Effects of Channel Estimation Errors on BPSK Systems Using Superimposed Pilots

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***Abstract*—Super-imposed training can be used as an alternative solution to estimating the channel state information (CSI). Since pilots are sent together with the data symbols, superimposed training improves the bandwidth efficiency. In this paper, we study the effects of the channel estimation errors on binary phase shift keying (BPSK) systems using superimposed pilots. Specifically, we derive an approximate bit error rate (BER) expression in low signal-to-noise ratio (SNR) region while in high SNR region, we derive the lower bound of the BER. Simulation results are then provided to corroborate the proposed studies.**

I. Introduction

The traditional way to obtain channel state information (CSI) is to send pilot symbols from the transmitter to the receiver. This process has to be carried out frequently in order to keep track of the channel changes, especially for time-varying channels [1], resulting in the so called pilot symbol assisted modulation (PSAM) [2]. An alternative method is to apply the super-imposed pilots, that are transmitted together with data symbols. By doing this, the bandwidth efficiency is naturally improved.

The idea of superimposed pilots first appeared in [3] for analog communication systems. Recently, superimposed pilots are further exploited for both synchronization and channel estimation in digital communication systems [4]–[10]. The ways to apply superimposed pilots can be divided into two categories: one is the linear precoding that can map pilots and data symbols into orthogonal space; the other is the first-order statistics that utilize the zero-mean property of both data symbols and noise. In this paper, we will only focus on the second case.

Most existing works only target at the channel estimation mean square error (MSE), while a more sophisticated criterion should be the bit error rate (BER) considering the channel estimation error. Such a performance analysis has not been addressed yet, and this motivates our present work. In this paper, we study the effects of the channel estimation error on BER of binary phase shift keying (BPSK) systems when superimposed pilots are applied for channel estimation. The difficulty exists in the facts that the estimated channel has relationship with several random variables such as channel information, pilot power, signal power, as well as the noise. For example, the amplitudes of the estimated channel and the real channel are treated as bivariate Rayleigh distribution [11], which entails complicated analysis. To make the analysis executable, we choose the simplest form of the superimposed training and look into the low signal-to-noise ratio (SNR) and the high SNR regions, where we derive approximation of the BER expression. In low SNR region, we give an approximate BER expression while in high SNR region, we prove the existence of BER lower bound and derive its expression. Simulations are then provided to corroborate the proposed studies.

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II. System Model

Let $p[n]$ denote the superimposed pilots with the period P_T and the power E_p, and let $s[n]$ denote the data symbols with the power E_s. Assume that the channel is flat fading and remains unchanged during the transmission of N symbols. The received symbol is expressed as

$$r[n] = h(\sqrt{E_s}s[n] + \sqrt{E_p}p[n]) + w[n], \tag{1}$$

where $h \sim CN(0, \sigma_h^2)$ represents the Rayleigh fading channel, and $w[n] \sim CN(0, \sigma_w^2)$ is the zero-mean white complex Gaussian noise. The total transmitter power is $E = E_p + E_s$. Let $\beta = E_p/E$ be the power allocation ratio which, in most cases, satisfies $0 \leq \beta < 0.5$.

After collecting N symbols, the receiver estimates the channel from the first-order statistics. If pilots take the simplest form

$$p[n] = \sum_k \delta[n - kP_T], \tag{2}$$

then the estimated channel is given by

$$\begin{aligned}\widehat{h} =& \frac{1}{Q\sqrt{E_p}}\sum_{k=1}^{Q} r[n + kP_T] \qquad (3)\\ =& h + \underbrace{\frac{1}{Q\sqrt{E_p}}\left(h\sqrt{E_s}\sum_{k=1}^{Q} s[n + kP_T] + \sum_{k=1}^{Q} w[n + kP_T]\right)}_{\phi},\end{aligned}$$

where $Q = N/P_T$ is an integer, and ϕ represents the channel estimation error. According to the central limit theorem (CLT), ϕ can be treated as a zero-mean complex Gaussian random variable [12], i.e., $\phi \sim CN(0, \sigma_\phi^2)$. From (3), it can be readily verified that

$$\sigma_\phi^2 = \frac{1}{E_p Q}(E_s \sigma_h^2 + \sigma_w^2). \quad (4)$$

After obtaining the CSI, the receiver will remove superimposed pilots as

$$r[n] - \widehat{h}\sqrt{E_p}p[n] = h\sqrt{E_s}s[n] + w[n] - \phi\sqrt{E_p}p[n]. \quad (5)$$

