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Generalized singular value decomposition for Coordinated Beamforming in MIMO systems

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***Abstract*—In this paper we examine the use of generalized singular value decomposition (GSVD) for coordinated beamforming in MIMO systems. GSVD facilitates joint decomposition of a class of matrices arising inherently in source-to-2 destination MIMO broadcast scenarios. GSVD allows two channels of suitable dimensionality to be jointly diagonalized, i.e. to be reduced to non-interfering virtual broadcast channels, through the use of jointly determined transmit precoding and receiver reconstruction matrices.**

Potential applications for GSVD-based beamforming can be found in MIMO broadcasting, as well as in MIMO relaying under all amplify-and-forward, decode-and-forward, and code-and-forward relay processing schemes. Several of them are highlighted here. We also present simulation-based performance analysis results to justify the use of GSVD for coordinated beamforming.

***Index Terms*—generalized singular value decomposition, beamforming, MIMO relaying, MIMO broadcasting, network coding**

I. Introduction

Beamforming techniques facilitate parallel, ideally non-interfering, virtual communication channels between two multiple-input multiple-output (MIMO) capable terminals. Coordinated beamforming techniques are required whenever more than two such terminals communicate simultaneously. The channel of our interest here, is the broadcast channel consisting of a single MIMO capable source terminal and two MIMO capable destinations. This scenario arises, obviously, in multi-user channels. Since certain phases of relaying involves broadcasting, it also appears in MIMO relaying contexts.

The phrase 'MIMO broadcast channel' is frequently used in a loose sense in the literature, to include point-to-multipoint unicast (i.e. 'private') channels carrying different messages from a single source to each of the multiple destinations (e.g. in multi-user MIMO). Its use in this paper is more specific, and denotes the presence of at least one 'common' virtual broadcast channel from the source to the destinations.

The use of iterative and non-iterative beamforming techniques to achieve point-to-multipoint private channels has been investigated, for instance in [1]–[9]. Their use for MIMO broadcasting, which requires common point-to-multipoint broadcast channels is not much attractive, given the fact that the total number of private and common channels is limited by the number of antennas the source has.

Wherever each receiver of a broadcast channel conveys what it receives orthogonally to the same destination, as in the case of MIMO amplify-and-forward (AF) relaying, the whole system can be envisaged as a single point-to-point MIMO channel. Beamforming techniques for point-to-point MIMO channels would be usable therein. However, the choice of optimal relay gains, although known for certain cases (e.g. [10], [11]), is not straightforward with this approach. Since the individual relays have no non-iterative means of decoding the signals, this approach cannot be used with decode-and-forward (DF), and code-and-forward (CF) relay processing schemes.

The use of zero-forcing at the destination has been examined [12], [13] as a mean of coordinated beamforming, since it does not require transmitter processing. The scheme scales to any number of destinations, but requires each destination to have no less antennas than the source.

Although not used as commonly as the singular value decomposition (SVD), *generalized singular value decomposition* (GSVD) [14, Thm. 8.7.4] is not unheard of in the wireless literature. It has been used in multi-user MIMO transmission [15], [16], MIMO secrecy communication [17], [18], and MIMO relaying [19]. Reference [19] uses GSVD in dual-hop AF relaying with arbitrary number of relays. Since it employs zero-forcing at the relay for the forward channel, its use of GSVD appears almost similar to the use of SVD in [1].

Despite GSVD being the natural generalization of SVD for two matrices, we are yet to see in the literature, a generalization of SVD-based beamforming to GSVD-based beamforming. Although the purpose and the use is somewhat different, the reference [17, p.1] appears to be the first to hint the possible use of GSVD for beamforming. In present work, we illustrate how GSVD can be used for coordinated beamforming in source-to-2 destination MIMO broadcasting; thus in AF, DF and CF MIMO relaying. We also present comparative, simulation-based performance analysis results to justify GSVD-based beamforming.

The paper is organized as follows: Section II presents the mathematical framework, highlighting how and under which constraints GSVD can be used for beamforming. Section III examines how GSVD-based beamforming can be applied in certain simple MIMO and MIMO relaying configurations. Performance analysis is conducted in section IV on one of these applications. Section V concludes with some final remarks.

Notations: Given a matrix $\mathbf{A}$ and a vector $\mathbf{v}$, (i) $\mathbf{A}(i,j)$ gives the i^{th} element on the j^{th} column of $\mathbf{A}$; (ii) $\mathbf{v}(i)$

