Performance Analysis of Wireless Systems from the MGF of the Reciprocal of the Signal-to-Noise Ratio

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***Abstract*—A class of wireless problems is characterized by the availability of the moment generating function (MGF) of the reciprocal of the signal-to-noise ratio. We show how to compute the average error rates and outage probability in this case. The result allows a simple, accurate numerical calculation of the average error rate by using the Gauss-Legendre numerical quadratures. We also derive the exact bit error rate of multihop relays for the special case where the fading index of each hop is an odd multiple of one-half.**

***Index Terms*—Average probability of error.**

I. Introduction

HOW does one compute the average bit error rate $\mathbb{E}\left[\mathcal{Q}\left(\sqrt{\gamma}\right)\right]$, where $\mathcal{Q}(x)$ is the Gaussian-Q function, $\mathbb{E}(\cdot)$ is the expected value and γ is the signal-to-noise ratio (SNR), from the moment generating function (MGF) of $1/\gamma$?[1] This problem arises in several applications. Consider a multihop communication network with channel-assisted (CA) relaying. where the end-to-end SNR γ can be represented [2] as

$$\frac{1}{\gamma}=\sum_{i=1}^{N}\frac{1}{\gamma_i}, \tag{1}$$

where γ_i is the SNR of the i^{th} hop and $N \geq 2$ is the number of hops. Exact closed-form bit error rate (BER) results are typically found only for the case $N = 2$; others require numerical evaluation. Alternatively, approximations have been devised by bounding γ, for instance, by the smallest of [3], [4], the largest of [5] or the geometric mean of [6] γ_i's. However, recently in [7], simple single integral exact error expressions were presented.

Assuming the γ_i's to be statistically independent, the MGF of the reciprocal of γ given by (1) can be obtained as $\mathcal{M}_{\frac{1}{\gamma}}(s)=\prod_{i=1}^{N}\mathcal{M}_{\frac{1}{\gamma_i}}(s)$, where $\mathcal{M}_X(\cdot)$ denotes the MGF of X. The MGF $\mathcal{M}_{\frac{1}{\gamma_i}}(s)$ is known for various fading distributions including Nakgami-m, Nakagami-q and Nakagami-n [5]. The key steps in the past attempts based on have been: (i) computing $\mathcal{M}_{1/\gamma}(s)$, (ii) computing $\mathcal{M}_{\gamma}(s)$ from $\mathcal{M}_{1/\gamma}(s)$, and (iii) computing the BER using the MGF approach [1]. The step (ii) above may require evaluating a double integral. Reference [5] seeks to reduce step (ii) to a single integration. Step (iii) requires evaluating yet another integral, for example, to compute the average of form $\mathbb{E}\left[\mathcal{Q}\left(\sqrt{\gamma}\right)\right]$ in the case of binary phase shift keying (BPSK).

Other instances of this problem include the analysis of multiple access interference in spread spectrum multiple access systems [8], ultrawide band radio, multiple antenna relaying with transmit antenna selection [9], and multiple-input multiple-output channels under the channel-inversion power allocation technique [10]. Another variant of the problem is evaluating the outage probability given the reciprocal MGF.

This letter contributes the following. First, the average $\mathbb{E}\left[\mathcal{Q}\left(\sqrt{\gamma}\right)\right]$ is expressed in terms of the MGF of $1/\gamma$. This expression allows simple numerical calculation of the average by using the Gauss-Legendre Quadrature (GLQ) techniques [11, 25.4.29]. Second, the closed-form exact BER and outage of multihop relays are derived for the special case where the fading index of each hop is an odd multiple of one-half.

The focus of our work overlaps with [7], where expressions similar to (2) for the BER were derived for several modulation schemes. But our work differs in two ways: (1) our method of derivation is based on the Gil-Pelaez approach [12] and (2) we derive closed-form expressions for the special case (Section III). Due to limited space, only the BPSK case is treated here, but following [7], other modulation formats can be analyzed readily.