Since the exact channel information is not available at the receiver, (5) can be rewritten as:

$$\begin{aligned} r[n] - \widehat{h}\sqrt{E_p}p[n] =& \widehat{h}\sqrt{E_s}s[n] + w[n] \\ &- \phi(\sqrt{E_p}p[n] + \sqrt{E_s}s[n]). \end{aligned} \quad (6)$$

III. BER AT LOW SNR REGION

Without loss of generality, we assume $P_T = 1$ in the following BER derivation. The only change made for a general case is the separate discussion for data symbols being and not being superimposed by pilots.

The detection SNR γ at the receiver can be expressed as

$$\gamma = \frac{|\widehat{h}|^2 E_s}{\sigma_w^2 + |\phi|^2 E} = \frac{\gamma_1 E_s}{\sigma_w^2 + \gamma_2 E}, \quad (7)$$

where $\gamma_1 = |\widehat{h}|^2$ and $\gamma_2 = |\phi|^2$. Both γ_1 and γ_2 can be assumed as exponential variables with parameters λ_1 and λ_2 given by

$$\lambda_1 = \sigma_h^2 + \sigma_\phi^2, \quad (8)$$

$$\lambda_2 = \sigma_\phi^2. \quad (9)$$

According to the expression of ϕ, γ_1 and γ_2 are correlated. Nonetheless, at low SNR we can make the approximation that γ_1 and γ_2 are independent. Our simulation results further verify the validity of such approximation.

The outage happens when the instant SNR value γ falls below a threshold γ_{th}. The outage probability is

$$\begin{aligned} P_{out} =& P(\gamma < \gamma_{th}) \\ =& \int_0^\infty P\left(\frac{\gamma_1 E_s}{\sigma_w^2 + \gamma_2 E} < \gamma_{th} | \gamma_2\right) p(\gamma_2) d\gamma_2 \\ =& \int_0^\infty \left[1 - \exp\left(-\frac{(\sigma_w^2 + \gamma_2 E)\gamma_{th}}{E_s \lambda_1}\right)\right] p(\gamma_2) d\gamma_2 \\ =& 1 - \frac{1}{\lambda_2} \exp\left(-\frac{\sigma_w^2 \gamma_{th}}{E_s \lambda_1}\right) \\ & \times \int_0^\infty \exp\left(-\frac{E\gamma_{th}\gamma_2}{E_s \lambda_1}\right) \exp\left(-\frac{\gamma_2}{\lambda_2}\right) d\gamma_2 \\ =& 1 - \frac{E_s \lambda_1}{E\lambda_2 \gamma_{th} + E_s \lambda_1} \exp\left(-\frac{\sigma_w^2 \gamma_{th}}{E_s \lambda_1}\right). \end{aligned} \quad (10)$$

With the instant SNR γ, the BER can be expressed as the Q-function $Q(\sqrt{2\gamma})$. Therefore, the BER on average can be computed as

$$\begin{aligned} P_b =& \int_0^\infty Q(\sqrt{2x}) p_\gamma(x) dx = \int_0^\infty Q(\sqrt{2x}) dP_{out} \\ =& Q(\sqrt{2x}) P_{out}|_0^\infty - \int_0^\infty P_{out} dQ(\sqrt{2x}) \\ =& \frac{1}{2} - \int_0^\infty \frac{E_s \lambda_1}{2\sqrt{\pi x}(E\lambda_2 x + E_s \lambda_1)} \\ & \times \exp\left(-\frac{\sigma_w^2 + E_s \lambda_1}{E_s \lambda_1} x\right) dx. \end{aligned} \quad (11)$$

