Probability Density Function of Logarithmic Ratio of Arithmetic Mean to Geometric Mean for Nakagami-m Fading Power

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Abstract—**In this work, we study the statistical properties of a parameter Δ, which is defined as the logarithmic ratio of the arithmetic mean to the geometric mean for the Nakagami-m fading power. This parameter is useful in studying the maximum-likelihood (ML) based estimation of the Nakagami m fading parameter. Closed-form expressions are derived for the moment generating function and the probability density function (PDF) of the parameter Δ. For large sample size, it is found that the PDF of the parameter Δ can be well approximated by a two-parameter Gamma PDF. This approximation is validated by the Kolmogorov-Smirnov test.**

I. Introduction

The Nakagami-m fading model is important in wireless communications research since it fits the empirical multipath fading measurements better than the other fading models [1]. Nakagami-m fading model is also desirable because error rate performance analysis with Nakagami fading often leads to closed-form analytical solutions.

The probability density function (PDF) of the Nakagami-m fading envelope *R* is given by [1]

$$f_R(r) = \frac{2}{\Gamma(m)}\left(\frac{m}{\Omega}\right)^m r^{2m-1}\exp\left(-\frac{m}{\Omega}r^2\right), \quad r \geq 0 \tag{1}$$

where $\Omega = E[R^2]$, and the parameter m is defined as

$$m = \frac{\Omega^2}{E[(R^2-\Omega)^2]}, \quad m \geq \frac{1}{2}. \tag{2}$$

In order to characterize wireless channel using the Nakagami-m distribution, it is crucial to determine or estimate the value of m from N random variates $R_1, R_2, \ldots, R_N$ drawn independently according to (1). Several methods for estimating the m parameter have been reported in the literature. The Greenwood-Durand estimator (GDE) [2], a maximum-likelihood (ML) based Gamma-shape parameter estimator, is given by

$$\hat{m}_{GDE} = \begin{cases} f_1(\Delta) & \Delta < 0.5772 \\ f_2(\Delta) & 0.5772 \leq \Delta \leq 17 \end{cases} \tag{3}$$

where

$$f_1(\Delta) = \frac{0.5000876 + 0.1648852\Delta - 0.0544274\Delta^2}{\Delta} \tag{4a}$$

$$f_2(\Delta) = \frac{8.898919 + 9.059950\Delta - 0.9775373\Delta^2}{(17.79728 + 11.968477\Delta + \Delta^2)\Delta} \tag{4b}$$

and

$$\Delta = \ln\left[\frac{1}{N}\sum_{i=1}^{N} R_i^2\right] - \frac{1}{N}\sum_{i=1}^{N}\ln R_i^2 = -\psi(\hat{m}) + \ln(\hat{m}) \tag{5}$$

in which $\psi(\cdot)$ is the digamma function defined as $\psi(x) = \Gamma'(x)/\Gamma(x)$. More recently, Cheng and Beaulieu [3] proposed to use the first-order and second-order approximations to $\psi(\cdot)$ and derived two approximate ML estimators for m as

$$\hat{m}_1 = \frac{1}{2\Delta} \tag{6}$$

and

$$\hat{m}_2 = \frac{6 + \sqrt{36 + 48\Delta}}{24\Delta}. \tag{7}$$

It was pointed out by Zhang [4] that estimators similar to (6) and (7) were reported earlier by Thom [5] in estimation problem for Gamma distribution in another discipline.

The ML-based estimators presented in (3), (6), and (7) are all functions of the parameter Δ. This immediately implies that if we know the PDF of the parameter Δ, we can assess the performance of ML-based estimators for Nakagami m parameter without performing intensive computer simulations.