Mathematical Notations: $K_n(\cdot)$ is the modified Bessel function of the second kind of order n [11, 9.6.1], and $\mathcal{D}_k(\cdot)$ [11, 19.3.1] denotes the parabolic cylinder function. $\mathcal{C}(\cdot)$ [11, 7.3.3] and $\mathcal{S}(\cdot)$ [11, 7.3.4] are the Fresnel cosine and sine integrals. $\mathrm{j}^2=-1$; $\Re(z)$ and $\Im(z)$ are the real and imaginary parts of z. The probability density function (pdf) and cumulative distribution function (cdf) of X are $f_X(\cdot)$ and $F_X(\cdot)$. If $X \sim \mathcal{G}(\alpha,\beta)$, the Gamma pdf is $f_X(x)=\frac{x^{\alpha-1}\mathrm{e}^{-\frac{x}{\beta}}}{\beta^{\alpha}\Gamma(\alpha)}$, $x \geq 0$, where $\Gamma(\cdot)$ is the Gamma function.

II. Theory

Proposition 1. *The average of $\mathcal{Q}\left(\sqrt{\gamma}\right)$ relates to the MGF of $1/\gamma$ as follows:*

$$\begin{aligned}P_e &\doteq \mathbb{E}\left[\mathcal{Q}\left(\sqrt{\gamma}\right)\right]\\ &= \frac{1}{2}-\frac{1}{\pi}\int_0^{\infty}\frac{\Re\left(\mathcal{M}_{1/\gamma}(\mathrm{j}\omega)\right)}{\omega}\mathrm{e}^{-\sqrt{\omega}}\sin\left(\sqrt{\omega}\right)\mathrm{d}\omega.\end{aligned} \tag{2}$$

where $\mathcal{M}_{1/\gamma}(s)=\mathbb{E}\left[\mathrm{e}^{s/\gamma}\right]=\int_0^{\infty}\mathrm{e}^{s/x}f_{\gamma}(x)\,\mathrm{d}x$. *Moreover,*

$$P_e=\frac{1}{2}-\frac{4}{\pi}\int_{-1}^{1}\frac{\Re\left(\mathcal{M}_{1/\gamma}\left(\mathrm{j}\left(\frac{1+t}{1-t}\right)^2\right)\right)}{1-t^2}\mathrm{e}^{-\frac{1+t}{1-t}}\sin\left(\frac{1+t}{1-t}\right)\mathrm{d}t. \tag{3}$$

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[1]Evaluation of this average directly from the MGF of γ is a well-known and ubiquitous problem in wireless research [1].

Proof: We need to compute

$$P_e = \int_0^\infty \mathcal{Q}\left(\frac{1}{\sqrt{x}}\right) f_{1/\gamma}(x)\,\mathrm{d}x. \tag{4}$$

By using integration by parts on (4), we can readily show that

$$P_e = \frac{1}{2} - \frac{1}{\sqrt{8\pi}} \int_0^\infty x^{-3/2} \mathrm{e}^{-1/2x} F_{1/\gamma}(x)\,\mathrm{d}x \tag{5}$$

where $F_{1/\gamma}(x)$ is the cdf of $1/\gamma$. Substituting the modified version of the Gil-Pelaez lemma for positive random variables [12, Eq. 5] for $F_{1/\gamma}(x)$ in (5), and integrating over x, one can prove Proposition 1. ■

Our extensive numerical experiments indicate that, it is better to convert the integration range to $[-1, 1]$; using $\omega = (1+t)^2/(1-t)^2$ in (2) yields (3). The integral (3) can be readily evaluated using the GLQ technique. In our numerical experiments, we found that 50- to 75-point GLQ sums are sufficient to achieve about 15-digit accuracy for all the cases that were tested. This level of accuracy is more than sufficient for practical applications.