$$
\begin{aligned}
\{\hat{\mathbf{y}}_\mathbf{1}\}_{\mathcal{R}(r+1,r+s)} &= \tilde{\mathbf{\Sigma}}\,\{\mathbf{x}\}_{\mathcal{R}(r+1,r+s)} + \left\{\mathbf{U}^H\mathbf{n_1}\right\}_{\mathcal{R}(r+1,r+s)},\\
\{\hat{\mathbf{y}}_\mathbf{2}\}_{\mathcal{R}(p-t+r+1,p-t+r+s)} &= \tilde{\mathbf{\Lambda}}\,\{\mathbf{x}\}_{\mathcal{R}(r+1,r+s)} + \left\{\mathbf{V}^H\mathbf{n_2}\right\}_{\mathcal{R}(p-t+r+1,p-t+r+s)},\\
\{\hat{\mathbf{y}}_\mathbf{1}\}_{\mathcal{R}(1,r)} &= \{\mathbf{x}\}_{\mathcal{R}(1,r)} + \left\{\mathbf{U}^H\mathbf{n_1}\right\}_{\mathcal{R}(1,r)},\\
\{\hat{\mathbf{y}}_\mathbf{2}\}_{\mathcal{R}(p-t+r+s+1,p)} &= \{\mathbf{x}\}_{\mathcal{R}(r+s+1,t)} + \left\{\mathbf{V}^H\mathbf{n_2}\right\}_{\mathcal{R}(p-t+r+s+1,p)}. \qquad (1)
\end{aligned}
$$

gives the element of $\mathbf{v}$ at the i^{th} position. $\{A\}_{\mathcal{R}(n)}$ and $\{A\}_{\mathcal{C}(n)}$ denote the sub-matrices consisting respectively of the first n rows, and the first n columns of $\mathbf{A}$. Let $\{A\}_{\mathcal{R}(m,n)}$ denote the sub-matrix consisting of the rows m through n of $\mathbf{A}$. The expression $\mathbf{A} = \text{diag}(a_1,\ldots,a_n)$ indicates that $\mathbf{A}$ is rectangular diagonal; and that first n elements on its main diagonal are $a_1,\ldots,a_n$. $\text{rank}(\mathbf{A})$ gives the rank of $\mathbf{A}$. The operators $(\cdot)^H$, and $(\cdot)^{-1}$ denote respectively the conjugate transpose and the matrix inversion. $\mathbb{C}^{m\times n}$ is the space spanned by $m\times n$ matrices containing possibly complex elements. The channel between the wireless terminals T_1 and T_2 in a MIMO system is designated $T_1 \rightarrow T_2$.

II. Mathematical framework

Let us examine GSVD to see how it can be used for beamforming. There are two major variants of GSVD in the literature (e.g. [20] vs. [21]). We use them both here to elaborate the notion of GSVD-based beamforming.

A. GSVD - Van Loan definition

Let us first look at GSVD as initially proposed by Van Loan [20, Thm. 2].

Definition 1: Consider two matrices, $\mathbf{H} \in \mathbb{C}^{m\times n}$ with $m \geq n$, and $\mathbf{G} \in \mathbb{C}^{p\times n}$, having the same number n of columns. Let $q = \min(p,n)$. $\mathbf{H}$ and $\mathbf{G}$ can be jointly decomposed as

$$\mathbf{H} = \mathbf{U}\mathbf{\Sigma}\mathbf{Q}, \quad \mathbf{G} = \mathbf{V}\mathbf{\Lambda}\mathbf{Q} \qquad (2)$$

where (i) $\mathbf{U} \in \mathbb{C}^{m\times m}, \mathbf{V} \in \mathbb{C}^{p\times p}$ are unitary, (ii) $\mathbf{Q} \in \mathbb{C}^{n\times n}$ non-singular, and (iii) $\mathbf{\Sigma} = \text{diag}(\sigma_1,\ldots,\sigma_n) \in \mathbb{C}^{m\times n}, \sigma_i \geq 0$; $\mathbf{\Lambda} = \text{diag}(\lambda_1,\ldots,\lambda_q) \in \mathbb{C}^{p\times n}, \lambda_i \geq 0$.

As a crude example, suppose that $\mathbf{G}$ and $\mathbf{H}$ above represent channel matrices of MIMO subsystems $S \rightarrow D_1$ and $S \rightarrow D_2$ having a common source S. Assume perfect channel-state-information (CSI) on $\mathbf{G}$ and $\mathbf{H}$ at all S, D_1, and D_2. With a transmit precoding matrix $\mathbf{Q}^{-1}$, and receiver reconstruction matrices $\mathbf{U}^H, \mathbf{V}^H$ we get q non-interfering virtual broadcast channels. The invertible factor $\mathbf{Q}$ in (2) facilitates joint-precoding for the MIMO subsystems; while the factors $\mathbf{U}, \mathbf{V}$ allow receiver reconstruction without noise enhancement. Diagonal elements 1 through q of $\mathbf{\Sigma}, \mathbf{\Lambda}$ represent the gains of these virtual channels. Since $\mathbf{Q}$ is non-unitary, precoding would cause the instantaneous transmit power to fluctuate. This is a drawback not present in SVD-based beamforming. Transmit signal should be normalized to maintain the average total transmit power at the desired level.

This is the essence of 'GSVD-based beamforming' for a single source and two destinations. As would be shown in Section III, this three-terminal configuration appears in various MIMO subsystems making GSVD-based beamforming applicable.

B. GSVD - Paige and Saunders definition

Before moving on to applications, let us appreciate GSVD-based beamforming in a more general sense, through another form of GSVD proposed by Paige and Saunders [21, (3.1)]. This version of GSVD relaxes the constraint $m \geq n$ present in (2).