II. Statistical Properties of Δ

A. Alternative Expression of Δ

We rewrite the expression of Δ in (5) as

$$\begin{aligned}\Delta &= \ln\left(\frac{1}{N}\sum_{i=1}^{N} R_i^2\right) - \ln\left[\left(\prod_{i=1}^{N} R_i^2\right)^{\frac{1}{N}}\right] \\ &= \ln\left[\frac{\frac{1}{N}\sum_{i=1}^{N} R_i^2}{(\prod_{i=1}^{N} R_i^2)^{\frac{1}{N}}}\right].\end{aligned} \tag{8}$$

We can observe from (8) that the parameter Δ is just the logarithmic ratio of the arithmetic mean to the geometric mean of N samples of the Nakagami-m fading power.

B. Nonnegative Property of Δ

According to the well-known Arithmetic-Geometric inequality, we have

$$\frac{1}{N}\sum_{i=1}^{N} R_i^2 \geq \left(\prod_{i=1}^{N} R_i^2\right)^{\frac{1}{N}} \tag{9}$$

and therefore we must have $\Delta > 0$. By recognizing the fact that when m approaches $+\infty$ the Nakagami PDF becomes an

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impulse function located at $\sqrt{\Omega}$, we arrive at

$$\lim_{m\to+\infty}\Delta=\lim_{m\to+\infty}\ln\left[\frac{\frac{1}{N}\sum_{i=1}^{N}R_i^2}{(\prod_{i=1}^{N}R_i^2)^{\frac{1}{N}}}\right]=\ln\left[\frac{\frac{1}{N}\sum_{i=1}^{N}\Omega}{(\prod_{i=1}^{N}\Omega)^{\frac{1}{N}}}\right]=0. \tag{10}$$

C. Moment Generating Function of Δ

To derive the moment generating function (MGF) of Δ, denoted by $\Phi_\Delta(s)$, we start with the definition and have

$$\begin{aligned}\Phi_\Delta(s)&=E\left[e^{s\Delta}\right]\\&=\underbrace{\int_0^{+\infty}\cdots\int_0^{+\infty}}_{N}\left[\frac{\left(\sum_{i=1}^{N}r_i^2\right)^s}{N^s\left(\prod_{i=1}^{N}r_i^2\right)^{\frac{s}{N}}}\right]\\&\quad\times\left[\frac{2}{\Gamma(m)}\left(\frac{m}{\Omega}\right)^m r_1^{2m-1}e^{-\frac{m}{\Omega}r_1^2}\right]\\&\quad\times\cdots\times\left[\frac{2}{\Gamma(m)}\left(\frac{m}{\Omega}\right)^m r_N^{2m-1}e^{-\frac{m}{\Omega}r_N^2}\right]dr_1\cdots dr_N\\&=\frac{\left[\frac{2}{\Gamma(m)}\left(\frac{m}{\Omega}\right)^m\right]^N}{N^s}\underbrace{\int_0^{+\infty}\cdots\int_0^{+\infty}}_{N}\prod_{i=1}^{N}r_i^{2m-\frac{2s}{N}-1}\\&\quad\times\left(\sum_{i=1}^{N}r_i^2\right)^s\exp\left(-\frac{m}{\Omega}\sum_{i=1}^{N}r_i^2\right)dr_1\cdots dr_N.\end{aligned} \tag{11}$$

If we let $d=m-s/N$, after a change of variable ($r_i^2=x_i$), we obtain

$$\begin{aligned}\Phi_\Delta(s)=\frac{\left[\frac{1}{\Gamma(m)}\left(\frac{m}{\Omega}\right)^m\right]^N}{N^s}\underbrace{\int_0^{+\infty}\cdots\int_0^{+\infty}}_{N}\prod_{i=1}^{N}x_i^{d-1}\\ \times\left(\sum_{i=1}^{N}x_i\right)^s\times\exp\left(-\frac{m}{\Omega}\sum_{i=1}^{N}x_i\right)dx_1\cdots dx_N.\end{aligned} \tag{12}$$