III. Exact performance of multi-hop AF relays

In general, the integrals (3) or (4) must be evaluated numerically. However, in the following, a closed-form evaluation is feasible. The multi-hop amplify and forward (AF) case has been investigated earlier, but exact closed-form solutions are not available. Let the i^{th} hop SNR be distributed as $\gamma_i \sim \mathcal{G}(m_i, \overline{\gamma}_i/m_i)$ for $i = 1, \ldots, N$. The MGF of the reciprocal of received SNR γ for this case is given by [2]

$$\mathcal{M}_{1/\gamma}(s) = \prod_{i=1}^{N} \frac{2}{\Gamma(m_i)} \left(\frac{-m_i s}{\overline{\gamma}_i}\right)^{m_i/2} K_{m_i}\left(2\sqrt{\frac{-m_i s}{\overline{\gamma}_i}}\right).$$

The Bessel function $K_m(x)$ can be expressed as a finite expression if m is an odd multiple of one-half [13, 8.468]. For the sake of brevity, in the following, we only consider the case of independent, identical fading statistics.

A. BER of BPSK

Suppose that all $m_i = m + \frac{1}{2}$, where $m \geq 0$ is an integer, and $\overline{\gamma}_i = \gamma$, for $i = 1, \ldots, N$. We can show using [13, 8.468] that

$$\mathcal{M}_{1/\gamma}(s) = \left(\frac{\sqrt{\pi}}{2^m \Gamma(m+1/2)} \mathrm{e}^{-z} \sum_{k=0}^{m} \frac{(m+k)! z^{m-k}}{k!\,(m-k)! 2^k}\right)^N = \mathrm{e}^{-Nz} \sum_{k=0}^{mN} b_k z^k, \tag{6}$$

where $z = \left(2\sqrt{\frac{-(m+\frac{1}{2})s}{\overline{\gamma}}}\right)$ and b_k are readily given by [13, 0.314]. From Proposition 1, and using [14, 2.5.31 (4)] and [14, 2.5.31 (11)], we get

$$P_e = \frac{1}{2} - \frac{2}{\pi} \Re\left(\sum_{k=0}^{mN} b_k \alpha^k A(k)\right), \tag{7}$$

where $\alpha = \left(2\sqrt{\frac{-\mathrm{j}(m+\frac{1}{2})}{\overline{\gamma}}}\right)$ and

$$A(k) = \begin{cases} \frac{\Gamma(k)}{((N\alpha+1)^2+1)^{\frac{k}{2}}} \sin\left(k \tan^{-1}\left(\frac{1}{N\alpha+1}\right)\right), & k \neq 0 \\ \cot^{-1}(N\alpha+1), & k = 0 \end{cases}. \tag{8}$$

Eq. (7) is a new closed-form BER expression, which holds for any $N \geq 2$.

B. Outage analysis

The MGF of the reciprocal of received SNR γ is given by (6). The outage, $\Pr(\gamma \leq x)$, can be expressed by using [12, Eq. (5)] as

$$F_\gamma(x) = 1 - \frac{2}{\pi} \int_0^\infty \frac{\Re\left(\mathcal{M}_{1/\gamma}(\mathrm{j}\omega)\right)}{\omega} \sin\left(\frac{\omega}{x}\right) \mathrm{d}\omega = 1 - \frac{2}{\pi} \sum_{k=0}^{mN} b_k\, \Re\left(\int_0^\infty \mathrm{e}^{-N\alpha\sqrt{\omega}} \alpha^k \omega^{k/2} \frac{\sin\left(\frac{\omega}{x}\right)}{\omega} \mathrm{d}\omega\right), \tag{9}$$

where $\alpha = \left(2\sqrt{\frac{-(m+\frac{1}{2})\mathrm{j}}{\overline{\gamma}}}\right) = \pm(1-\mathrm{j})\sqrt{\frac{2m+1}{\overline{\gamma}}}$. By substituting $t = \sqrt{\omega}$ in (10), we get

