[8] I. Korn, "Error probability of offset differential phase shift keying with intersymbol and adjacent channel interference," *Proc. Inst. Elect. Eng.*, vol. 135, no. 2, pp. 175–182, Apr. 1988.

[9] I. Korn, "Differential phase shift keying in two-path Rayleigh channel with adjacent channel interference," *IEEE Trans. Veh. Technol.*, vol. 40, no. 2, pp. 461–471, May 1991.

[10] R. Y. Hougui and I. Kalet, "MSK-type signals in the presence of adjacent channel interference and white noise," *IEEE Trans. Commun.*, vol. 45, no. 4, pp. 404–407, Apr. 1997.

[11] D. P. Bouras, P. T. Mathiopoulos, and D. Makrakis, "Neural-net-based receiver structures for single- and multitude bandlimited signals in CCI and ACI channels," *IEEE Trans. Veh. Technol.*, vol. 46, no. 3, pp. 791–798, Aug. 1997.

[12] P. T. Mathiopoulos, G. K. Karagiannidis, and J. S. Toor, "Effects of ACI and nonlinearities on the performance of differentially detected GMSK signals," *Proc. Inst. Elect. Eng.—Commun.*, vol. 151, no. 2, pp. 163–169, Apr. 2004.

[13] D. Subasinghe-Dias and K. Feher, "Baseband pulse shaping for $\pi/4$ FQPSK in nonlinearly amplified mobile channels," *IEEE Trans. Commun.*, vol. 42, no. 10, pp. 2843–2852, Oct. 1994.

[14] S. Ariyavisitakul and T.-P. Liu, "Characterizing the effects of nonlinear amplifiers on linear modulation for digital portable radio communications," *IEEE Trans. Veh. Technol.*, vol. 39, no. 4, pp. 383–389, Nov. 1990.

[15] J. Li and Q. Liu, "PSK communications systems using fully saturated power amplifiers," *IEEE Trans. Aerosp. Electron. Syst.*, vol. 42, no. 2, pp. 464–477, Apr. 2006.

[16] A. Chini, Y. Wu, M. El-Tanany, and S. Mahmoud, "Hardware nonlinearities in digital TV broadcasting using OFDM modulation," *IEEE Trans. Broadcast.*, vol. 44, no. 1, pp. 12–21, Mar. 1998.

[17] Q. Shi, "OFDM in bandpass nonlinearity," *IEEE Trans. Consum. Electron.*, vol. 42, no. 3, pp. 253–258, Aug. 1996.

[18] G. T. Zhou and J. S. Kennedy, "Predicting spectral regrowth of nonlinear power amplifiers," *IEEE Trans. Commun.*, vol. 50, no. 5, pp. 718–722, May 2002.

[19] C.-P. Liang, J.-H. Jong, W. E. Stark, and J. R. East, "Nonlinear amplifier effects in communication systems," *IEEE Trans. Microw. Theory Tech.*, vol. 47, no. 8, pp. 1461–1466, Aug. 1999.

[20] O. Hammi, S. Carichner, B. Vassilakis, and F. M. Ghannouchi, "Power amplifiers' model assessment and memory effects intensity quantification using memoryless post-compensation technique," *IEEE Trans. Microw. Theory Tech.*, vol. 56, no. 12, pp. 3170–3179, Dec. 2008.

[21] N. C. Beaulieu and A. A. Abu-Dayya, "Analysis of equal-gain diversity on Nakagami fading channels," *IEEE Trans. Commun.*, vol. 39, no. 2, pp. 225–234, Feb. 1991.

[22] A. Annamalai, C. Tellambura, and K. Bhargava, "Equal-gain diversity receiver performance in wireless channels," *IEEE Trans. Commun.*, vol. 48, no. 10, pp. 1732–1745, Oct. 2000.

[23] N. C. Sagias, "Closed-form analysis of equal-gain diversity in wireless radio networks," *IEEE Trans. Veh. Technol.*, vol. 56, no. 1, pp. 173–182, Jan. 2007.

[24] V. A. Aalo and G. P. Efthymoglou, "Another look at the performance of MRC schemes in Nakagami-m fading channels with arbitrary parameters," *IEEE Trans. Commun.*, vol. 53, no. 12, pp. 2002–2005, Dec. 2005.

[25] A. Annamalai, C. Tellambura, and K. Bhargava, "Exact evaluation of maximal-ratio and equal-gain diversity receivers for M-ary QAM on Nakagami fading channels," *IEEE Trans. Commun.*, vol. 47, no. 9, pp. 1335–1344, Sep. 1999.

[26] N. C. Beaulieu, C. C. Tan, and M. O. Damen, "A 'better than' Nyquist pulse," *IEEE Commun. Lett.*, vol. 5, no. 9, pp. 367–368, Sep. 2001.

[27] H. Harada and R. Prasad, *Simulation and Software Radio for Mobile Communications*. Norwood, MA: Artech House, 2002.

[28] G. P. White, A. G. Burr, and T. Javornik, "Modeling of nonlinear distortion in broadband fixed wireless access systems," *Electron. Lett.*, vol. 39, no. 8, pp. 686–687, Apr. 17, 2003.

[29] C. Rapp, "Effects of HPA-nonlinearity on a 4-DQPSK/OFDM signal for a digital sound broadcasting system," in *Proc. 2nd Eur. Conf. Satellite Commun.*, Liege, Belgium, Oct. 1991, pp. 179–184.

[30] RF impairment models for 60 GHz-band SYS/PHY simulation, IEEE 802.15-06-0477-01-03c, Nov. 2006.

[31] J. H. van Lint, *An Introduction to Coding Theory,* 2nd ed. New York: Springer-Verlag, 1992.