The multiple N integrals in (12) can be reduced into a single integral by invoking the following useful integral identity [7]

$$\begin{aligned}&\underbrace{\int_0^{+\infty}\cdots\int_0^{+\infty}}_{N}x_1^{\alpha_1-1}x_2^{\alpha_2-1}\cdots x_n^{\alpha_n-1}f(\sum_{i=1}^{n}x_i)dx_1\cdots dx_n\\&=\frac{\Gamma(\alpha_1)\Gamma(\alpha_2)\cdots\Gamma(\alpha_n)}{\Gamma(\alpha_1+\alpha_2+\cdots+\alpha_n)}\int_0^{+\infty}u^{\alpha_1+\alpha_2+\cdots+\alpha_{n-1}}f(u)du.\end{aligned} \tag{13}$$

Letting $\alpha_1=\alpha_2=\ldots=\alpha_N=d$ and $f(x)=x^s\exp\left(-\frac{m}{\Omega}x\right)$, we can obtain a compact form for the MGF of Δ as

$$\begin{aligned}\Phi_\Delta(s)&=\frac{\left[\frac{2}{\Gamma(m)}\left(\frac{m}{\Omega}\right)^m\right]^N}{N^s}\cdot\frac{[\Gamma(d)]^N}{\Gamma(Nd)}\\&\quad\times\int_0^{+\infty}u^{Nd-1}u\exp\left(-\frac{m}{\Omega}u\right)du\\&=\frac{\Gamma(mN)[\Gamma(m-s/N)]^N}{N^s[\Gamma(m)]^N\Gamma(mN-s)}\end{aligned} \tag{14}$$

where in obtaining the last step we have used the definition of Gamma function.

D. Probability Density Function of Δ

The PDF of Δ can be obtained from its MGF by applying an inverse Laplace transform as

$$\begin{aligned}f_\Delta(\delta)&=\frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty}\Phi_\Delta(-s)e^{s\delta}ds\\&=\frac{\Gamma(mN)}{[\Gamma(m)^N]^N}\cdot\frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty}\frac{N^s[\Gamma(m+s/N)]^N}{\Gamma(mN+s)}e^{s\delta}ds\end{aligned} \tag{15}$$

where $j^2=-1$ and c is a suitably chosen positive constant which ensures that the contour path is in the region of convergence. The integration is taken along the vertical line $\Re\{s\}=c$ in the complex plane such that c is greater than the real part of any singularity of $\Phi_\Delta(-s)$.

If we now let $y=s/N$, the PDF becomes

$$f_\Delta(\delta)=\frac{\Gamma(mN)}{[\Gamma(m)^N]^N}\cdot\frac{N}{2\pi j}\int_{c'-j\infty}^{c'+j\infty}\frac{N^{Ny}[\Gamma(m+y)]^N}{\Gamma[N(m+y))]}e^{Ny\delta}dy \tag{16}$$

where $c'=c/N$ is another positive constant. With the aid of Gauss multiplication theorem [6]

$$\Gamma(nx)=(2\pi)^{\frac{1-n}{2}}n^{nx-\frac{1}{2}}\prod_{k=0}^{n-1}\Gamma\left(x+\frac{k}{n}\right) \tag{17}$$

we arrive at

$$\begin{aligned}f_\Delta(\delta)&=N\cdot\frac{\Gamma(mN)}{(2\pi)^{\frac{1-N}{2}}N^{Nm-\frac{1}{2}}[\Gamma(m)]^N}\cdot\frac{1}{2\pi j}\\&\quad\times\int_{c'-j\infty}^{c'+j\infty}\frac{\prod_{k=1}^{N}\Gamma[1-(1-m)+y]}{\prod_{k=0}^{N-1}\Gamma\left[1-(1-m-\frac{k}{N})+y\right]}(e^{N\delta})^y dy.\end{aligned} \tag{18}$$

