, are treated. Closed-form lower bounds for the outage probability and the average symbol error rate (SER) are also derived. These bounds are **asymptotically-exact** (Corollary 3). Numerical results are presented to compare the proposed bounds with the existing bounds [2], [9]. Monte-Carlo simulation results are provided to verify the accuracy of our analytical results. Two applications of our results are demonstrated for the multi-hop multi-branch relay networks.

The rest of this latter is organized as follows. In Section II, the system model and the channel model are presented. Sections III and V present the performance analysis and numerical results. Section VI concludes this letter.

Notations: $\mathcal{K}_\nu(z)$ is the Modified Bessel function of the second kind of order ν [13, Eq. (8.407.1)]. ${}_2\mathcal{F}_1(\alpha, \beta; \gamma; z)$ is the Gauss Hypergeometric function [13, Eq. (9.14.1)]. $\mathcal{E}_\Lambda\{.\}$ denotes the expected value over the random variable Λ. $\mathcal{Q}(z)$ denotes the Gaussian Q-function[14, Eq. (26.2.3)]. $X \sim \mathcal{G}(\alpha, \beta)$ means X is distributed with Gamma(α, β) PDF. $\mathbb{Z}^+$ is the set of positive integers. $\lceil z \rceil$ denotes the smallest integer not less than z.

II. System And Channel Models

We consider a multi-hop relay network with N hops, source (S), destination (D) and $N-1$ AF relays. Only single-antenna terminals are used. The relays are CA-AF type [3], [15], [16]. The gain of a CA-AF relay n is $G_n = \sqrt{\frac{\mathcal{P}_n}{\mathcal{P}_n |h_n|^2 + N_{0,n}}}$ [3], [15], where $\mathcal{P}_n$ is the average energy per symbol used at the n-th relay, $|h_n|$ is the fading amplitude of the preceding hop, and $N_{0,n}$ is the variance of the zero-mean additive white Gaussian noise at the input of the n-th receiver. The SNR γ_{eq} of a multi-hop CA-AF relay network is given by [3] $\gamma_{\text{eq}} = \left[\prod_{n=1}^{N}\left(1 + \frac{1}{\gamma_n}\right) - 1\right]^{-1}$, where $\gamma_n = \mathcal{P}_n |h_n|^2 / N_{0,n}$ is the SNR of the n-th hop. Since the exact distribution function of γ_{eq} is mathematically intractable, reference [3] shows that γ_{eq} can be tightly upper bounded by a more tractable form as follows:

$$\gamma_{\text{eq,ideal}} = \left[\sum_{n=1}^{N} \frac{1}{\gamma_n}\right]^{-1}. \tag{1}$$

Then the gain of the n-th relay is given by $G_n = 1/|h_n|$ and this gain corresponds to an ideal CA-AF relay, which is capable of inverting the channel of the previous hop (regardless of the fading state of that hop) [3]. The performance measures of multi-hop relay networks with ideal CA-AF relays serve as benchmarks for systems with various practical relays.

In order to analyze the system performance, statistics for the SNR (1) are required. However, the probability distribution of (1) is not mathematically tractable, particularly for $N \geq 3$. Thus, in order to develop a more accurate performance analysis framework, we propose a new upper bound for (1). The key idea is to partition the set of $\gamma_n|_{n=1}^{N}$ into two groups. The minimum of γ_n of each group is then used to bound (1) as follows:

$$\gamma_{\text{eq,ideal}} \leq \gamma_{\text{eq}}^{\text{ub}} = \left[\frac{1}{\min\limits_{1 \leq n \leq N-P}(\gamma_n)} + \frac{1}{\min\limits_{N-P+1 \leq n \leq N}(\gamma_n)}\right]^{-1}, \tag{2}$$

where $0 \leq P \leq N$. The SNR bound $\gamma_{\text{eq}}^{\text{ub}}$ in (2) is related to the harmonic mean of the minimum of SNR of the first P hops and the minimum of the next $N - P$ hops. Intuitively, we expect the tightness of the bound to increase as P gets closer to $N - P$. Thus, $P = \lceil \frac{N}{2} \rceil$ is a good choice.

Interestingly, when $P = 0$ or $P = N$, (2) reduces to the bound given by Hasna [9, Eq. (11)]. Note that (2) with $N = 2$ and $P = 1$ reduces to the exact SNR for the case of dual-hop systems with ideal CA-AF relays.