[32] [Online]. Available: http://www.ieee802.org/15/pub/TG3c.html

[33] C. S. Sum, R. Funada, J. Wang, T. Baykas, M. A. Rahman, and H. Harada, "Error performance and throughput evaluation of a multi-Gbps millimeter-wave WPAN system in the presence of adjacent and co-channel interference," *IEEE J. Sel. Areas Commun.*, vol. 27, no. 8, pp. 1433–1442, Oct. 2009.

[34] S. Moriguchi, K. Udagawa, and S. Hitotsumatsu, *Iwanami Tsugaku Koushiki II*. Tokyo, Japan: Iwanami, 1987 (in (Japanese).

[35] S. Okui, *Denki Tsushin Kogaku no Tame no Tokushu Kansu to Sono Ohyou*. Tokyo, Japan: Morikita, 1997 (in (Japanese).

Unified Exact Performance Analysis of Two-Hop Amplify-and-Forward Relaying in Nakagami Fading

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***Abstract*—We present a general two-parameter received signal-to-noise ratio (SNR) model for two-hop amplify-and-forward (AF) relaying. It encompasses AF schemes that select the relay gain as the reciprocal of a linear combination of the channel gain and the noise power of the incoming link, including all channel-noise-assisted, channel-assisted, and blind relay schemes. Moreover, the model is flexible enough to represent independent source and relay power allocations. A unified performance analysis is then developed for AF relaying over independent but nonidentically distributed Nakagami-m faded links, where m is an integer. Exact analytical expressions are derived for the cumulative-distribution function (cdf), probability density function (pdf), and moment-generating function (mgf) of the received SNR. Monte Carlo simulation results are provided to verify the results.**

***Index Terms*—Amplify-and-forward (AF) relaying, blind relaying, channel-assisted relaying, channel-noise-assisted relaying.**

I. Introduction

User cooperation via signal relaying provides benefits such as wider coverage, transmit power saving, and reduced interference [1]–[3]. Cooperative relaying can be broadly categorized as decode-and-forward (DF) (regenerative) and amplify-and-forward (AF) (non-regenerative). AF relays may be categorized as *Blind/fixed gain* [4], [5] (includes semiblind [6] relaying), *channel assisted* [7], and *channel noise assisted* [1], [8], [9], based on how source-to-relay channel state information and noise statistics are used in determining the relay gains. This correspondence aims to provide a unified analysis of two-hop relay networks, covering all these cases, as well as additional ones. We propose the general model

$$\Lambda = \frac{\gamma_1\gamma_2}{a\gamma_1 + \gamma_2 + b} \tag{1}$$

for the received SNR Λ of a dual-hop AF system, where a, $b \geq 0$, and γ_1 and γ_2 are the hop SNRs, which corresponds to selecting the relay gain in [10, eq. (1)] as $G_n^2 = 1/a\alpha_n^2 + bN_{0,n}$. Thus, the proposed model encompasses the standard schemes: blind, channel assisted, and channel noise assisted as special cases. All of them are

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TABLE I
COMPARISON WITH PREVIOUS WORK

citation	parameters					
	a	b	α_1	β_1	α_2	β_2
[4]	0	C	1	α_1	1	α_2
[7], [15]	1	0	1	$\bar{\gamma}_1$	1	$\bar{\gamma}_2$
[8]	1	$c \in \{0,1\}$	m_1	γ_1	m_2	γ_2
[16]	1 & 0	1	m_1	$\frac{\bar{\gamma}_1}{m_1}$	m_2	$\frac{\bar{\gamma}_2}{m_2}$

"extreme" configurations corresponding, respectively, to constant relay gain, disregarding noise statistics, and constant instantaneous relay output power. With parameters (a, b), (1) can model nonstandard configurations as well.

The model has another implication with respect to power allocation. Suppose $G_n^2 = P_R/a\alpha_n^2 + bN_{0,n}$ in [10, eq. (1)] so that the power gain of the relay is proportional to the average[1] relay power P_R. Let distributions γ_1 and γ_2 be link SNRs corresponding to "unit average source transmit power and unit average relay transmit power." The received SNR for the average source and relay transmit powers P_s and P_r can be modeled with coefficients $((a/P_s), b)$ in (1), provided that the distributions of γ_1 and γ_2 are modified to have their average powers scaled, respectively, by P_s and P_r.

The performance of an AF system can fully be characterized if the distribution of the received SNR is analytically known. Our analysis of the proposed model assumes independent but nonidentically distributed Nakagami-m fading, where m is an integer, and derives exact analytical expressions for the cumulative-distribution function (cdf), probability density function (pdf), and moment-generating function (mgf) of the received SNR. The case of noninteger m, which typically requires infinite series representations, is not treated for brevity. Thus, the unified SNR model of (1) is analyzed for $\gamma_1 \sim \text{Gamma}(\alpha_1, \beta_1)$, $\gamma_2 \sim \text{Gamma}(\alpha_2, \beta_2)$, and $\alpha_1, \alpha_2 \in \mathbb{Z}^+$. Special cases of our model that have already appeared in the literature are shown in Table I.

The correspondence is organized as follows. Section II presents the system model. Expressions derived for the pdf, cdf, and mgf of the received SNR are presented in Section III. Section IV provides simulation results verifying the derived expressions. Section V provides the concluding remarks. Proofs of the results are given in the Appendix.