Now applying the definition of Meijer's G-function [8] to (18), we can simply write the PDF of Δ as

$$f_\Delta(\delta)=N\xi\cdot G_{N,N}^{0,N}\left(e^{N\delta}\left|\begin{matrix}1-m & \cdots & 1-m\\ 1-m & \cdots & 1-m-\frac{N-1}{N}\end{matrix}\right.\right) \tag{19}$$

where

$$\xi=\frac{\Gamma(mN)}{(2\pi)^{\frac{1-N}{2}}N^{Nm-\frac{1}{2}}(\Gamma(m))^N}. \tag{20}$$

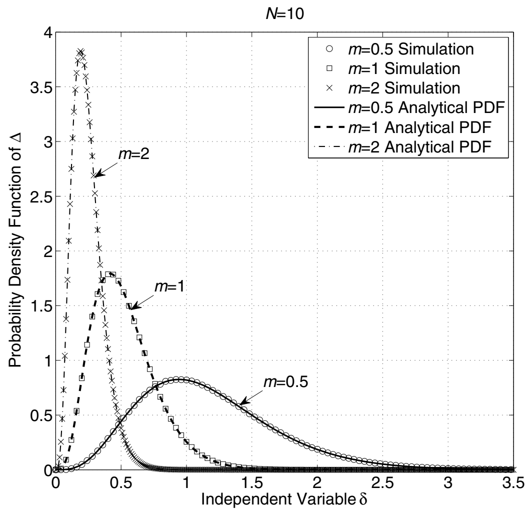

Fig. 1. Comparison of empirical PDFs and analytical PDFs of Δ for $m = 0.5, 1, 2$ with $N = 10$.

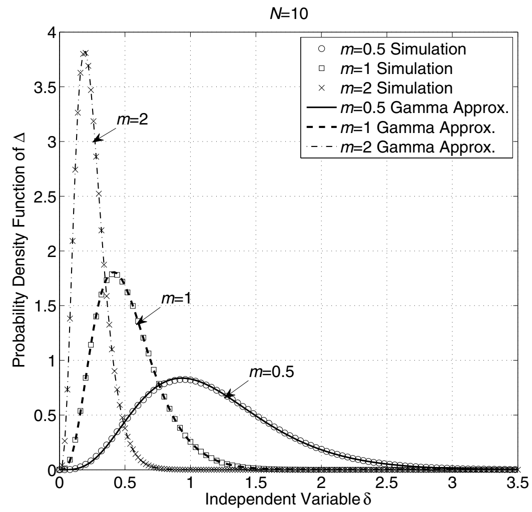

Fig. 2. Comparison of empirical PDFs and Gamma approximated PDFs of Δ for $m = 0.5, 1, 2$ with $N = 10$

Computer simulations were carried out to generate empirical PDFs of Δ for different m and N values, and to compare them with the analytical PDFs obtained from (19). The number of trials used for generating each empirical PDF was 2,000,000. Fig.1 shows the analytical and empirical PDFs of Δ for $N = 10$ when $m = 0.5,\ 1,\ 2$. It is shown that the analytical PDFs of Δ have excellent agreement with the empirical ones.

When the sample size N becomes large, the latest version of commercial software such as MAPLE and MATHEMATICA are incapable of evaluating our analytical PDF expression in (19).

III. Gamma Approximation

To avoid the high computational complexity associated with the Meijer's G-function for large N, we are motivated to approximate the PDF of Δ using another PDF which can be easily evaluated and is analytically tractable.

A. Gamma Approximation for PDF of Δ

From the nonnegative property discussed in Section II, we know that Δ is defined on [0, $+\infty$). We propose to use a two-parameter Gamma PDF, which is also defined on [0, $+\infty$), to approximate the PDF of Δ. To determine the parameters θ and k in the two-parameter Gamma PDF

$$f_X(x) = \frac{x^{k-1}e^{-x/\theta}}{\theta^k \Gamma(k)} \qquad x \geq 0;\ \theta, k > 0 \tag{21}$$

we can simply match the mean and variance of the two-parameter Gamma distribution to the mean and variance of Δ.