III. Performance Analysis

This section presents the performance analysis of multi-hop relay networks by using (2). By first finding the distribution of $\gamma_{\text{eq}}^{\text{ub}}$ for i.n.i.d. Rayleigh fading and i.i.d. Nakagami-m fading, $m \in \mathbb{Z}^+$, we derive the outage probability and the average SER.

A. Statistical characterization of the SNR

The CDF and the MGF of $\gamma_{\text{eq}}^{\text{ub}}$ in i.i.d. Nakagami-m fading are given by Theorem 1.

Theorem 1: Let $\gamma_n \sim \mathcal{G}(m, \frac{\bar{\gamma}}{m})$, $n = 1, \ldots, N$, be independent hop SNRs. The CDF of γ_{eq}^{ub} is then given by

$$\begin{aligned} F_{\gamma_{\text{eq}}^{\text{ub}}}(x) &= 1 - \sum_{j=0}^{P(m-1)} \sum_{k=0}^{(N-P-1)(m-1)} \sum_{l=0}^{m+j+k-1} \frac{2}{\Gamma(m)} \\ &\times \beta_{j,P}\beta_{k,N-P-1}\binom{m+j+k-1}{l}\frac{P^{\frac{l-j+1}{2}}}{(N-P)^{\frac{l-j-1}{2}}} \\ &\times \left(\frac{mx}{\bar{\gamma}}\right)^{m+j+k} \exp\left(-\frac{mNx}{\bar{\gamma}}\right) \\ &\times \mathcal{K}_{l-j+1}\left(\frac{2m}{\bar{\gamma}}\sqrt{P(N-P)}\,x\right), \end{aligned} \tag{3}$$

where

$$\beta_{k,N} = \sum_{i=k-m+1}^{k} \frac{\beta_{i,N-1}}{(k-i)!} I_{[0,(N-1)(m-1)]}(i). \tag{4}$$

Here, $I_{[a,c]}(b) = \begin{cases} 1, & a \leq b \leq c \\ 0, & \text{otherwise} \end{cases}$, $\beta_{0,0} = \beta_{0,N} = 1$, $\beta_{k,1} = 1/k!$, and $\beta_{1,N} = N$.

The MGF of γ_{eq}^{ub} is given by

$$\begin{aligned} M_{\gamma_{\text{eq}}^{\text{ub}}}(s) &= 1 - \sum_{j=0}^{P(m-1)} \sum_{k=0}^{(N-P-1)(m-1)} \sum_{l=0}^{m+j+k-1} \frac{2}{\Gamma(m)} \\ &\times \beta_{j,P}\beta_{k,N-P-1}\binom{m+j+k-1}{l}\frac{P^{\frac{l-j+1}{2}}}{(N-P)^{\frac{l-j-1}{2}}} \\ &\times \left(\frac{m}{\bar{\gamma}}\right)^{m+j+k} s\,\mathbb{I}(\mu, \nu, \alpha, \beta), \end{aligned} \tag{5}$$

where

$$\mathbb{I}(\mu,\nu,\alpha,\beta) = \frac{\sqrt{\pi}(2\beta)^{\nu}\Gamma(\mu+\nu)\Gamma(\mu-\nu)}{\Gamma(\mu+\frac{1}{2})(\alpha+\beta)^{\mu+\nu}} \times {}_2\mathcal{F}_1\left(\mu+\nu,\nu+\frac{1}{2};\mu+\frac{1}{2};\frac{\alpha-\beta}{\alpha+\beta}\right). \quad (6)$$

Here, $\mu = m+j+k+1$, $\nu = l-j+1$, $\alpha = s+\frac{mN}{\bar{\gamma}}$ *and* $\beta = \frac{2m}{\bar{\gamma}}\sqrt{P(N-P)}$.