Definition 2: Consider two matrices, $\mathbf{H} \in \mathbb{C}^{m\times n}$ and $\mathbf{G} \in \mathbb{C}^{p\times n}$, having the same number n of columns. Let $\mathbf{C}^H = \left(\mathbf{H}^H, \mathbf{G}^H\right) \in \mathbb{C}^{n\times(m+p)}$, $t = \text{rank}(\mathbf{C})$, $r = t - \text{rank}(\mathbf{G})$ and $s = \text{rank}(\mathbf{H}) + \text{rank}(\mathbf{G}) - t$.
$\mathbf{H}$ and $\mathbf{G}$ can be jointly decomposed as

$$
\begin{aligned}
\mathbf{H} &= \mathbf{U}\,(\mathbf{\Sigma}\ \mathbf{0_1})\,\mathbf{Q} = \mathbf{U}\,\mathbf{\Sigma}\,\{\mathbf{Q}\}_{\mathcal{R}(t)},\\
\mathbf{G} &= \mathbf{V}\,(\mathbf{\Lambda}\ \mathbf{0_2})\,\mathbf{Q} = \mathbf{V}\,\mathbf{\Lambda}\,\{\mathbf{Q}\}_{\mathcal{R}(t)}, \qquad (3)
\end{aligned}
$$

where (i) $\mathbf{U} \in \mathbb{C}^{m\times m}, \mathbf{V} \in \mathbb{C}^{p\times p}$ are unitary, (ii) $\mathbf{Q} \in \mathbb{C}^{n\times n}$ non-singular, (iii) $\mathbf{0_1} \in \mathbb{C}^{m\times(n-t)}$, $\mathbf{0_2} \in \mathbb{C}^{p\times(n-t)}$ zero matrices, and (iv) $\mathbf{\Sigma} \in \mathbb{C}^{m\times t}, \mathbf{\Lambda} \in \mathbb{C}^{p\times t}$ have structures

$$\mathbf{\Sigma} \triangleq \begin{pmatrix} \mathbf{I_H} & & \\ & \tilde{\mathbf{\Sigma}} & \\ & & \mathbf{0_H} \end{pmatrix}$$

and

$$\mathbf{\Lambda} \triangleq \begin{pmatrix} \mathbf{0_G} & & \\ & \tilde{\mathbf{\Lambda}} & \\ & & \mathbf{I_G} \end{pmatrix}.$$

$\mathbf{I_H} \in \mathbb{C}^{r\times r}$ and $\mathbf{I_G} \in \mathbb{C}^{(t-r-s)\times(t-r-s)}$ are identity matrices. $\mathbf{0_H} \in \mathbb{C}^{(m-r-s)\times(t-r-s)}$, and $\mathbf{0_G} \in \mathbb{C}^{(p-t+r)\times r}$ are zero matrices possibly having no rows or no columns. $\tilde{\mathbf{\Sigma}} = \text{diag}(\sigma_1,\ldots,\sigma_s), \tilde{\mathbf{\Lambda}} = \text{diag}(\lambda_1,\ldots,\lambda_s) \in \mathbb{C}^{s\times s}$ such that $1 > \sigma_1 \geq \ldots \geq \sigma_s > 0$, and $\sigma_i^2 + \lambda_i^2 = 1$ for $i \in \{1,\ldots,s\}$.

Let us examine (3) in the MIMO context. It is not difficult to see that a common transmit precoding matrix $\left\{\mathbf{Q}^{-1}\right\}_{\mathcal{C}(t)}$ and receiver reconstruction matrices $\mathbf{U}^H, \mathbf{V}^H$ would jointly diagonalize the channels represented by $\mathbf{H}$ and $\mathbf{G}$.

For broadcasting, only the columns $(r+1)$ through $(r+s)$ of $\mathbf{\Sigma}$ and $\mathbf{\Lambda}$ are of interest. Nevertheless, other $(t-s)$ columns, when they are present, may be used by the source S to privately communicate with the destinations D_1 and

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configuration	# common channels $S \to \{D_1, D_2\}$	# private channels $S \to D_1$	$S \to D_2$
$m > n, p \leq n$	p	$n - p$	0
$m \leq n, p > n$	m	0	$n - m$
$m \geq n, p \geq n$	n	0	0
$m < n, p < n,$ $(m+p) > n$	$m + p - n$	$n - p$	$n - m$
$n \geq (m+p)$	0	m	p

TABLE I
NUMBERS OF COMMON CHANNELS AND PRIVATE CHANNELS FOR DIFFERENT CONFIGURATIONS

D_2. It is worthwhile to compare this fact with [22], and appreciate the similarity and the conflicting objectives GSVD-based beamforming for broadcasting has with MIMO secrecy communication.

Thus we can get $\hat{\mathbf{y}}_\mathbf{1} \in \mathbb{C}^{m\times 1}, \hat{\mathbf{y}}_\mathbf{2} \in \mathbb{C}^{p\times 1}$ as in (1) at the detector input, when $\mathbf{x} \in \mathbb{C}^{t\times 1}$ is the symbol vector transmitted. It can also be observed from (1) that the private channels always have *unit gains*; while the gains of common channels are smaller.