From the MGF of Δ in (14), the first two moments of Δ can thus be obtained by taking the first and the second derivatives of the MGF with respect to s and evaluating the results at $s = 0$. It is straightforward to show that the first two moments of Δ are given by

$$\mu_1 = -\psi(m) - \ln(N) + \psi(mN) \tag{22a}$$

$$\begin{aligned}\mu_2 = {} & [\psi(m)]^2 + [\ln(N)]^2 + [\psi(mN)]^2 + 2\psi(m)\ln(N) \\ & - 2\psi(m)\psi(mN) - 2\psi(mN)\ln(N) \\ & + \frac{1}{N}\psi'(m) - \psi'(mN).\end{aligned} \tag{22b}$$

Setting the mean and variance of the two-parameter Gamma distribution equal to the mean and variance of Δ

$$\mu_1 = k\theta \tag{23a}$$

$$\mu_2 - (\mu_1)^2 = k\theta^2 \tag{23b}$$

we arrive at

$$\theta = \frac{\frac{1}{N}\psi'(m) - \psi'(mN)}{-\psi(m) - \ln(N) + \psi(mN)} \tag{24a}$$

$$k = \frac{\frac{1}{N}\psi'(m) - \psi'(mN)}{[-\psi(m) - \ln(N) + \psi(mN)]^2}. \tag{24b}$$

The two-parameter Gamma approximation is quite desirable since this PDF has a simple exponential form, which can be easily evaluated and manipulated in practice.

B. Validating the Gamma Approximation

Computer simulations were also carried out to compare the two-parameter Gamma approximate PDFs of Δ with the empirical PDFs. The number of trials conducted for each empirical PDF was also 2,000,000.

Fig.2 shows the comparison between the empirical PDFs and the Gamma approximated PDFs of Δ for $m = 0.5$, 1, and 2 with $N = 10$. Fig.3 presents the comparison between the empirical PDFs and the corresponding Gamma PDFs for $m = 0.5$, 1, and 2 with $N = 100$. Both Figs. 2 and 3 demonstrate that the two-parameter Gamma PDF is a good candidate for approximating the PDF of Δ.

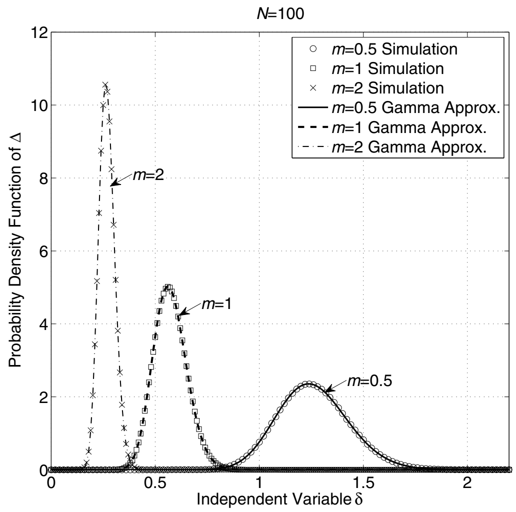

Fig. 3. Comparison of empirical PDFs and Gamma approximated PDFs of Δ for $m = 0.5, 1, 2$ with $N = 100$.

TABLE I
KOLMOGOROV-SMIRNOV TEST FOR GAMMA APPROXIMATION

	$n = 100$			$n = 1,000$		
	D_{max}	D_{avg}	*Acpt.* %	D_{max}	D_{avg}	*Acpt.* %
$N = 10$						
$m = 0.5$	0.226	0.085	99.03%	0.067	0.027	99.05%
$m = 1$	0.218	0.085	98.87%	0.069	0.027	98.83%
$m = 2$	0.226	0.085	99.01%	0.072	0.027	99.00%
$m = 5$	0.232	0.085	99.11%	0.078	0.027	99.01%
$N = 100$						
$m = 0.5$	0.212	0.085	98.95%	0.075	0.027	99.00%
$m = 1$	0.222	0.085	99.11%	0.066	0.027	98.91%
$m = 2$	0.211	0.085	99.00%	0.074	0.027	99.03%
$m = 5$	0.232	0.085	99.01%	0.068	0.027	98.88%