Notation: $K_\nu(\cdot)$ is the *modified Bessel function of the second kind* [11, 9.6] of order ν. $\mathcal{W}_{\mu,\nu}(\cdot)$ denotes the *Whittaker W function* [11, 13.1]. ${}_2\mathcal{F}_1(\alpha, \beta; \gamma; \cdot)$ represents the *Gauss hypergeometric function* [11, 15.1]. $f_\Lambda(\cdot)$, $F_\Lambda(\cdot)$, $\bar{F}_\Lambda(\cdot)$, and $M_\Lambda(\cdot)$ denote, respectively, the pdf, cdf, complementary cdf (ccdf), and mgf of a continuous random variable Λ. The mgf is defined as the Laplace transform of the pdf. $\mathbb{Z}^+$ denotes the set of positive integers.

II. SYSTEM AND CHANNEL MODEL

Our analysis considers a two-hop relayed path $S \to R \to D$ in a cooperative wireless network, where node S acts as the data source AF relaying data through another node R to a third node D, which is the destination. The direct path and other relayed paths that may lead from S to D are not considered in this analysis.

The source-to-relay and relay-to-destination links undergo independent Nakagami-m fading characterized by integer fading parameters. Thus, received SNRs are Gamma distributed. The Gamma(α, β) pdf is given by

$$f_\gamma(x) = \frac{x^{\alpha-1} e^{-\frac{x}{\beta}}}{\Gamma(\alpha)\beta^\alpha}, \quad x \geq 0$$

where $\alpha \in \mathbb{Z}^+$, $\beta > 0$ are, respectively, the shape and scale parameters.

The received SNR is modeled as (1), whose parameters (a, b) reflect the relay configuration and source–relay power allocation in use. Standard configurations, namely, blind, channel-assisted, and channel-noise-assisted relay configurations, are represented, respectively, with $(a, b) \in \{(0, C), (1, 0), (1, 1)\}$, where C is a constant. Other values (e.g., $a = 0.5$, $b = 1$) represent nonstandard configurations.

III. PERFORMANCE ANALYSIS

Given Λ in (1), which is the random variable denoting the received SNR at destination node D, we derive exact expressions for the cdf, pdf, and mgf.

A. CDF of Λ

We begin with the derivation of the cdf (or, equivalently, ccdf) because differentiation is guaranteed to yield a closed-form result for the pdf.

Theorem 1: CCDF of Λ: Let γ_1 and γ_2 be distributed Gamma(α_1, β_1) and Gamma(α_2, β_2), respectively. The ccdf $\bar{F}_\Lambda(x)$ of Λ in (1) is given for $x \geq 0$ by

$$\bar{F}_\Lambda(x) = 2e^{-\left(\frac{1}{\beta_1} + \frac{a}{\beta_2}\right)x} \sum_{n=0}^{\alpha_1-1} \sum_{k=0}^{\alpha_2-1} \sum_{m=0}^{k} C_1(n,k,m) \times K_{n-m+1}\left(2\sqrt{\frac{x(ax+b)}{\beta_1\beta_2}}\right) \left(a + \frac{b}{x}\right)^{\frac{n+m+1}{2}} x^{\alpha_1+k} \quad (2)$$

where

$$C_1(n,k,m) = \frac{a^{k-m} \beta_1^{\frac{n-m+1-2\alpha_1}{2}} \beta_2^{\frac{m-n-1-2k}{2}}}{m!\,(k-m)!\,n!\,(\alpha_1-n-1)!}.$$

For the case $a = 0$, (2) reduces to

$$\bar{F}_\Lambda(x) = 2e^{-\frac{x}{\beta_1}} \sum_{n=0}^{\alpha_1-1} \sum_{k=0}^{\alpha_2-1} C_2(n,k) \times K_{n-k+1}\left(2\sqrt{\frac{bx}{\beta_1\beta_2}}\right) x^{\frac{2\alpha_1+k-n-1}{2}} \quad (3)$$

where

$$C_2(n,k) = \frac{\beta_1^{\frac{n-k+1-2\alpha_1}{2}} \left(\frac{b}{\beta_2}\right)^{\frac{n+k+1}{2}}}{k!\,n!\,(\alpha_1-n-1)!}.$$

The cdf for each case is given by $F_\Lambda(x) = 1 - \bar{F}_\Lambda(x)$.

Proof: See the Appendix. ■

B. PDF of Λ

The pdf $f_\Lambda(x)$ can be obtained by differentiating the ccdf and applying a negative sign.

Theorem 2: PDF of Λ: Let γ_1 and γ_2 be distributed Gamma(α_1, β_1) and Gamma(α_2, β_2), respectively. The pdf $f_\Lambda(x)$ of Λ in (1) is given for $x \geq 0$ by

$$f_\Lambda(x) = 2e^{-\left(\frac{1}{\beta_1} + \frac{a}{\beta_2}\right)x} \sum_{n=0}^{\alpha_1-1} \sum_{k=0}^{\alpha_2-1} \sum_{m=0}^{k} C_1(n,k,m) \times \mathbb{I}_1(n,k,m) \left(a + \frac{b}{x}\right)^{\frac{n+m-1}{2}} x^{\alpha_1+k-2} \quad (4)$$

[1]Only a subclass of schemes that includes channel-noise-assisted relaying has the relay operating at constant instantaneous power P_r. In others, one may have to compare the "average power."

where

$$\mathbb{I}_1(n,k,m) = \left((ax+b)\left(\left(\frac{1}{\beta_1}+\frac{a}{\beta_2}\right)x-(\alpha_1+k-n-1)\right)-amx\right) \times K_{n-m+1}\left(2\sqrt{\frac{x(ax+b)}{\beta_1\beta_2}}\right) + (2ax+b)\sqrt{\frac{x(ax+b)}{\beta_1\beta_2}}K_{n-m}\left(2\sqrt{\frac{x(ax+b)}{\beta_1\beta_2}}\right).$$