Since, σ_is are in descending order, while the λ_is ascend with i, selecting a subset of the available s broadcast channels (say $k \leq s$ channels) is somewhat challenging. This highlights the need to further our intuition on GSVD.

C. GSVD-based beamforming

Any two MIMO subsystems having a common source and channel matrices $\mathbf{H}$ and $\mathbf{G}$ can be effectively reduced, depending on their ranks, to a set of common (broadcast) and private (unicast) virtual channels. The requirement for having common channels is $\boxed{\text{rank}(\mathbf{H}) + \text{rank}(\mathbf{G}) > \text{rank}(\mathbf{C})}$ where $\mathbf{C} = \left(\mathbf{H}^H, \mathbf{G}^H\right)^H$.

When the matrices have full rank, which is the case with most MIMO channels (key-hole channels being an exception), this requirement boils down to having $\boxed{m + p > n}$. Table I indicates how the numbers of common channels and private channels vary in full-rank MIMO channels. It can be noted that the cases $(m > n, p \leq n)$ and $(m \geq n, p \geq n)$ correspond to the form of GSVD discussed in the Subsection II-A. Further, the case $n \geq (m + p)$ which produces only private channels with unit gains, can be seen identical to zero forcing at the transmitter. Thus, GSVD-based beamforming is also a generalization of zero-forcing.

Based on Table I, it can be concluded that the full-rank $\min(n, m + p)$ of the combined channel always gets split between the common and private channels.

D. MATLAB implementation

A general discussion on the computation of GSVD is found in [23]. Let us focus here on what it needs for simulation: namely its implementation in the MATLAB computational environment, which extends [14, Thm. 8.7.4] and appears as less restrictive as [21].

The command $[\mathtt{V}, \mathtt{U}, \mathtt{X}, \mathtt{Lambda}, \mathtt{Sigma}] = \mathtt{gsvd}(\mathtt{G}, \mathtt{H})$; gives[1] a decomposition similar to (3). Its main deviations from (3) are,

[1]Reverse order of arguments in and out of 'gsvd' function should be noted.

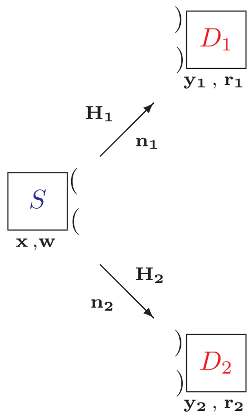

Fig. 1. Source-to-2 destination MIMO broadcast system

- $\mathbf{Q}^H = \mathbf{X} \in \mathbb{C}^{n\times t}$ is not square when $t < n$. Precoding for such cases would require the use of the pseudo-inverse operator.
- $\boldsymbol{\Sigma}$ has the same block structure as in (3). But the structure of $\boldsymbol{\Lambda}$ has the block $\mathbf{0_G}$ shifted to its bottom as follows:

$$\boldsymbol{\Lambda} \triangleq \begin{pmatrix} & \tilde{\boldsymbol{\Lambda}} & \\ & & \mathbf{I_G} \\ \mathbf{0_G} & & \end{pmatrix}.$$

This can be remedied by appropriately interchanging the rows of $\boldsymbol{\Lambda}$ and the columns of $\mathbf{V}$. However, restructuring $\boldsymbol{\Lambda}$ is not a necessity, since the column position of the block $\tilde{\boldsymbol{\Lambda}}$ within $\boldsymbol{\Lambda}$ is what matters in joint precoding. Following MATLAB code snippet for example jointly diagonalizes $\mathbf{H}, \mathbf{G}$ to obtain the s common channels (3) would have given.

MATLAB code

```
% channel matrices
   H = (randn(m,n)+i*randn(m,n))/sqrt(2);
   G = (randn(p,n)+i*randn(p,n))/sqrt(2);
% D1, D2: diagonalized channels
   [V,U,X,Lambda,Sigma] = gsvd(G,H);
   w = X*inv(X'*X); C = [H' G']'; t = rank(C);
   r = t - rank(G); s = rank(H)+rank(G)-t;

   D1 = U(:,r+1:r+s)'*H*w(:,r+1:r+s);
   D2 = V(:,1:s)'*G*w(:,r+1:r+s);
```

III. Applications

Let us look at some of the possible applications of GSVD-based beamforming. We assume the Van Loan form of GSVD for simplicity, having taken for granted that the dimensions are such that the constraints hold true. Nevertheless, the Paige and Saunders form should be usable as well.

A. Source-to-2 destination MIMO broadcast system

Consider the MIMO broadcast system shown in Fig. 1, where the source S broadcasts to destinations D_1 and D_2. MIMO subsystems $S \to D_1$ and $S \to D_2$ are modeled to have channel matrices $\mathbf{H_1}, \mathbf{H_2}$ and additive complex Gaussian noise vectors $\mathbf{n_1}, \mathbf{n_2}$. Let $\mathbf{x} = [x_1, \ldots, x_n]^T$

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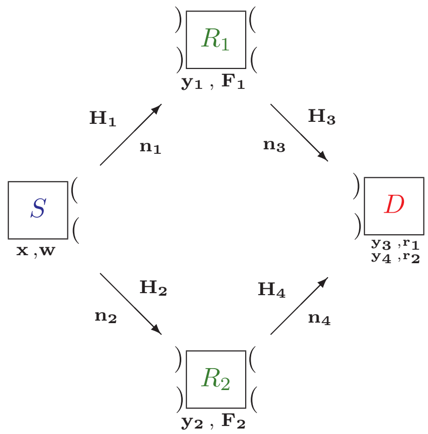

Fig. 2. MIMO relay system with two 2-hop-branches

be the signal vector desired to be transmitted over $n \leq \min(\operatorname{rank}(\mathbf{H_1}), \operatorname{rank}(\mathbf{H_2}))$ virtual-channels. The source employs a precoding matrix $\mathbf{w}$.