Proof: See Appendix. ■

Remark: III.1: The CDF (3) and the MGF (5) for i.i.d. Nakagami-m fading do not hold for $P=0$ or $P=N$. Thus, the CDF of $\gamma_{\text{eq}}^{\text{ub}}$ for $P=0$ or $P=N$ is derived explicitly as

$$F_{\gamma_{\text{eq}}^{\text{ub}}}(x) = 1-\exp\left(-\frac{mNx}{\bar{\gamma}}\right)\sum_{k=0}^{N(m-1)}\beta_{k,N}\left(\frac{mx}{\bar{\gamma}}\right)^{k}. \quad (7)$$

The corresponding MGF of $\gamma_{\text{eq}}^{\text{ub}}$ for $P=0$ or $P=N$ is derived as

$$M_{\gamma_{\text{eq}}^{\text{ub}}}(s) = 1-\sum_{k=0}^{N(m-1)}\frac{\beta_{k,N}\Gamma(k+1)\,\bar{\gamma}s}{m}\left(\frac{m}{mN+\bar{\gamma}s}\right)^{k+1}. \quad (8)$$

Further, for the sake of completeness, the CDF and the MGF of $\gamma_{\text{eq}}^{\text{ub}}$ in i.n.i.d. Rayleigh fading are also given as follows: The CDF of $\gamma_{\text{eq}}^{\text{ub}}$ is given by

$$F_{\gamma_{\text{eq}}^{\text{ub}}}(x) = 1-2\sqrt{\lambda_1\lambda_2}\;x\exp(-\lambda_0 x)\,\mathcal{K}_1\left(2x\sqrt{\lambda_1\lambda_2}\right), \quad (9)$$

where $\lambda_1 = \sum_{n=1}^{N-P}\frac{1}{\bar{\gamma}_n}$, $\lambda_2 = \sum_{n=N-P+1}^{N}\frac{1}{\bar{\gamma}_n}$ and $\lambda_0 = \lambda_1+\lambda_2$. The MGF of $\gamma_{\text{eq}}^{\text{ub}}$ is given by

$$M_{\gamma_{\text{eq}}^{\text{ub}}}(s) = 1-\frac{64}{3}\lambda_1\lambda_2 s\frac{{}_2\mathcal{F}_1\left(3,\frac{3}{2};\frac{5}{2};\frac{s+\lambda_0-2\sqrt{\lambda_1\lambda_2}}{s+\lambda_0+2\sqrt{\lambda_1\lambda_2}}\right)}{\left(s+\lambda_0+2\sqrt{\lambda_1\lambda_2}\right)^{3}}. \quad (10)$$

The PDF of $\gamma_{\text{eq}}^{\text{ub}}$ can easily be derived by differentiating the CDF with the help of [13, Eq. 8.486.12]. However, for the sake of brevity, the PDF results are omitted.

For direct insight, we derive the asymptotic outage probability from (3) as follows:

Corollary 1: *Let* $\gamma_n \sim \mathcal{G}(m,\frac{\bar{\gamma}}{m})$, $n=1,\ldots,N$, *be independent hop SNRs. The asymptotic outage probability obtained by using* γ_{eq}^{ub} *as* $\bar{\gamma}\to\infty$ *is then given by*

$$P_{\text{out}}^{\infty} = \frac{Nm^{m}}{\Gamma(m+1)}\left(\frac{\gamma_{th}}{\bar{\gamma}}\right)^{m} + o\left(\bar{\gamma}^{-(m+1)}\right). \quad (11)$$

Proof: The behavior of the CDF for large $\bar{\gamma}$ is equivalent to the behavior of $F_{\gamma_{\text{eq}}^{\text{ub}}}(y)$ around $y=0$ [17]. By substituting $x=\bar{\gamma}y$ into (3), and by using the Taylor series expansions of the Exponential and Bessel functions [13, Eq. (1.211) and Eq. (8.446)], the CDF can be approximated near the origin. Then the asymptotic outage probability can be obtained by evaluating the asymptotic CDF at γ_{th}. ■

The asymptotic outage probability of i.n.i.d. Rayleigh fading can be given as a special case of the Corollary 1 as

$$P_{\text{out}}^{\infty} = \left(\sum_{n=1}^{N}\frac{1}{C_n}\right)\frac{\gamma_{th}}{\bar{\gamma}} + o\left(\bar{\gamma}^{-2}\right), \quad (12)$$

where $C_n|_{n=1}^{N} = \frac{\bar{\gamma}_n}{\bar{\gamma}}$.

Because our bounds are asymptotically-exact (Section V), P_{out}^{∞} in (11) provides the exact asymptotic outage probability for multi-hop AF relay networks. Our asymptotic outage analysis also reveals that the diversity orders of multi-hop relay networks over i.i.d. Nakagami-m fading and i.n.i.d. Rayleigh fading are m and 1, respectively.