For the case $a=0$, it reduces to

$$f_\Lambda(x) = 2e^{-\frac{x}{\beta_1}}\sum_{n=0}^{\alpha_1-1}\sum_{k=0}^{\alpha_2-1} C_2(n,k) \times \mathbb{I}_2(n,k)\left(\frac{b}{x}\right)^{\frac{n+k-1}{2}} x^{\alpha_1+k-2} \quad (5)$$

where

$$\mathbb{I}_2(n,k) = b\left(\frac{1}{\beta_1}x-(\alpha_1+k-n-1)\right)K_{n-k+1}\left(2\sqrt{\frac{bx}{\beta_1\beta_2}}\right) + b\sqrt{\frac{bx}{\beta_1\beta_2}}K_{n-k}\left(2\sqrt{\frac{bx}{\beta_1\beta_2}}\right).$$

Proof: Given the ccdf $\bar{F}_\Lambda(\lambda)$, the pdf can be obtained by using the product rule of differentiation and the identity [12, 8.486.12] that gives the derivative $(d/dz)K_n(z)$. ∎

C. MGF of Λ

Given the ccdf $\bar{F}_\Lambda(x)$, the mgf can be expressed via integration by parts as

$$M_\Lambda(s) = \int_0^\infty e^{-sx}f_\Lambda(x)dx = 1 - s\int_0^\infty e^{-sx}\bar{F}_\Lambda(x)dx. \quad (6)$$

Cases $a=0$ and $a\neq 0$ are separately treated to make the problem mathematically tractable. Unlike with the cdf and pdf results, the mgf result for the case $a=0$ is not a direct reduction of that for the case $a\neq 0$.

Theorem 3: MGF of Λ: Let γ_1 and γ_2 be distributed Gamma(α_1,β_1) and Gamma(α_2,β_2), respectively. The mgf $M_\Lambda(s)$ of Λ in (1) is given by the following.

Case $a\neq 0$:

$$M_\Lambda(s) = 1-2s\sum_{n=0}^{\alpha_1-1}\sum_{k=0}^{\alpha_2-1}\sum_{m=0}^{k} C_1(n,k,m)a^{\frac{n+m+1}{2}} \times \sum_{q=0}^{n+m+2}\binom{n+m+2}{q}\left(\frac{b}{a}\right)^q \mathbb{J}_1(n,k,m,q) \quad (7)$$

where

$$\mathbb{J}_1(n,k,m,q) = \frac{\sqrt{a\beta_1\beta_2}\Gamma(n+2)\Gamma(m+1)}{2b(-1)^{\alpha_1+k-q+1}}\frac{d^{\alpha_1+k-q+1}}{dt^{\alpha_1+k-q+1}} \times \left\{e^{\frac{bt}{2a}}\mathcal{W}_{-\frac{n+m+2}{2},\frac{n-m+1}{2}}\left(\frac{b\left(t-\sqrt{t^2-\frac{4a}{\beta_1\beta_2}}\right)}{2a}\right) \times \mathcal{W}_{-\frac{n+m+2}{2},\frac{n-m+1}{2}} \times \left(\frac{b\left(t+\sqrt{t^2-\frac{4a}{\beta_1\beta_2}}\right)}{2a}\right)\right\}\Bigg|_{t\doteq s+\frac{1}{\beta_1}+\frac{a}{\beta_2}}.$$

The mgf can be obtained in a more compact form as follows whenever $(a\neq 0, b=0)$:

$$M_\Lambda(s) = 1-2s\sum_{n=0}^{\alpha_1-1}\sum_{k=0}^{\alpha_2-1}\sum_{m=0}^{k}C_1(n,k,m)\mathbb{J}_3(n,k,m)a^{\frac{n+m+1}{2}} \quad (8)$$

where

$$\mathbb{J}_3(n,k,m) = \frac{\sqrt{\pi}\Gamma(\alpha_1+k+n-m+2)\Gamma(\alpha_1+k-n+m)}{\Gamma\left(\alpha_1+k+\frac{3}{2}\right)} \times \left(\frac{16a}{\beta_1\beta_2}\right)^{\frac{n-m+1}{2}} \times \frac{{}_2\mathcal{F}_1\left(\alpha_1+k+n-m+2, n-m+\frac{3}{2}; \alpha_1+k+\frac{3}{2}; \bar{s}\right)}{\left(s+\left(\sqrt{\frac{1}{\beta_1}}+\sqrt{\frac{a}{\beta_2}}\right)^2\right)^{\alpha_1+k+n-m+2}}$$

and $\bar{s} = (s+(\sqrt{1/\beta_1}-\sqrt{a/\beta_2})^2)/(s+(\sqrt{1/\beta_1}+\sqrt{a/\beta_2})^2)$.

Case $a=0$:

$$M_\Lambda(s) = 1-2s\sum_{n=0}^{\alpha_1-1}\sum_{k=0}^{\alpha_2-1}C_2(n,k)\mathbb{J}_2(n,k) \quad (9)$$

where

$$\mathbb{J}_2(n,k) = \frac{\Gamma(\alpha_1+1)\Gamma(\alpha_1+k-n)}{2\sqrt{\frac{b}{\beta_1\beta_2}}}\left(s+\frac{1}{\beta_1}\right)^{\frac{n-k-2\alpha_1}{2}} \times e^{\frac{b}{2\beta_2(1+\beta_1 s)}}\mathcal{W}_{\frac{n-2\alpha_1-k}{2},\frac{n-k+1}{2}}\left(\frac{b}{\beta_2(1+\beta_1 s)}\right).$$

Proof: See the Appendix. ∎

Although (7) appears complicated, it is in closed form, because higher order derivatives of arbitrary order are known for the Whittaker W function. The expression can be expanded without much difficulty for given α_1, β_1, α_2, β_2, a, and b, using the *generalized Leibniz rule* for higher derivatives. Moreover, widely available technical computing software such as MAPLE, MATHEMATICA, and MATLAB readily provides tools for straightforward symbolic evaluation of (7). The numerical results for this paper were produced through the symbolic evaluation of (7) by using MATHEMATICA.