The input $\mathbf{y_1}, \mathbf{y_2}$ and output $\hat{\mathbf{y}}_1, \hat{\mathbf{y}}_2$ at the receiver filters $\mathbf{r_1}, \mathbf{r_2}$ at D_1 and D_2 are given by

$$\begin{aligned} \mathbf{y_1} &= \mathbf{H_1}\,\mathbf{w}\,\mathbf{x} + \mathbf{n_1}\,;\ \hat{\mathbf{y}}_1 = \mathbf{r_1}\,\mathbf{y_1}\,, \\ \mathbf{y_2} &= \mathbf{H_2}\,\mathbf{w}\,\mathbf{x} + \mathbf{n_2}\,;\ \hat{\mathbf{y}}_2 = \mathbf{r_2}\,\mathbf{y_2}\,. \end{aligned}$$

Applying GSVD we get $\mathbf{H_1} = \mathbf{U_1}\,\mathbf{\Sigma_1}\,\mathbf{V}$ and $\mathbf{H_2} = \mathbf{U_2}\,\mathbf{\Sigma_2}\,\mathbf{V}$. Choose the precoding matrix $\mathbf{w} = \alpha\left\{\mathbf{V}^{-1}\right\}_{\mathcal{C}(n)}$; and receiver reconstruction matrices $\mathbf{r_1} = \left\{\mathbf{U_1}^H\right\}_{\mathcal{R}(n)}$, $\mathbf{r_2} = \left\{\mathbf{U_2}^H\right\}_{\mathcal{R}(n)}$. The constant α normalizes the total average transmit power.

Then we get,

$$\begin{aligned} \hat{\mathbf{y}}_1(i) &= \alpha\mathbf{\Sigma_1}(i,i)\,\mathbf{x}(i) + \tilde{\mathbf{n}}_1(i)\,, \\ \hat{\mathbf{y}}_2(i) &= \alpha\mathbf{\Sigma_2}(i,i)\,\mathbf{x}(i) + \tilde{\mathbf{n}}_2(i)\,, \quad i \in \{1 \dots n\}, \end{aligned}$$

where $\tilde{\mathbf{n}}_1, \tilde{\mathbf{n}}_2$ have the same noise distributions as $\mathbf{n_1}, \mathbf{n_2}$.

B. MIMO relay system with two 2-hop-branches (3 time-slots)

Fig. 2 shows a simple MIMO AF relay system where a source S communicates a symbol vector $\mathbf{x}$ with a destination D via two relays R_1 and R_2. MIMO channels $S \to R_1$, $S \to R_2$, $R_1 \to D$ and $R_2 \to D$ are denoted: $\mathbf{H_i}, i \in \{1,2,3,4\}$. Corresponding channel outputs and additive complex Gaussian noise vectors are $\mathbf{y_i}$, $\mathbf{n_i}$ for $i \in \{1,2,3,4\}$. Assume relay operations to be linear, and modeled as matrices $\mathbf{F_1}$ and $\mathbf{F_2}$.

Assume orthogonal time-slots for transmission. The source S uses $\mathbf{w}$ as the precoding matrix. Destination D uses different reconstruction matrices $\mathbf{r_1}, \mathbf{r_2}$ during the time slots 2 and 3. Then we have:

<u>Time slot 1</u>: $\mathbf{y_1} = \mathbf{H_1}\,\mathbf{w}\,\mathbf{x} + \mathbf{n_1}, \quad \mathbf{y_2} = \mathbf{H_2}\,\mathbf{w}\,\mathbf{x} + \mathbf{n_2}$

<u>Time slot 2</u>: $\mathbf{y_3} = \mathbf{H_3}\,\mathbf{F_1}\,\mathbf{y_1} + \mathbf{n_3}$

<u>Time slot 3</u>: $\mathbf{y_4} = \mathbf{H_4}\,\mathbf{F_2}\,\mathbf{y_2} + \mathbf{n_4}$

Let $\hat{\mathbf{y}} = \mathbf{r_1}\,\mathbf{y_3} + \mathbf{r_2}\,\mathbf{y_4}$ be the input to the detector. Suppose $n \leq \min_i(\operatorname{rank}(\mathbf{H_i}))$ virtual-channels are in use.