B. Outage probability

The outage is the probability that the instantaneous SNR γ_{eq} falls below a certain target value γ_{th}. Thus, the lower bounds for the outage probability P_{out} for i.n.i.d. Rayleigh and i.i.d. Nakagami-m fading can immediately be obtained by using the results given in (9) and (3): $P_{out} = \Pr(\gamma_{\text{eq}}^{\text{ub}} \le \gamma_{th}) = F_{\gamma_{\text{eq}}^{\text{ub}}}(\gamma_{th})$.

C. Average error rate

The average SER is one of the most widely used performance metrics of digital communication systems. The conditional error probability (CEP) $P_e|\gamma$ in this case is averaged over the PDF of $\gamma_{\text{eq}}^{\text{ub}}$. For example, the CEP of coherent binary frequency shift keying (C-BFSK) and M-ary pulse amplitude modulation (PAM) can be expressed as $P_e|\gamma = a\mathcal{Q}(\sqrt{b\gamma})$, where a and b are modulation-dependent constants. The SER can be simplified by integrating by parts as $\bar{P}_e = \frac{a}{2}-\frac{a}{2}\sqrt{\frac{b}{2\pi}}\int_0^{\infty}x^{-\frac{1}{2}}\exp\left(-\frac{bx}{2}\right)\bar{F}_{\gamma_{\text{eq}}^{\text{ub}}}(x)\mathrm{d}x$, where $\bar{F}_{\gamma_{\text{eq}}^{\text{ub}}}(x)$ is the complementary cumulative distribution function (CCDF) of $\gamma_{\text{eq}}^{\text{ub}}$ defined by $1-F_{\gamma_{\text{eq}}^{\text{ub}}}(x)$. The average SER bound for i.i.d. Nakagami-m fading is given by Corollary 2.

Corollary 2: *Let* $\gamma_n \sim \mathcal{G}(m,\frac{\bar{\gamma}}{m})$, $n=1,\ldots,N$, *be independent hop SNRs. The average SER bound obtained by using* γ_{eq}^{ub} *is then lower bounded by*

$$\bar{P}_e = \frac{a}{2}-a\sqrt{\frac{b}{2\pi}}\sum_{j=0}^{P(m-1)}\sum_{k=0}^{(N-P-1)(m-1)}\sum_{l=0}^{m+j+k-1}\frac{1}{\Gamma(m)} \times \beta_{j,P}\beta_{k,N-P-1}\binom{m+j+k-1}{l}\frac{P^{\frac{l-j+1}{2}}}{(N-P)^{\frac{l-j-1}{2}}} \times \left(\frac{m}{\bar{\gamma}}\right)^{m+j+k}\mathbb{I}(\mu,\nu,\alpha,\beta), \quad (13)$$

where $\mu = m+j+k+\frac{1}{2}$, $\nu = l-j+1$, $\alpha = \frac{b}{2}+\frac{mN}{\bar{\gamma}}$, $\beta = \frac{2m}{\bar{\gamma}}\sqrt{P(N-P)}$ *and* $\mathbb{I}(\mu,\nu,\alpha,\beta)$ *is defined in* (5).

Proof: The average SER (13) can be derived by substituting (3) into the integral representation of $\bar{P}_e$ in Section III-C and solving the resulting integral by using [13, Eq. (6.621.3)]. ■

Remark: III.2: The average SER bound in i.i.d. Nakagami-m fading (13) does not hold for $P=0$ or $P=N$. Thus, it is derived explicitly as

$$\bar{P}_e = \frac{a}{2}-\frac{a}{2}\sqrt{\frac{b}{2\pi}}\beta_{k,N}\Gamma\left(k+\frac{1}{2}\right)\left(\frac{m}{\bar{\gamma}}\right)^{k}\left(\frac{2\bar{\gamma}}{b\bar{\gamma}+2mN}\right)^{k+\frac{1}{2}}. \quad (14)$$

For the sake of completeness, the average SER for i.n.i.d. Rayleigh fading is given as follows:

$$\bar{P}_e = \frac{a}{2} - 3a\pi\sqrt{\frac{b}{2}}\lambda_1\lambda_2 \frac{{}_2\mathcal{F}_1\left(\frac{5}{2},\frac{3}{2};2;\frac{\frac{b}{2}+\lambda_0-2\sqrt{\lambda_1\lambda_2}}{\frac{b}{2}+\lambda_0+2\sqrt{\lambda_1\lambda_2}}\right)}{\left(\frac{b}{2}+\lambda_0+2\sqrt{\lambda_1\lambda_2}\right)^{\frac{5}{2}}}. \quad (15)$$

The lower bound for the average SER obtained by using our proposed upper bound for the SNR is asymptotically exact (see Section V). To prove this claim, we provide the following Corollary.