The integration in (11) can be evaluated by Eq.(3.383.10), Eq.(8.338.2) and Eq.(8.359.3) in [13], yielding

$$\begin{aligned} P_b =& \frac{1}{2} - \frac{1}{2}\sqrt{\frac{E_s \lambda_1}{E\lambda_2}} \exp\left(\frac{E_s \lambda_1 + \sigma_w^2}{E\lambda_2}\right) \Gamma\left(\frac{1}{2}, \frac{E_s \lambda_1 + \sigma_w^2}{E\lambda_2}\right) \\ =& \frac{1}{2} - \frac{1}{2}\sqrt{\frac{E_s \lambda_1}{E\lambda_2}} \exp\left(\frac{E_s \lambda_1 + \sigma_w^2}{E\lambda_2}\right) \text{erfc}\left(\sqrt{\frac{E_s \lambda_1 + \sigma_w^2}{E\lambda_2}}\right). \end{aligned} \quad (12)$$

Combing (8), (9) and (12) gives

$$\begin{aligned} P_b \;=\; & \frac{1}{2} - \frac{\sqrt{\pi}}{2}\sqrt{\frac{E_s(\sigma_h^2 + \sigma_\phi^2)}{E\sigma_\phi^2}} \exp\left(\frac{\sigma_w^2 + E_s(\sigma_h^2 + \sigma_\phi^2)}{E\sigma_\phi^2}\right) \\ & \times \text{erfc}\left(\sqrt{\frac{\sigma_w^2 + E_s(\sigma_h^2 + \sigma_\phi^2)}{E\sigma_\phi^2}}\right). \end{aligned} \quad (13)$$

Since $E_p = \beta E$ and $E_s = (1-\beta)E$, (13) can be rewritten as

$$P_b = \frac{1}{2} - \frac{\sqrt{\pi}}{2}\sqrt{\frac{M}{\sigma_\phi^2}} \exp\left(\frac{\sigma_w^2 + EM}{E\sigma_\phi^2}\right) \text{erfc}\left(\sqrt{\frac{\sigma_w^2 + EM}{E\sigma_\phi^2}}\right), \quad (14)$$

where $M = (1-\beta)(\sigma_h^2 + \sigma_\phi^2)$.

IV. LOWER BOUND AT HIGH SNR REGION

From the expression of ϕ, we know $\widehat{h}$ and ϕ are correlated random variables. At high SNR region, we can ignore the influence of $\frac{1}{Q\sqrt{E_p}} \sum w[n]$ in ϕ. We then introduce a real Gaussian random variable $\mu \sim N(0, \sigma_\mu^2)$ with

$$\phi = h\mu, \quad (15)$$

whose variance is computed as

$$\sigma_\mu^2 = \frac{E_s}{QE_p} = \frac{1-\beta}{Q\beta}. \quad (16)$$

Clearly, μ is independent of h With this approximation, we can find the lower bound of the BER.

We first consider the case of $n = kP_T$ and then the case of $n = (k-1)P_T + i$[1], $1 \le k \le Q$, $1 \le i \le P_T - 1$. Let T denote the the sufficient statistics for receiver detection. For BPSK system, the detection criterion is $\Re(T) > 0$ for $s[n] = 1$ and $\Re(T) < 0$ for $s[n] = -1$.

[1]Here we assume $P_T \ge 2$. If $P_T = 1$, then we require $i = 0$, which means only one case to be considered.