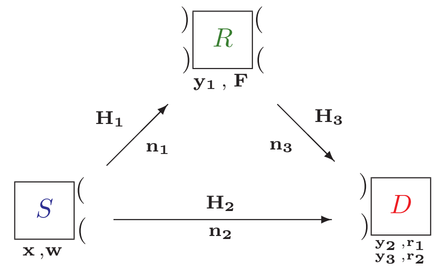

Fig. 3. MIMO relay system having a direct path and a relayed path

Applying GSVD on the broadcast channel matrices we get $\mathbf{H_1} = \mathbf{U_1}\,\mathbf{\Sigma_1}\,\mathbf{Q}$ and $\mathbf{H_2} = \mathbf{U_2}\,\mathbf{\Sigma_2}\,\mathbf{Q}$. Through SVD we obtain $\mathbf{H_3} = \mathbf{V_1}\,\mathbf{\Lambda_1}\,\mathbf{R_1}^H$ and $\mathbf{H_4} = \mathbf{V_2}\,\mathbf{\Lambda_2}\,\mathbf{R_2}^H$. Choose $\mathbf{w} = \alpha\left\{\mathbf{Q}^{-1}\right\}_{\mathcal{C}(n)}$; $\mathbf{F_1} = \mathbf{R_1}\,\mathbf{U_1}^H$; $\mathbf{F_2} = \mathbf{R_2}\,\mathbf{U_2}^H$; $\mathbf{r_1} = \left\{\mathbf{V_1}^H\right\}_{\mathcal{R}(n)}$; $\mathbf{r_2} = \left\{\mathbf{V_2}^H\right\}_{\mathcal{R}(n)}$. The constant α normalizes the total average transmit power. Then we get $\hat{\mathbf{y}}$ to be

$$\left\{\alpha\{(\mathbf{\Lambda_1}\mathbf{\Sigma_1} + \mathbf{\Lambda_2}\mathbf{\Sigma_2})\}_{\mathcal{C}(n)}\mathbf{x} + \mathbf{\Lambda_1}\tilde{\mathbf{n}}_1 + \mathbf{\Lambda_2}\tilde{\mathbf{n}}_2 + \tilde{\mathbf{n}}_3 + \tilde{\mathbf{n}}_4\right\}_{\mathcal{R}(n)},$$

where each $\tilde{\mathbf{n}}_i$ has the same noise distribution as $\mathbf{n_i}$.

Remarks:

- The matrices $\mathbf{U_1}^H$ and $\mathbf{U_2}^H$ can be used by the relays to extract (and if necessary decode) each channel passing through them. Hence, the same beamforming matrices can be used with the DF and CF schemes.
- The relay operations $\mathbf{F_i}$ can be modeled more generally as $\mathbf{R_i}\,\mathbf{P_i}\,\mathbf{U_i}^H, i \in \{1,2\}$ with the diagonal matrices $\mathbf{P_i}$ governing power allocation among the virtual-channels.

C. MIMO relay system having a direct path and a relayed path (2 time-slots)

Fig. 3 depicts a MIMO relay system having 3 nodes: source S, relay R and destination D. The $S \to R$, $S \to D$ and $R \to D$ MIMO channels are $\mathbf{H_1}, \mathbf{H_2}$ and $\mathbf{H_3}$. Corresponding channel outputs are $\mathbf{y_1}, \mathbf{y_2}$ and $\mathbf{y_3}$; additive complex Gaussian noise vectors are $\mathbf{n_1}, \mathbf{n_2}$ and $\mathbf{n_3}$. Relay operation is linear and represented by a matrix $\mathbf{F}$. For a transmit symbol vector $\mathbf{x}$ we get:

<u>Time slot 1</u>: $\mathbf{y_1} = \mathbf{H_1}\,\mathbf{w}\,\mathbf{x} + \mathbf{n_1}, \ \mathbf{y_2} = \mathbf{H_2}\,\mathbf{w}\,\mathbf{x} + \mathbf{n_2}$

<u>Time slot 2</u>: $\mathbf{y_3} = \mathbf{H_3}\,\mathbf{F}\,\mathbf{y_1} + \mathbf{n_3}$

Let $\hat{\mathbf{y}} = \mathbf{r_1}\,\mathbf{y_2} + \mathbf{r_2}\,\mathbf{y_3}$ be the input to the detector.

Assume $n \leq \min_i(\operatorname{rank}(\mathbf{H_i}))$ virtual-channels to be in use. Applying GSVD on the broadcast-phase channel matrices we get $\mathbf{H_1} = \mathbf{U_1}\,\mathbf{\Sigma_1}\,\mathbf{Q}$ and $\mathbf{H_2} = \mathbf{U_2}\,\mathbf{\Sigma_2}\,\mathbf{Q}$. Applying SVD we obtain $\mathbf{H_3} = \mathbf{V}\,\mathbf{\Lambda}\,\mathbf{R}^H$. Choose $\mathbf{w} = \alpha\left\{\mathbf{Q}^{-1}\right\}_{\mathcal{C}(n)}$; $\mathbf{F} = \mathbf{R}\,\mathbf{U_1}^H$; $\mathbf{r_1} = \left\{\mathbf{U_2}^H\right\}_{\mathcal{R}(n)}$; and $\mathbf{r_2} = \left\{\mathbf{V}^H\right\}_{\mathcal{R}(n)}$. The constant α normalizes the total average transmit power. Then we get $\hat{\mathbf{y}}$ to be

$$\left\{\alpha\left\{(\mathbf{\Lambda}\,\mathbf{\Sigma_1} + \mathbf{\Sigma_2})\right\}_{\mathcal{C}(n)}\mathbf{x} + \mathbf{\Lambda}\,\tilde{\mathbf{n}}_1 + \tilde{\mathbf{n}}_2 + \tilde{\mathbf{n}}_3\right\}_{\mathcal{R}(n)}, \tag{4}$$

where each $\tilde{\mathbf{n}}_i$ has the same noise distribution as $\mathbf{n_i}$.