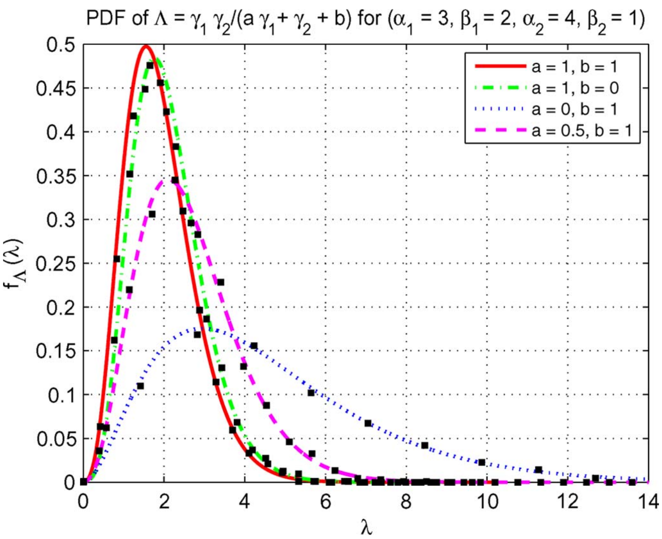

Fig. 1. PDF of the received SNR λ $(\alpha_1 = 3, \beta_1 = 2, \alpha_2 = 4, \beta_2 = 1)$. Legend: channel-noise assisted $(a=1, b=1)$, channel assisted $(a=1, b=0)$, blind $(a = 0, b = 1)$, and nonstandard $(a = 0.5, b = 1)$.

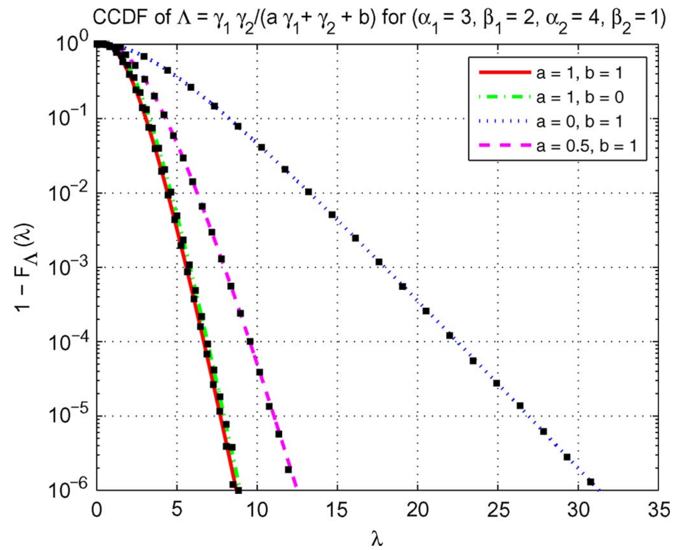

Fig. 2. CCDF of the received SNR λ $(\alpha_1 = 3, \beta_1 = 2, \alpha_2 = 4, \beta_2 = 1)$. Legend: channel-noise assisted $(a=1, b=1)$, channel assisted $(a=1, b=0)$, blind $(a = 0, b = 1)$, and nonstandard $(a = 0.5, b = 1)$.

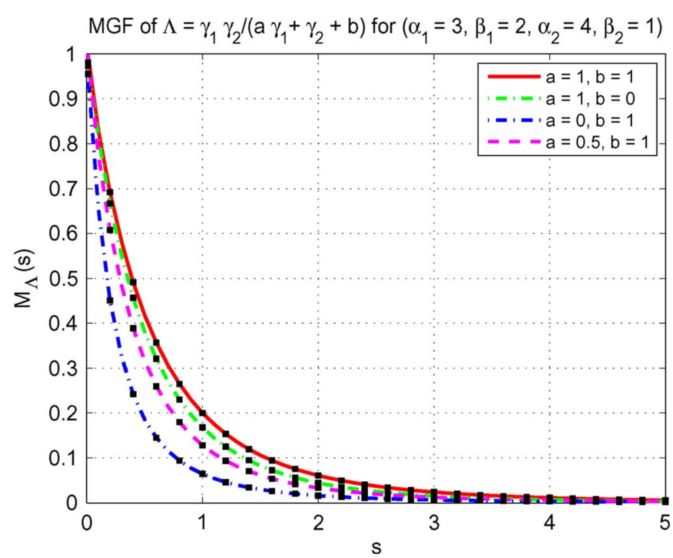

Fig. 3. MGF of the received SNR $(\alpha_1 = 3, \beta_1 = 2, \alpha_2 = 4, \beta_2 = 1)$. Legend: channel-noise assisted $(a=1, b=1)$, channel assisted $(a=1, b=0)$, blind $(a = 0, b = 1)$, and nonstandard $(a = 0.5, b = 1)$.