When transmitted signal is $s[n] = 1$, we have

$$\begin{aligned} T =& (r[n] - \widehat{h}\sqrt{E_p})\widehat{h}^* \\ =& |h + \mu h|^2\sqrt{E_s} + w\widehat{h}^* - \mu h(\sqrt{E_p} + \sqrt{E_s})\widehat{h}^*. \end{aligned}$$

To get the correct detection, it requires

$$\Re\left\{ |h + \mu h|^2 + \frac{w\widehat{h}^*}{\sqrt{E_s}} - |h|^2(\mu + |\mu|^2)\left(\sqrt{\frac{\beta}{1-\beta}} + 1\right)\right\} > 0. \tag{17}$$

As $\sqrt{E_s}$ increases, the second item can be neglected to give

$$\Re\left\{ |1 + \mu|^2 - (\mu + |\mu|^2)\left(\sqrt{\frac{\beta}{1-\beta}} + 1\right)\right\} > 0, \tag{18}$$

which can be further simplified as

$$(\mu + 1)\left(\sqrt{\frac{1-\beta}{\beta}} - \mu\right) > 0. \tag{19}$$

The error rate is then

$$P_1 = P(\mu < -1) + P\left(\mu > \sqrt{\frac{1-\beta}{\beta}}\right). \tag{20}$$

When $s[n] = -1$ is transmitted, the sufficient statistics is similarly obtained as

$$\begin{aligned} T =& (r[n] - \widehat{h}\sqrt{E_p})\widehat{h}^* \\ =& -|h + \mu h|^2\sqrt{E_s} + w\widehat{h}^* - \mu h(\sqrt{E_p} - \sqrt{E_s})\widehat{h}^*. \end{aligned} \tag{21}$$

We require $\Re(T) \leq 0$, which can be simplified to

$$(\mu + 1)\left(\mu + \sqrt{\frac{1-\beta}{\beta}}\right) > 0. \tag{22}$$

The error rate can be found

$$P_2 = P\left(-\sqrt{\frac{1-\beta}{\beta}} < \mu < -1\right). \tag{23}$$

For the case of $n = (k-1)P_T + i$, when $s[n] = 1$ is transmitted, the sufficient statistics for correct detection is:

$$T = r[n]\widehat{h}^* = |\widehat{h}|^2\sqrt{E_s} + w\widehat{h}^* - h\mu\widehat{h}^*\sqrt{E_s}. \tag{24}$$

From $\Re(T) > 0$, the error rate is obtained as

$$P_3 = P(\mu < -1). \tag{25}$$

Similarly when $s[n] = -1$ is transmitted, the error rate can be found as

$$P_4 = P(\mu < -1). \tag{26}$$

Finally, the overall BER can be expressed as

$$\begin{aligned} P_b \geq& P(n = kP_T)[P(s[n] = 1)P_1 + P(s[n] = -1)P_2] \\ &+ P(n \neq kP_T)[P(s[n] = 1)P_3 + P(s[n] = -1)P_4] \\ =& \frac{P_1 + P_2 + (Q-1)(P_3 + P_4)}{2Q}. \end{aligned} \tag{27}$$

Lower Bound	Q=60	Q=80
β=0.10	0.0049	0.0014
β=0.15	5.6905×10^{-4}	8.5863×10^{-5}
β=0.20	5.3756×10^{-5}	3.8721×10^{-6}

TABLE I
BER LOWER BOUND FOR BPSK SYSTEMS

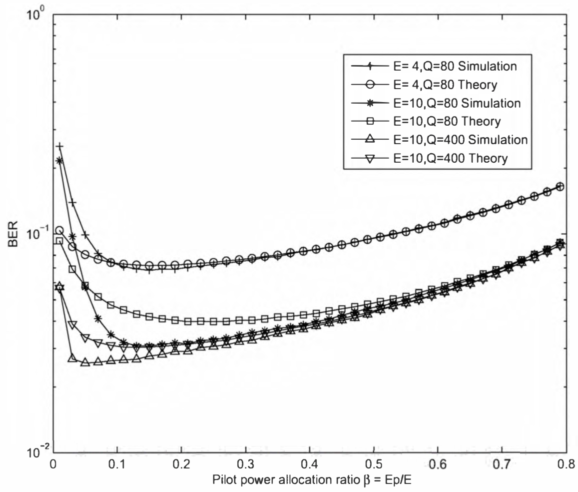

Fig. 1. Compare BER by changing β

Combing (20), (23), (25), (26), and (27), we can obtain

$$P_b \geq \Phi\left(-\sqrt{\frac{\beta Q}{(1-\beta)}}\right) = P_{b_0}, \tag{28}$$

where $\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} \exp\left(-\frac{t^2}{2}\right)dt$. Since (28) has no relation with E, so P_{b_0} means an error floor.