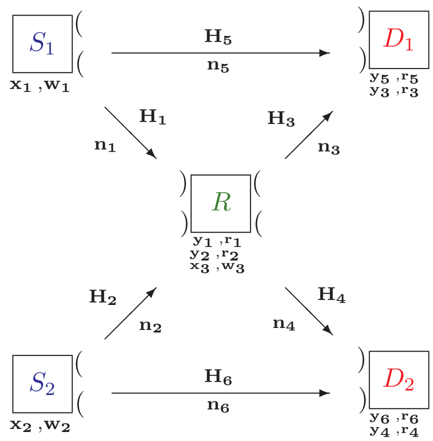

Fig. 4. CF relaying with Network Coding

D. CF relaying with Network Coding (3 time-slots)

Network coding schemes that code 2-messages at a time (e.g. those based on XOR operation), require broadcasting information to 2 destinations. The simplest network to support CF relaying (see Fig. 4), for instance, has three such broadcast phases, each of which can be exploited via GSVD-based beamforming.

Let S_1, S_2 be the sources; D_1, D_2 the destinations; and R the CF relay. MIMO channels $S_1 \to R$, $S_2 \to R$, $R \to D_1$, $R \to D_2$, $S_1 \to D_1$, and $S_2 \to D_2$ are denoted respectively $\mathbf{H_i}, i \in \{1, \ldots, 6\}$. Corresponding channel outputs and additive complex Gaussian noise vectors are $\mathbf{y_i}$, and $\mathbf{n_i}, i \in \{1, \ldots, 6\}$.

Output of receiver filters $\hat{\mathbf{y}}_\mathbf{i} = \mathbf{r_i}\,\mathbf{y_i}, i \in \{1, \ldots, 6\}$, are used to decode the signals at R, D_1 and D_2. The sources S_1, S_2 transmit the codewords $\mathbf{x_1}, \mathbf{x_2}$ respectively in the 1^{st} and 2^{nd} time-slots. The relay XORs what it decodes from $\hat{\mathbf{y}}_\mathbf{1}$ and $\hat{\mathbf{y}}_\mathbf{3}$ to form $\mathbf{x_3}$, and transmits it in the 3^{rd} time-slot.

Applying GSVD, separately for each time-slot, provides the transmit precoding and receiver reconstruction matrices for diagonalizing all 6 channels.

IV. Performance Analysis

This section evaluates the performance of GSVD-based beamforming, comparing it with that of SVD-based beamforming for a specific example: the MIMO AF relay system outlined in Subsection III-C.

SVD-based beamforming is also possible for this case since (i) AF relaying is used; and (ii) the system has a single source-destination pair. Define $\mathbf{y} = \left(\mathbf{y_2}^H, \mathbf{y_3}^H\right)^H$. Then we have,

$$\mathbf{y} = \underbrace{\begin{pmatrix} \mathbf{H_3\,F\,H_1} \\ \mathbf{H_2} \end{pmatrix}}_{\hat{\mathbf{H}}} \mathbf{w\,x} + \begin{pmatrix} \mathbf{H_3\,F\,n_1} \\ \mathbf{0} \end{pmatrix} + \begin{pmatrix} \mathbf{n_3} \\ \mathbf{n_2} \end{pmatrix} \quad (5)$$

Suppose $\hat{\mathbf{H}} = \hat{\mathbf{U}}\,\hat{\mathbf{\Sigma}}\,\hat{\mathbf{V}}^H$ is the SVD. The channel can be diagonalized by choosing the transmit precoding matrix $\mathbf{w} = \left\{\hat{\mathbf{V}}\right\}_{\mathcal{C}(n)}$ and receiver reconstruction matrix $\mathbf{r} = \left\{\hat{\mathbf{U}}^H\right\}_{\mathcal{R}(n)}$.

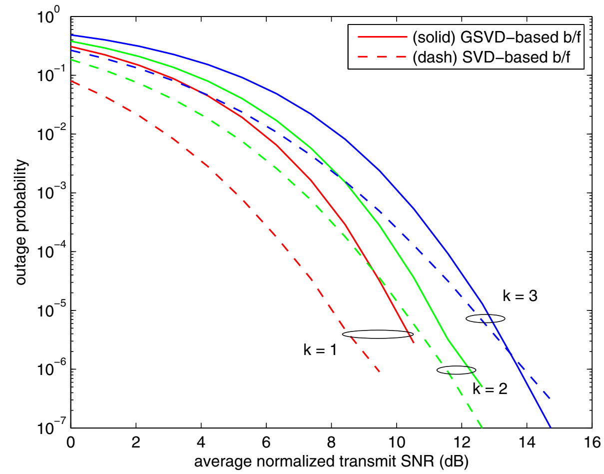

Fig. 5. Outage performance of GSVD-based beamforming vs. SVD-based beamforming, for MIMO AF relaying with $N_s = 4, N_r = 3, N_d = 5$, for $n = 3$ common channels.