Corollary 3: *Let $\gamma_n \sim \mathcal{G}(1,\bar{\gamma}_n)$, $n = 1,\dots,N$, be independent hop SNRs. The asymptotic average SER obtained by using γ_{eq}^{ub} as $\bar{\gamma}_n \to \infty$ is then given by*

$$\bar{P}_e^{\infty} = \frac{a}{2b}\sum_{n=1}^{N}\frac{1}{C_n\bar{\gamma}} + o\left(\bar{\gamma}^{-2}\right). \quad (16)$$

Proof: The value at zero of the PDF of the random variable $\Gamma = \frac{1}{\Gamma_1}+\frac{1}{\Gamma_2}$ can be express as in [10, Eq. (16)]: $f_\Gamma(0) = f_{\Gamma_1}(0) + f_{\Gamma_2}(0)$. The PDF of Γ_1 and Γ_2 in i.n.i.d. Rayleigh fading are given by $f_{\Gamma_1}(x) = \left(\sum_{n=1}^{N-P}\frac{1}{\bar{\gamma}_n}\right)\exp\left(-\sum_{n=1}^{N-P}\frac{x}{\bar{\gamma}_n}\right)$ and $f_{\Gamma_2}(x) = \left(\sum_{n=N-P+1}^{N}\frac{1}{\bar{\gamma}_n}\right)\exp\left(-\sum_{n=N-P+1}^{N}\frac{x}{\bar{\gamma}_n}\right)$, respectively. Thus, $f_\Gamma(0)$ can easily be obtained as $f_\Gamma(0) = \sum_{n=1}^{N}\frac{1}{\bar{\gamma}_n}$. The asymptotic average SER (16) can then be derived by using [10, Eq. (10)]. ■

Note that Eq. (16) exactly agrees with the asymptotic exact average SER for multi-hop relay networks [10, Eq. (39)]. Numerical results in Section V too confirm this asymptotic-exactness. The corresponding asymptotic average SER for i.i.d. Nakagami-m fading can be derived by using (11) as

$$\bar{P}_e^{\infty} = \frac{aNm^m 2^{m-1}\Gamma(m+\frac{1}{2})}{\sqrt{\pi}(b\bar{\gamma})^m} + o(\bar{\gamma}^{-(m+1)}). \quad (17)$$

IV. Applications

In this section, two applications are presented to depict the usefulness of our proposed bounds.

A. *Outage probability of multi-hop multi-branch relay networks with the best branch selection*

Consider a multi-hop multi-branch relay network with L branches and $N_l|_{l=1}^{L}$ hops per branch. In this system, the source-to-destination communication is facilitated by $N_R = \sum_{l=1}^{L} N_l$ ideal CA-AF relays. We consider the best branch selection, where the destination selects the best branch with multiple hops having the largest instantaneous SNR. Now, we use our proposed upper bound of the SNR given in (2) to obtain an upper bound as $\gamma_{sc} \le \gamma_{sc}^{ub} = \max\{\gamma_{eq,1}^{ub},\gamma_{eq,2}^{ub},\dots,\gamma_{eq,L}^{ub}\}$, where $\gamma_{eq,l}^{ub}\Big|_{l=1}^{L}$ is the SNR of l-th multi-hop branch given in (2) with N replaced by N_l. For independent multiple branches, the CDF of γ_{sc}^{ub} can be derived as

$$F_{\gamma_{sc}^{ub}}(x) = \prod_{l=1}^{L} F_{\gamma_{eq,l}^{ub}}(x), \quad (18)$$

where $F_{\gamma_{eq,l}^{ub}}(x)\Big|_{l=1}^{L}$ is the CDF of $\gamma_{eq,l}^{ub}$ and can readily be obtained by using (3) and (9) for i.i.d. Nakagami-m and i.n.i.d. Rayleigh fading, respectively.