IV. Numerical Results

Theorems 1–3 are verified through Monte Carlo simulation. We consider a nonidentical fading case $(\alpha_1 = 3, \beta_1 = 2, \alpha_2 = 4, \beta_1 = 1)$ under four AF relaying schemes: channel-noise assisted $(a = 1, b = 1)$, channel assisted $(a = 1, b = 0)$, blind $(a = 0, b = 1)$, and nonstandard $(a = 0.5, b = 1)$. The nonstandard scheme we have considered here is a scheme chosen to perform in between channel-assisted and blind relaying.

The mgf for the case $(a \neq 0, b \neq 0)$ was implemented in MATHEMATICA, while all other computations were carried out using MATLAB. Monte Carlo simulations were carried out by averaging over 10^7 sample points.

Fig. 1 depicts the pdf of the received SNR. The pdf's of channel-assisted and channel-noise-assisted schemes closely agree. This justifies the approximation of the latter by the former [13]. Blind relaying shows the highest dispersion.

The cdf of the received SNR is shown in Fig. 2. Channel-noise-assisted relaying, which is more conservative with respect to transmit power, has the highest probability of outage; while blind relaying with $b = 1$ suffers the least outage.

Fig. 3 shows the mgf for the same schemes. As observed before in the cases of the pdf and the ccdf, the mgf curves for channel-assisted and channel-noise-assisted relaying closely agree. This justifies the use of the mgf for channel-assisted relaying to approximate that for channel-noise-assisted relaying when analyzing maximal-ratio-combined multiple-relayed branch systems. Blind relaying, which shows the highest dispersion of Λ, exhibits, as expected, the fastest rate of decay in the mgf. Figs. 1–3 show the curve for the case $(a = 0.5, b = 1)$ lying in between those of blind and channel-assisted relaying.

All analytical results are observed to agree well with the corresponding Monte Carlo simulations, supplementing the mathematical proofs and reinforcing the correctness of the theorems.

V. Conclusion

This paper has presented a general received SNR model for two-hop AF relaying that encompasses much analyzed conventional channel-noise-assisted, channel-assisted, and blind relay configurations as special cases. Moreover, the model is flexible and general enough so that independent power allocations for source and relay can be treated. Based on the model, a unified performance analysis has been provided for the case of nonidentical Nakagami-m fading links. The exact closed-form cdf, pdf, and mgf expressions have been derived. The results have been analytically proved and verified through Monte Carlo simulation. For brevity, our analysis only consider the case of integer m. The results may prove useful not only in performance analysis problems but in power allocation and other topics for relay systems as well.

Appendix
Proofs of the Results

The proofs of Theorems 1 and 3 are presented here in a concise form. Let random variable Λ be denoted by (1), where

$\gamma_1 \sim \text{Gamma}(\alpha_1, \beta_1)$ and $\gamma_2 \sim \text{Gamma}(\alpha_2, \beta_2)$, such that $\alpha_1, \alpha_2 \in \mathbb{Z}^+$, and $\beta_1, \beta_2 > 0$. Their pdf and ccdf are given for $i \in \{1, 2\}$ by

$$f_{\gamma_i}(x) = \frac{x^{\alpha_i-1} e^{-\frac{x}{\beta_i}}}{\Gamma(\alpha_i)\beta_i^{\alpha_i}}; \quad \bar{F}\gamma_i(x) = e^{-\frac{x}{\beta_i}} \sum_{k=0}^{\alpha_i-1} \frac{\left(\frac{x}{\beta_i}\right)^k}{k!}. \tag{10}$$

A) Proof: CDF of Λ:

Theorem 1: Let $\mathcal{P}\{A\}$ denote the probability of an event A. The cdf can be derived as follows:

$$\begin{aligned} F_\Lambda(\lambda) &= \mathcal{P}\left\{\frac{\gamma_1\gamma_2}{a\gamma_1 + \gamma_2 + b} \le \lambda\right\} \\ &= \int_0^\lambda \underbrace{\mathcal{P}\left\{\gamma_2 \ge \frac{(ax+b)\lambda}{x-\lambda}\right\}}_{=1} \cdot f_{\gamma_1}(x)dx \\ &\quad + \int_\lambda^\infty \mathcal{P}\left\{\gamma_2 \le \frac{(ax+b)\lambda}{x-\lambda}\right\} \cdot f_{\gamma_1}(x)dx \\ &= 1 - \int_\lambda^\infty \bar{F}_{\gamma_2}\left(\frac{(ax+b)\lambda}{x-\lambda}\right) \cdot f_{\gamma_1}(x)dx. \end{aligned}$$

Substituting from (10) and simplifying with $u = x - \lambda$, we find

$$\bar{F}_\Lambda(\lambda) = \frac{e^{-\left(\frac{1}{\beta_1}+\frac{a}{\beta_2}\right)\lambda} \lambda^{\alpha_1-1}}{\Gamma(\alpha_1)\beta_1^{\alpha_1}} \sum_{k=0}^{\alpha_2-1} \frac{\left(\frac{\lambda}{\beta_2}\right)^k}{k!} \times \int_0^\infty \left(a + \frac{a\lambda+b}{u}\right)^k \left(1+\frac{u}{\lambda}\right)^{\alpha_1-1} e^{-\left(\frac{u}{\beta_1}+\frac{\lambda(a\lambda+b)}{\beta_2 u}\right)} du. \tag{11}$$

Case $a \ne 0$: Binomial expansion of the terms $(a + (a\lambda + b/u))^k$ and $(1 + (u/\lambda))^{\alpha_1-1}$ reduces (11) to a nested sum of manageable integrals

$$\begin{aligned} \bar{F}_\Lambda(\lambda) &= \frac{e^{-\left(\frac{1}{\beta_1}+\frac{a}{\beta_2}\right)\lambda} \lambda^{\alpha_1-1}}{\Gamma(\alpha_1)\beta_1^{\alpha_1}} \sum_{k=0}^{\alpha_2-1} \frac{\left(\frac{\lambda}{\beta_2}\right)^k}{k!} \\ &\quad \times \sum_{m=0}^{k} \binom{k}{m} a^{k-m}(a\lambda+b)^m \sum_{n=0}^{\alpha_1-1} \binom{\alpha_1-1}{n} \lambda^{-n} \\ &\quad \times \int_0^\infty u^{n-m} e^{-\left(\frac{u}{\beta_1}+\frac{\lambda(a\lambda+b)}{\beta_2 u}\right)} du. \end{aligned}$$

Substituting $\nu \doteq n - m + 1$, $\beta \doteq \lambda(a\lambda + b)/\beta_2$, and $\gamma \doteq 1/\beta_1$ in [12, 3.471.9] yields (2) after some simplification.