To numerically validate the feasibility of approximating Δ as a Gamma random variable, we use the Kolmogorov-Smirnov (K-S) test for goodness-of-fit. The basic idea of the K-S test is to compare the empirical CDF with the CDF of the hypothesized distribution. The test statistic D_n for test sample volume n is defined as the supremum of the absolute difference between the theoretical CDF $F(x)$ and the empirical CDF $F_n(x)$

$$D_n \equiv \sup_x |F(x) - F_n(x)|. \tag{25}$$

If the test statistic D_n is less than a critical value D_n^α, which is determined by both the test sample volume n (degree of freedom) and a prescribed significance level α, the theoretical distribution is acceptable at a confidence level of $1 - \alpha$.

Case studies were conducted using test sample volume $n =$ 100 and 1,000 for $m =$ 0.5, 1, 2, and 5 with $N =$ 10 and 100. The significance level α was chosen to be 0.01, giving a 99.00% confidence level for the K-S test.

Table I shows the maximum test statistics D_{max} and the average test statistics D_{avg} obtained from 10,000 experiments conducted in our study. According to [9], the critical values for test sample volume $n =$ 100 and 1,000 at significance level 0.01 are $D_{100}^{0.01} = 1.63/\sqrt{100} = 0.163$ and $D_{1,000}^{0.01} = 1.63/\sqrt{1000} = 0.0515$ respectively. We observe that in each case of our case studies, about 99% of the experiments accepted the hypothesis that the random variable Δ can be modeled as a Gamma random variable at a confidence level of 99.00%. Table I also shows that D_{max} values, under which the hypothesis is rejected, are slightly greater than the critical values; and the average test statistic D_{avg} values are significantly below the corresponding values. In summary, the K-S test concludes that the two-parameter Gamma PDF can be used to accurately approximate the PDF of Δ.

IV. CONCLUSION

By investigating statistical properties of parameter Δ, which is defined as the logarithmic ratio of the arithmetic mean to geometric mean for the Nakagami-m fading power, the MGF and the exact PDF of Δ have been derived. Gamma approximation of the PDF of Δ, which avoids computational complexity of the exact PDF for large sample size N has also been proposed. The validity of using the two-parameter Gamma PDF to approximate the PDF of Δ has been established using the Kolmogorov-Smirnov test.

REFERENCES

[1] M. Nakagami, "The m-distribution, a general formula of intensity distribution of rapid fading," *Statistical Methods in Radio Wave Propagation*, W. G. Hoffman, Ed. Oxford, England: Pergamon, 1960.
[2] J. A. Greenwood and D. Durand, "Aids for fitting the Gamma distribution by maximum likelihood," *Technometrics*, vol. 2, no. 1, pp. 55-64, Feb. 1960.
[3] J. Cheng and N. C. Beaulieu, "Maximum-Likelihood based estimation of the Nakagami m parameter," *IEEE Commun. Lett.*, vol.5, pp. 101-103, Mar. 2001.
[4] Q.T. Zhang, "A Note on the Estimation of Nakagami-m Fading Parameter," *IEEE Commun. Lett.*, vol.6, pp. 237-238, Jun. 2002.
[5] H. C. Thom, "A note on the Gamma distribution," *Monthly Weather Rev.*, vol. 86, no. 4, pp.117-122, 1958.
[6] E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis*, 4th ed., Cambridge University Press, Cambridge, 1962.
[7] G. A. Gibson, *Advanced Calculus*, Macmillan, London, 1931.
[8] A. M. Mathai and R. K. Saxena, *Generalized Hypergeometric Functions With Applications In Statistics And Physical Sciences*, Springer-Verlag, New York, 1973.
[9] A. H-S. Ang and W. H. Tang, *Probability Concepts in Engineering*, 2nd ed., John Wiley & Sons, Hoboken, NJ, 2007.