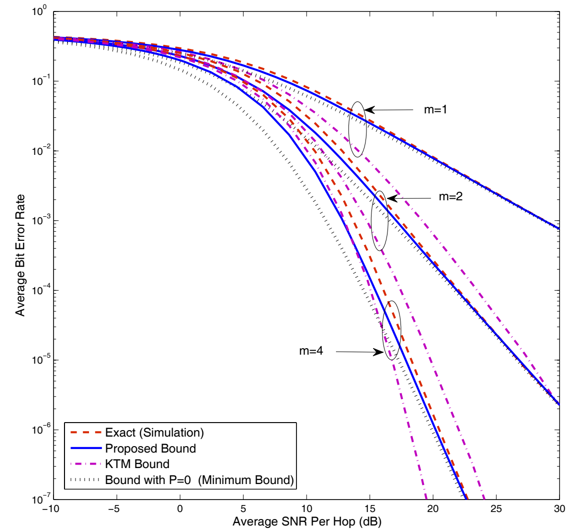

Fig. 1. A comparison of the effect of severity of fading on average BER bounds for a multi-hop relay network in i.i.d. Nakagami-m fading. $N = 3$ and $P = 2$.

B. *Average SER of multi-hop multi-branch relay networks with MRC Reception*

We consider the same network set-up in Section IV-A, however, in this case, the destination combines the signals received via all branches by using maximal ratio combining (MRC). We obtain an upper bound for the output SNR γ_{MRC} as follows: $\gamma_{\text{MRC}} = \sum_{l=1}^{L}\gamma_{\text{eq},l} \le \gamma_{\text{MRC}}^{\text{ub}} = \sum_{l=1}^{L}\gamma_{\text{eq},l}^{\text{ub}}$. For independent signals received via multiple branches at the destination, the MGF of $\gamma_{\text{MRC}}^{\text{ub}}$ can be expressed as $M_{\gamma_{\text{MRC}}^{\text{ub}}}(s) = \prod_{l=1}^{L} M_{\gamma_{\text{eq},l}^{\text{ub}}}(s)$, where $M_{\gamma_{\text{eq},l}^{\text{ub}}}(s)\Big|_{l=1}^{L}$ is the MGF of the upper bounded SNR of the l-th branch and can readily be obtained by using (5) and (10). By using [18], a compact closed-form approximation for the average SER can be derived as

$$\bar{P}_e = \frac{a}{2N_p}\sum_{j=1}^{N_p} M_{\gamma_{\text{MRC}}^{\text{ub}}}\left(\frac{b}{2}\sec^2(\theta_j)\right) + R_{N_p}, \quad (19)$$

where N_p is a small positive integer, $\theta_j = \frac{(2j-1)\pi}{4N_p}$ and R_{N_p} is the remainder term. R_{N_p} becomes negligible as N_p increases, even for small values such as 10 (see Section V).

V. Numerical Results

Numerical and simulation results are provided to investigate the tightness of the proposed performance bounds. Accordingly, they are compared with the existing multi-hop performance bounds [2], [9], [12] and exact simulations.

In Fig. 1, the proposed lower bounds of the binary phase shift keying (BPSK) average BER (by letting $a = 1$ and $b = 2$ in (13)) of a three-hop relay network operating over i.i.d. Nakagami-m fading are plotted. The BPSK average BER bound of [2, Eq. (24)] is plotted for comparison purposes. This bound is named the "KTM". Moreover, the proposed average BER bound with $P = 0$, which simplifies to the bound in [9, Eq. (11)] and [12, Ch. 3] ("Minimum bound") is also plotted. As expected, the proposed bound is tight, particularly

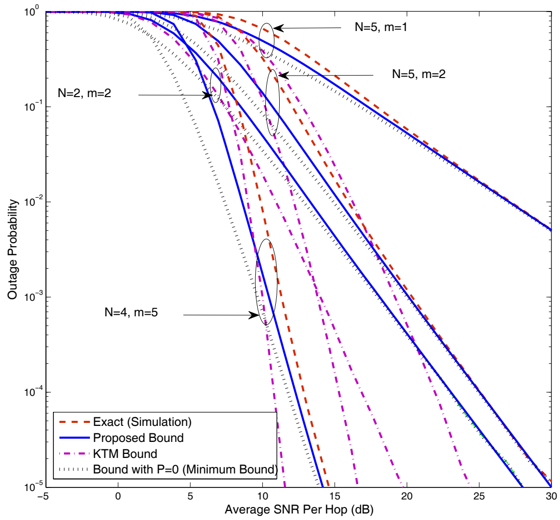

Fig. 2. A comparison of outage probability bounds of multi-hop relay network over i.i.d. Nakagami-m fading channels. $P = \left\lceil \frac{N}{2} \right\rceil$.