Case $a = 0$: Equation (11) reduces to

$$\bar{F}_\Lambda(\lambda) = \frac{e^{-\frac{\lambda}{\beta_1}} \lambda^{\alpha_1-1}}{\Gamma(\alpha_1)\beta_1^{\alpha_1}} \sum_{k=0}^{\alpha_2-1} \frac{\left(\frac{\lambda}{\beta_2}\right)^k}{k!} \times \int_0^\infty \left(\frac{b}{u}\right)^k \left(1 + \frac{u}{\lambda}\right)^{\alpha_1-1} e^{-\left(\frac{u}{\beta_1}+\frac{b\lambda}{\beta_2 u}\right)} du.$$

Binomial expansion of the term $(1 + (u/\lambda))^{\alpha_1-1}$ and the use of [12, 3.471.9] with $\nu \doteq n - k + 1$, $\beta \doteq b\lambda/\beta_2$, and $\gamma \doteq 1/\beta_1$ produce (3). ■

B) Proof: MGF of Λ:

Theorem 3: The mgf can be derived using (6) and the ccdf results in Theorem 1.

Case $a \ne 0$:

$$\begin{aligned} M_\Lambda(s) &= 1 - 2s \sum_{n=0}^{\alpha_1-1} \sum_{k=0}^{\alpha_2-1} \sum_{m=0}^{k} C_1(n, k, m) \\ &\quad \times \int_0^\infty e^{-\left(s+\frac{1}{\beta_1}+\frac{a}{\beta_2}\right)x} \left(a + \frac{b}{x}\right)^{\frac{n+m+1}{2}} \\ &\quad \times x^{\alpha_1+k} K_{n-m+1}\left(2\sqrt{\frac{x(ax+b)}{\beta_1\beta_2}}\right) dx. \end{aligned} \tag{12}$$

Since (12) is not mathematically tractable, its terms are rearranged by separating out a factor $(a + (b/x))^{n+m+2}$ in the integrand and performing binomial expansion on it. It is then arranged into a Laplace transformation that is simplified using the identities in [14, 4.1.6] and [14, 4.17.20].

Case $a \ne 0$, $b = 0$: By setting $b = 0$, (12) can be reduced to a manipulable form. Applying [12, 6.621.3] with $\mu \doteq \alpha_1 + k + 1$, $\alpha \doteq s + (1/\beta_1) + (a/\beta_2)$, $\nu \doteq n - m + 1$, and $\beta \doteq 2\sqrt{a/\beta_1\beta_2}$, one gets (8).

Case $a = 0$: Setting $a = 0$ in (12) and applying [12, 6.621.3] with $\alpha \doteq s + (1/\beta_1)$, $\nu \doteq n - k + 1$, $\mu \doteq 2\alpha_1 + k - n$, and $\beta \doteq 2\sqrt{b/\beta_1\beta_2}$, we get (9). ■

REFERENCES

[1] J. N. Laneman and G. W. Wornell, "Energy-efficient antenna sharing and relaying for wireless networks," in *Proc. IEEE WCNC*, Chicago, IL, Sep. 2000, vol. 1, pp. 7–12.
[2] J. N. Laneman, D. N. C. Tse, and G. W. Wornell, "Cooperative diversity in wireless networks: Efficient protocols and outage behavior," *IEEE Trans. Inf. Theory*, vol. 50, no. 12, pp. 3062–3080, Dec. 2004.
[3] M. Uysal and M. M. Fareed, *Cooperative Diversity Systems for Wireless Communication*. Singapore: World Scientific, 2008, ser. Handbook on Information and Coding Theory.
[4] M. O. Hasna and M. S. Alouini, "A performance study of dual-hop transmissions with fixed gain relays," *IEEE Trans. Wireless Commun.*, vol. 3, no. 6, pp. 1963–1968, Nov. 2004.
[5] G. K. Karagiannidis, "Performance bounds of multihop wireless communications with blind relays over generalized fading channels," *IEEE Trans. Wireless Commun.*, vol. 5, no. 3, pp. 498–503, Mar. 2006.
[6] Y. Song, Z. I. Sarkar, and H. Shin, "Cooperative diversity with blind relays in Nakagami-m fading channels: MRC analysis," in *Proc. IEEE VTC—Spring*, Marina Bay, Singapore, May 2008, pp. 1196–1200.
[7] M. O. Hasna and M. S. Alouini, "Performance analysis of two-hop relayed transmissions over Rayleigh fading channels," in *Proc. IEEE 56th VTC—Fall*, 2002, vol. 4, pp. 1992–1996.
[8] R. H. Y. Louie, Y. Li, and B. Vucetic, "Performance analysis of beamforming in two hop amplify and forward relay networks," in *Proc. IEEE ICC*, Beijing, China, May 2008, pp. 4311–4315.
[9] M. O. Hasna and M. S. Alouini, "End-to-end outage probability of multihop transmission over lognormal shadowed channels," *Arab. J. Sci. Eng.*, vol. 28, no. 2C, pp. 35–44, Dec. 2003.
[10] M. O. Hasna and M. S. Alouini, "Outage probability of multihop transmission over Nakagami fading channels," *IEEE Commun. Lett.*, vol. 7, no. 5, pp. 216–218, May 2003.
[11] M. Abramowitz and I. Stegun, *Handbook of Mathematical Functions*. New York: Dover, 1970.
[12] I. Gradshteyn and I. Ryzhik, *Table of Integrals, Series, and Products*. New York: Academic, 2000.
[13] M. Di Renzo, F. Graziosi, and F. Santucci, "On the performance of CSI-assisted cooperative communications over generalized fading channels," in *Proc. IEEE ICC*, Beijing, China, May 2008, pp. 1001–1007.