The Table I shows the BER lower bound (28) at different configuration of Q and β.

V. Simulation Results

In this section, we provide various numerical examples to corroborate the proposed studies. We consider Rayleigh fading channels, i.e. $h \sim CN(0,1)$. Meanwhile, we set $w \sim CN(0,1)$ and $P_T = 1$.

A. BER at Low SNR Region

Fig. 1 shows the simulated BER versus power allocation ratio β for $E = 4$ dB and $E = 10$ dB, respectively. The analytical BER curves are also included for comparison. It is seen that the BER firstly decreases and then increases when β increases. The analytical BER resulted from (14) becomes close to the simulated curve at relatively large β value. Better approximation can be obtained with the increase of Q.

We then set $\beta = 0.2$ and display the both the simulated BER and theoretical BER versus E in Fig. 2. Two different values of Q are taken as 100 and 400, respectively. Again, the analytical BER is a good approximation for the real BER result.

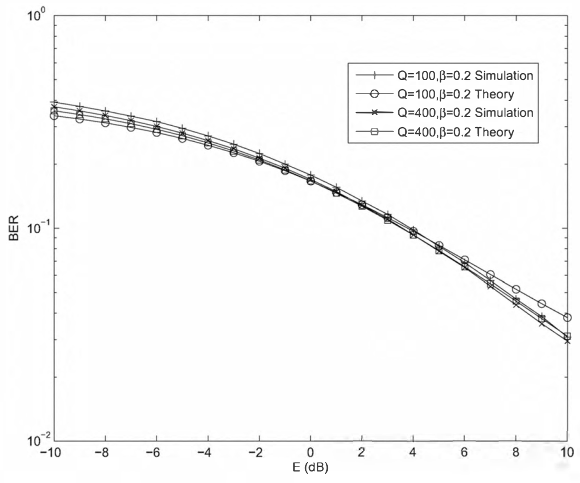

Fig. 2. Compare BER by changing E

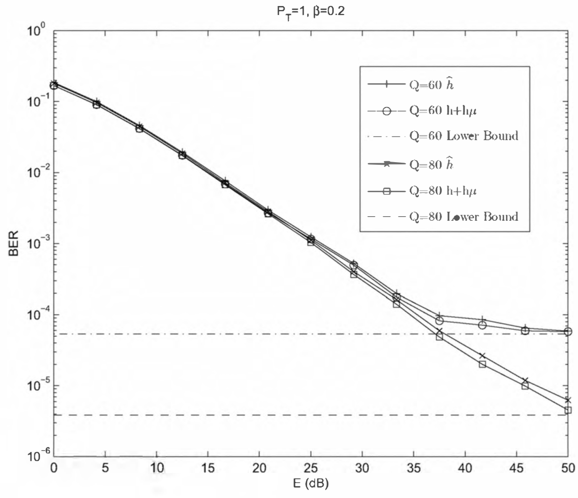

Fig. 4. BER and lower bound when $\beta = 0.2, P_T = 1$

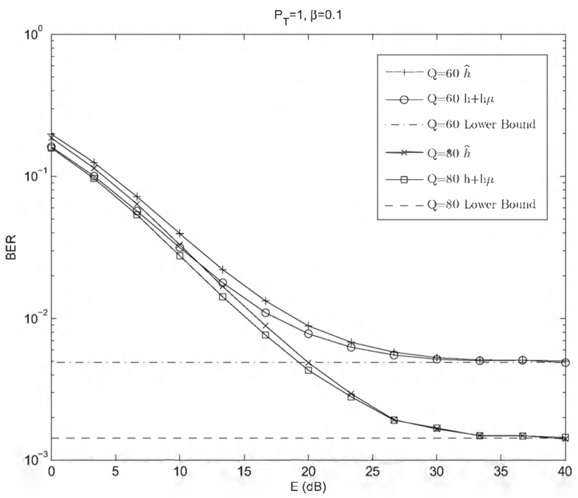

Fig. 3. BER and lower bound when $\beta = 0.1, P_T = 1$

B. BER Lower Bound at High SNR Region

We first take $Q = 60, 80$, $\beta = 0.1$, and plot the BER versus SNR curves in the Fig.3. The high SNR region is the main focus in this example. We then change to $\beta = 0.2$ and repeat the above process. The resultant simulation curves are shown in Fig.4. It can be seen that the simulated BER curves are lower bounded by our analytical results in both Fig. 3 and Fig. 4. And when β increases from 0.1 to 0.2, the lower bound will be smaller while only appear at higher SNR.

VI. CONCLUSION

In this paper, we study the influence of channel estimation errors on the BER performance in a BPSK system using superimposed pilots. We find the approximated BER expression for low SNR region, while at high SNR region, we prove that the channel estimation errors result in an error floor, whose closed-form expression is also derived. Finally, simulation results are provided to corroborate the proposed studies.

REFERENCES

[1] X. Ma, G. B. Giannakis, and S. Ohno, "Optimal training for block transmissions over doubly selective wireless fading channels," *IEEE Trans. on Signal Processing*, vol. 51, pp.1531-1366, May 2003.