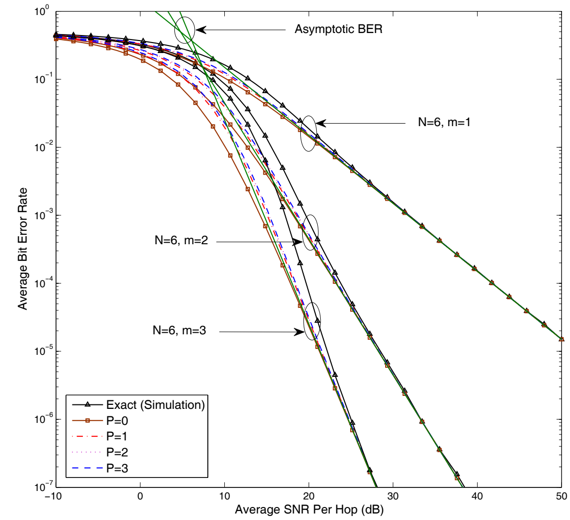

Fig. 3. The effect of P, number of hops and severity of fading on proposed BER bounds of a multi-hop relay network in i.i.d. Nakagami-m fading.

in medium-to-high SNR regime compared to KTM and Minimum bound. Specifically, our bound converges to the exact average BER curve for high SNRs. The KTM bound is quite loose for most SNRs ($\bar{\gamma} > 0\,\text{dB}$) and weakens progressively as the average per hop SNR increases. Although the proposed bound outperforms the KTM bound for moderate-to-high SNRs, the latter is tighter for low-to-moderate SNRs for less severe fading environments (approximately $m > 5$). However, the KTM bound significantly deviates from the exact BER for high SNRs. This fact is not surprising because our proposed bounds are asymptotically exact.

In Fig. 2, the multi-hop outage probability over Nakagami-m fading is plotted. Although the proposed bound loosens as N and m increase, it is tighter at moderate-to-high SNR than the KTM bound. Similar to the case of the BER bounds, the KTM bound is tighter than our bound for less severe fading conditions (e.g., $m > 5$) for low-to-moderate SNRs. The curve for $N = 2$ is plotted to verify that our proposed bound reduces to the exact outage probability of dual-hop system with ideal CA-AF relays. The proposed outage bound always outperforms the bound with $P = 0$ ("Minimum bound"). Similar to BER bound, our outage bound is asymptotically-exact.

In Fig. 3, as a function of P, we compare the tightness of different bounds for average BER of BPSK (13). For a six-hop relay network in Nakagami-m fading, four different BER bounds are obtained by assigning $P = 0$, $P = 1$, $P = 2$ and $P = 3$. As expected, the bound with $P = 0$, which is equivalent to the bound in [9, Eq. (11)] and [12, Ch. 3] is significantly weaker than the others. The tightness of the bounds is increased as P is closer to $N - P$. This happens because the criteria $P \approx N - P$ ensures the symmetry of (2). Moreover, the gaps between bounds with $P = 1$, $P = 2$ and $P = 3$ are insignificant for severe fading. The asymptotic average BER curves are also plotted to verify our high SNR analysis and to demonstrate the asymptotically-exactness of the proposed bounds.

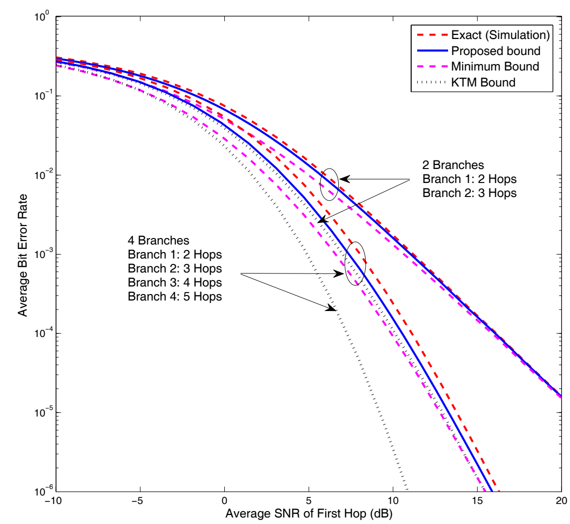

Fig. 4. The BPSK average BER bounds of multi-hop multi-branch relay network with MRC at the destination. System operates over i.n.i.d. Rayleigh fading channels. $P_l = \left\lceil \frac{N_l}{2} \right\rceil$.