$$F_\gamma(x) = 1 - \frac{4}{\pi} \sum_{k=0}^{mN} b_k \left(\frac{2m+1}{\gamma}\right)^{k/2} \Re\left((\pm(1-\mathrm{j}))^k\, \mathbb{I}(k)\right), \tag{10}$$

where $\mathbb{I}(k) = \int_0^\infty \mathrm{e}^{-pt} t^{k-1} \sin\left(\frac{t^2}{x}\right) \mathrm{d}t$. To ensure convergence, we pick $p = N(1-\mathrm{j})\sqrt{\frac{2m+1}{\overline{\gamma}}}$.

Case $k = 0$, (using [14, 2.5.41(8)], with $b \doteq \frac{1}{x}, \delta \doteq 1$)

$$\mathbb{I}(k) = \frac{\pi}{2}\left(\frac{1}{2} - \mathcal{C}\left(\sqrt{\frac{x}{2\pi}}p\right)\right)^2 + \frac{\pi}{2}\left(\frac{1}{2} - \mathcal{S}\left(\sqrt{\frac{x}{2\pi}}p\right)\right)^2 \tag{11}$$

Case $k \neq 0$,(using [14, 2.5.41(5)], with $b \doteq \frac{1}{x}, \alpha \doteq k, \delta \doteq 1$)

$$\mathbb{I}(k) = \frac{\mathrm{j}\Gamma(k)}{2}\left(\frac{x}{2}\right)^{k/2} \left(\mathrm{e}^{-\mathrm{j}\left(\frac{k\pi}{4}+\frac{p^2 x}{8}\right)} \mathcal{D}_{-k}\left(\frac{p\sqrt{x}}{\sqrt{2}} \mathrm{e}^{-\pi\mathrm{j}/4}\right) - \mathrm{e}^{\mathrm{j}\left(\frac{k\pi}{4}+\frac{p^2 x}{8}\right)} \mathcal{D}_{-k}\left(\frac{p\sqrt{x}}{\sqrt{2}} \mathrm{e}^{\pi\mathrm{j}/4}\right)\right) \tag{12}$$

These special functions are readily available in common mathematical software such as Mathematica and Maple.

IV. Numerical Results

Fig. 1 compares (i) numerical (computed with 75 point GLQ rule on (3)), (ii) simulation (computed semi-analytically using $\sim 10^6$ samples), and (iii) analytical BER results for 2-Hop CA relaying over Nakagami-m faded links. BPSK modulation is assumed. The exact closed-form expression (7) holds true only when Nakagami parameter $m = n + 0.5$, $n \in \mathcal{Z}^+$. Asymptotic result of [15] is used for other m values. Accuracy of numerical evaluation can be appreciated given its close agreement with (i) exact closed-form results (where available, e.g. case $m = 4.5$), (ii) simulation results (at low avg. SNR), and (iii) asymptotic results (at high avg. SNR, e.g. case $m = 3$). Fig. 2 shows a similar comparison for 5-Hop CA relaying. Asymptotic slope of the curves agrees with the fact that diversity order is equal to Nakagami parameter m.

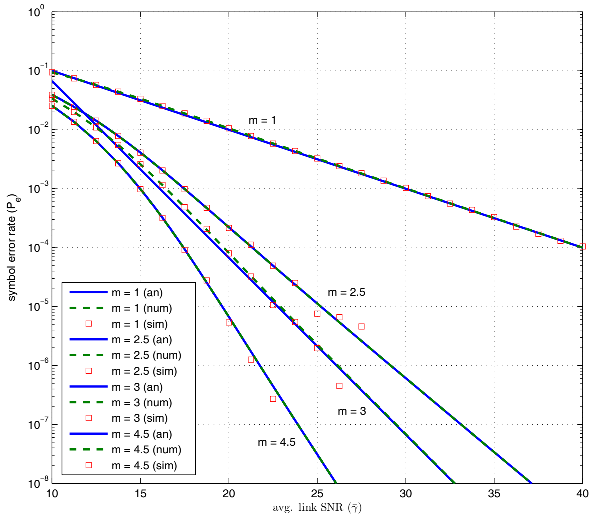

Fig. 1. BER for 2-Hop CA relaying over Nakagami-m faded links, for BPSK modulation. (i) solid (—): analytical results; exact for $m \in \{2.5, 4, 5\}$; asymptotic for $m \in \{1, 3\}$, (ii) dashed ($--$): numerical results from (3), (iii) marker ($\square$): simulation results.