[2] J. Cavers, "An analysis of pilot symbol assisted modulation for rayleigh fading channels [mobile radio]," *IEEE Trans. Veh. Technol.*, vol. 40, pp. 683-693, Nov. 1991.

[3] C. Kastenholz and W. Birkemeier, "A simultaneous information transfer and channel-sounding modulation technique for wide-band channels," *IEEE Trans. on Communications*, vol. 12, pp.162-165, June 1965.

[4] M. Ghogho, D. McLernon, E. Alameda-Hernandez, and A. Swami, "Channel estimation and symbol detection for block transmission using data-dependent superimposed training," *IEEE Signal Processing Letters*, vol. 12, pp. 226-229, March 2005.

[5] X. Meng, J. Tugnait and S. He, "Iterative joint channel estimation and data detection using superimposed training: Algorithms and performance analysis," *IEEE Trans. Veh. Technol.*, vol.56, pp. 1873-1880, July 2007.

[6] G. Wang and C. Tellambura, "Super-imposed pilot-aided channel estimation and power allocation for relay systems," in *Proc. IEEE WCNC*, Budapest, Hungary, April 2009, pp. 1-6.

[7] J. Tugnait and S. He, "Doubly-selective channel estimation using data-dependent superimposed training and exponential basis models," *IEEE Trans. on Wireless Communications*, vol. 6, pp. 3877-3883, Nov. 2007.

[8] S. He and J.Tugnait, "On doubly selective channel estimation using superimposed training and discrete prolate spheroidal sequences," *IEEE Trans. on Signal Processing*, vol. 56, pp.3214-3228, July 2008.

[9] V. Nguyen, H. D. Tuan, H. H. Nguyen, and N. N. Tran, "Optimal superimposed training design for spatially correlated fading MIMO channels," *IEEE Trans. on Wireless Communications*, vol. 7, pp. 3206-3217, Aug. 2008.

[10] C. Carrasco-Alvarez, R. Parra-Michel, A. Orozco-Lugo, and J. Tugnait, "Enhanced channel estimation using superimposed training based on universal basis expansion," *IEEE Trans. on Signal Processing*, vol. 57, pp. 1217-1222, March 2009.

[11] X. Tang, M. S. Alouini, and A. J. Goldsmith, "Effect of channel estimation error on m-QAM BER performance in rayleigh fading," *IEEE Trans. on Communications*, vol. 47, pp. 1856-1864, Dec. 1999.

[12] L. Cao and N. Beaulieu, "Exact error-rate analysis of diversity 16-QAM with channel estimation error," *IEEE Trans. on Communications*, vol. 52, pp. 1019-1029, June 2004.

[13] I. S. Cradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products*, 7th ed. San Diego, CA: Academic, 2007.