In Fig. 4, we plot the average BER bounds for a multi-hop multi-branch system with MRC reception over i.i.d. Nakagami-m fading. The proposed bounds for the average BER are tighter at moderate-to-high SNR. As expected, all BER bounds deteriorate as the number of hops per branch increases. In evaluating (19), we use only ten points ($N_p = 10$). Thus, this result illustrates the accuracy and efficiency of (19) for the average BER of multi-hop multi-branch systems. The proposed bound outperforms both KTM and Minimum bounds. The outage probability bound comparison of multi-hop multi-branch networks with the best branch selection demonstrate a similar behavior to that of BER bounds of multi-hop multi-branch system with MRC reception and numerical results are omitted for the sake of brevity.

VI. CONCLUSION

This letter proposed a new class of SNR upper bounds of multi-hop CA-AF relay networks. The parameter $0 \leq P \leq N$ specifies this class, and $P = \lceil \frac{N}{2} \rceil$ is a reasonably optimal choice. The closed-form CDF and MGF expressions for i.n.i.d. Rayleigh fading and for i.i.d. Nakagami-m fading, $m \in \mathbb{Z}^+$ were derived, leading to the average SER and the outage lower bounds. It is important to mention that these bounds are asymptotically-exact. These bounds were used to study the multi-hop multi-branch relay networks. Their asymptotic-exactness may render them useful for other applications; e.g., multiple-antenna beamforming relay networks [19] and optimal power allocation.

VII. APPENDIX

The sketches of the proof of the theorem I are presented here. Let the random variable Γ be $\Gamma = \frac{\Gamma_1\Gamma_2}{\Gamma_1+\Gamma_2}$, where $\Gamma = \gamma_{\text{eq}}^{\text{ub}}$, $\Gamma_1 = \min(\gamma_1, \gamma_2..., \gamma_{N-P})$ and $\Gamma_2 = \min(\gamma_{N-P+1}, \gamma_{N-P+2}..., \gamma_N)$. The CCDF of Γ can be expressed as [16]

$$\bar{F}_\Gamma(x) = \int_0^\infty \bar{F}_{\Gamma_1}\left(\frac{(z+x)x}{z}\right) f_{\Gamma_2}(z+x)\mathrm{d}z. \quad (20)$$

(Proof of Theorem 1): The CCDF of Γ_1 for i.i.d. Nakagami-m fading with integer m can be obtained by expanding $\left(\frac{\Gamma\left(m, \frac{mx}{\bar{\gamma}}\right)}{\Gamma(m)}\right)^P$ by using [13, Eq. (8.352.2)] and [20, Eq. (44)] as follows:

$$\bar{F}_{\Gamma_1}(x) = \exp\left(-\frac{mPx}{\bar{\gamma}}\right) \sum_{k=0}^{P(m-1)} \beta_{k,P}\left(\frac{mx}{\bar{\gamma}}\right)^k, \quad (21)$$

where $\beta_{k,P}$ is defined in (4). The PDF of Γ_2 for i.i.d. Nakagami-m fading is given by

$$f_{\Gamma_2}(x) = \frac{(N-P)}{\Gamma(m)} \sum_{k=0}^{(m-1)(N-P-1)} \beta_{k,N-P-1}\left(\frac{m}{\bar{\gamma}}\right)^{m+k} \times x^{m+k-1}\exp\left(-\frac{m(N-P)x}{\bar{\gamma}}\right). \quad (22)$$

By substituting (21) and (22) into (20), and by evaluating the resulting integral by using [13, Eq. (3.471.9)], the desired result given in (3) can be derived.

The MGF of Γ can be derived by substituting (3) into $M_\Gamma(s) = \mathcal{E}_\Gamma\{\exp(-sx)\} = 1 - \int_0^\infty s\bar{F}_\Gamma(x)\exp(-sx)\mathrm{d}x$, and by evaluating the resulting integral by using [13, Eq. (6.621.3)]. ■

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