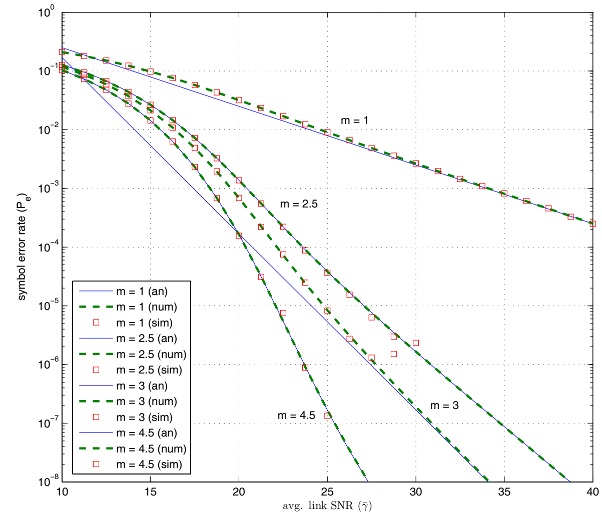

Fig. 2. BER for 5-Hop CA relaying over Nakagami-m links and BPSK. (i) solid (—): analytical results; exact for $m \in \{2.5, 4, 5\}$; asymptotic for $m \in \{1, 3\}$, (ii) dashed ($--$): numerical results from (3), (iii) marker ($\square$): simulation results.

Fig. 3 depicts the cdf of received SNR (10) for N-hop CA relaying over Nakagami-m faded links. As expected, the outage probability (given by $F(x_0)$ for a threshold x_0) increases with N. Semi-analytic Monte-Carlo simulation verifies the accuracy of (10).

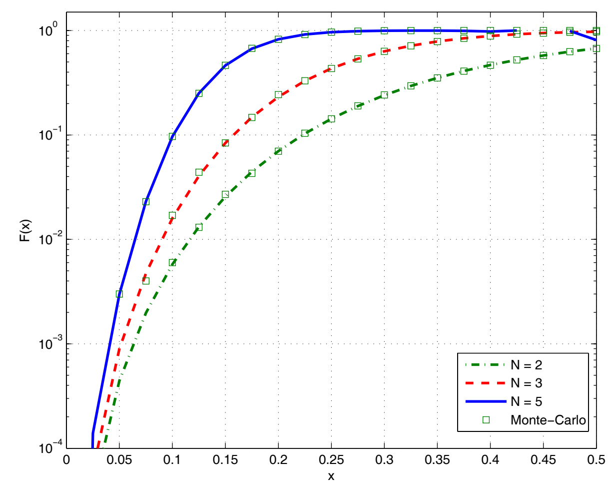

Fig. 3. The outage for N-Hop, $N \in \{2, 3, 5\}$, CA relaying over Nakagami-m ($m = 3.5$) faded links; analytic vs. simulation (marker $\square$).

V. Conclusion

Performance analysis of digital modulations when the MGF of the reciprocal of the SNR is available has been considered. The error rate was derived as a single integral expression that can be efficiently evaluated by the Gauss-Legendre formulas. For brevity, only the BPSK case was treated. The exact BER and outage of multihop CA relaying were also derived for the special case where the fading index of each hop is an odd multiple of one-half. Single-integral error rate expressions derived in [7] and in this letter will find applications including relay networks, multiple access systems and ultra wide band systems.

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