However, with this approach, the choice of $\mathbf{F}$ is not straightforward.

- An apparent choice is selecting $\mathbf{F}$ to invert $\mathbf{H_3}$; which is essentially zero-forcing in the forward direction.
- Another is to choose $\mathbf{F} = \mathbf{V_3}\,\mathbf{U_1}^H$, where $\mathbf{H_i} = \mathbf{U_i}\,\mathbf{\Sigma_i}\,\mathbf{V_i}^H, i \in \{1, 3\}$ as governed by the SVD. It's optimality, reasoned out for slightly different configurations in [11, Eqn. (22)], and [24, Eqn. (7)], may be appreciated in the light that aligning the eigenmodes of input and output channels is almost the best the relay can do towards improving the signal-to-noise ratio (SNR) at the destination. This form of $\mathbf{F}$ is assumed here.

Fig. 5 compares the outage performance of the three common channels[2] of a MIMO AF relay configuration having $N_s = 4, N_r = 3, N_d = 5$ source, relay and destination antennas, for both GSVD-based beamforming and SVD-based beamforming. Fig. 6 shows the average symbol error rate (SER) of quadrature phase shift keying (QPSK) modulation for the same configuration. Monte-Carlo simulation based on (4) and (5), has been employed with 10^7 simulation points. The constant α too was found through simulation.

As expected, the first channel ($k = 1$) shows better outage and SER performance than the other two. GSVD-based beamforming fares within $3\,\text{dB}$ of SVD-based beamforming for moderate SNR. Interestingly, GSVD-based beamforming appears to have higher diversity order for this case. This observation is yet to be established theoretically.

Incidentally, the instantaneous per channel received SNR $\gamma_i^{(\text{gsvd})}, i \in \{1, \ldots, n\}$ for GSVD-beamforming can be written from (4) as

$$\gamma_i^{(\text{gsvd})} = \frac{\left(\mathbf{\Lambda}(i,i)\;\mathbf{\Sigma_1}(i,i) + \mathbf{\Sigma_2}(i,i)\right)^2}{\mathbf{\Lambda}(i,i)^{\;2} + 2}\alpha^2 P, \quad (6)$$

[2]GSVD-based beamforming over this MIMO configuration yields 3 common channels and a single source-to-destination private channel. The private channel is not considered here. Performance over the others are compared against the best 3 of the 4 channels SVD-based beamforming produces.

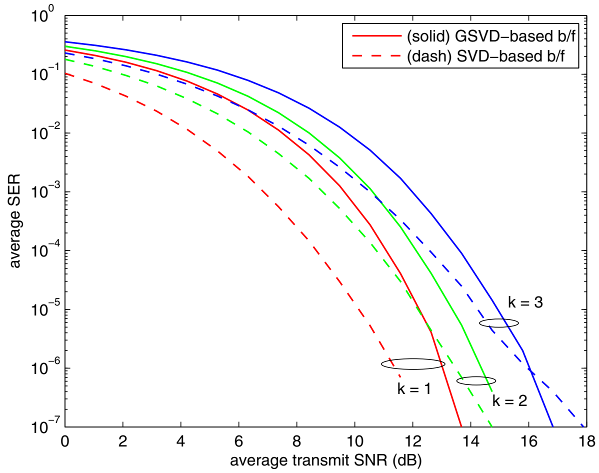

Fig. 6. The average SER of GSVD-based beamforming vs. SVD-based beamforming, for MIMO AF relaying with $N_s = 4, N_r = 3, N_d = 5$, for $n = 3$ common channels.

where P is the transmit SNR. SVD-based beamforming would not give an as concise form since $\mathbf{r}$ and $\mathbf{U_3}$ are not generally orthogonal.

More interestingly GSVD-beamforming allows the symbols to be decoded at the relay; and the received SNR over the i^{th} virtual channel would be $\alpha^2 P \left(\mathbf{\Sigma_1}(i,i)\right)^2$. Perfect decoupling of the virtual channels at both the relay and the destination makes GSVD-based beamforming usable with all AF, DF, and CF relay processing schemes. This is a feat not achievable with SVD-based beamforming.

V. Conclusion

The use of generalize singular value decomposition (GSVD) for coordinated beamforming in MIMO systems has been examined. Several applications of GSVD-based beamforming have been summarized. Performance of one of them was evaluated; and seen to perform within 3 dB of SVD-based beamforming. This, combined with the applicability with DF and CF relay processing schemes makes GSVD-beamforming promising. However, further analysis on different MIMO configurations is required to assess its usefulness. From a theoretical point-of-view, incorporating GSVD into random matrix theory is vital to accurately characterize GSVD-based beamforming. An interesting design problem would be seeking the ways of utilizing the common and private virtual channels in hybrid.

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