[14] A. Erdelyi, W. Magnus, F. Oberhettinger, and F. G. Tricomi, *Tables of Integral Transforms*, vol. 1. New York: McGraw-Hill, 1954, ser. Bateman Manuscript Project.

[15] M. O. Hasna and M. S. Alouini, "End-to-end performance of transmission systems with relays over Rayleigh-fading channels," *IEEE Trans. Wireless Commun.*, vol. 2, no. 6, pp. 1126–1131, Nov. 2003.

[16] T. A. Tsiftsis, G. K. Karagiannidis, P. T. Mathiopoulos, and S. A. Kotsopoulos, "Nonregenerative dual-hop cooperative links with selection diversity," *EURASIP J. Wirel. Commun. Netw.*, vol. 2006, no. 2, pp. 34–34, Apr. 2006.

An Asymptotic Maximum Likelihood for Joint Estimation of Nominal Angles and Angular Spreads of Multiple Spatially Distributed Sources

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***Abstract*—This paper proposes a large-sample approximation of the maximum likelihood (ML) criterion for the joint estimation of nominal directions and angular spreads in the presence of multiple spatially spread sources. The key idea is the concentration on the exact likelihood function by replacing the parametric nuisance estimate, which depends on all unknown parameters at the critical point, by another estimate relying on only the angles of interest, such as nominal angles and angular spreads. Rather than the $(3N_S + 1)$-dimensional optimization required by the exact ML estimator, the proposed large-sample approximation allows $2N_S$-dimensional search, where N_S is the number of sources. To demonstrate the proposed estimator, numerical results are conducted for the illustration of estimation error variance. In the non-asymptotic region, the proposed estimator outperforms previous approaches adopting the $2N_S$-dimensional search.**

***Index Terms*—Direction finding, local scattering, maximum likelihood (ML) estimator.**

I. Introduction

Sensor array processing plays a prominent role in the propagation of plane waves through a medium. The problem of finding the directions impinging on an array antenna or sensor array, namely, direction finding or direction-of-arrival (DoA) estimation, has been of interest for several decades [1]. This is because the direction is a useful parameter for several systems, such as wireless communications, radar, navigation, etc. Several works in the DoA estimation are based on the maximum likelihood (ML) criterion because this method can provide optimal performance in terms of asymptotic unbiasedness and statistical efficiency [1]–[3]. At the critical point of the likelihood function, an optimization search seems computationally intensive. To reduce the optimization task, the weighted least square (WLS) criterion is considered instead of the likelihood function [4]. Such a replacement idea stems from the fact that both ML and WLS methods yield the same asymptotic performance (see, e.g., [5, pp. 127–128] and [6, pp. 566–567]). Since the solution of the WLS criterion is often separable, the nuisance parameter elimination can reduce the effort of the optimization.

In the presence of local scattering around the vicinity of a source, each incoming signal consists of three individual parameters, such as nominal direction, angular spread, and the power observed by the sensor array, which render $(3N_S + 1)$ parameters, including the noise variance in the model, where N_S is the number of sources. Since the exact likelihood function cannot explicitly be derived in a concentrated form [7], the large dimension of the optimization search possibly makes the implementation infeasible. Based on this restriction, an autoregressive model, which amounts to $2N_S$ dimensions [8] (see also [9] and [10] for wideband signals and [11]–[14] for single source), is considered to reduce the computation. Although the dimension of the parameter space in [8] can be reduced, there exists the mismodeled parameterization of spatial fading. The exponential model is therein preassigned to model the underlying coherent loss function, which cannot cope with most angle deviation models whose spatial fading correlation is, in some environments, not exponential as in [8] and [15]. Therefore, the mismodeled estimation results in a performance that deviates from the inherent accuracy limitation. Regarding some relevant approaches, the techniques formulating the least square (LS) fit of array covariance matrices are proposed to account for a multiple-source case [16], [17]. However, the model is based on wavefront perturbation, which is not a function of the angular spread. Furthermore, the error variance of the LS estimator is worse than that of the WLS estimator and cannot asymptotically attain the Cramér–Rao bound (CRB) [9], [15].

Herein, the source-localization problem is the estimation of the nominal angles and their associating angular spreads, whereas the nuisance parameters are the received signal powers (channel gains and signal powers) and the noise variance. The angular spreads are included in the estimation because they provide an important impact on the link quality in wireless communications [18]. By replacing the exact likelihood in [7], which is computationally intensive, by its asymptotically consistent estimate, the localization of multiple spatially distributed sources can be conducted. It can be observed that the nuisance parameter depends on all parameters in the considered model. Since it is difficult to compute the exact solution at this point, an approximate value of the nuisance parameter is preferred in the aspect of the computational complexity. As a consequence, it is reasonable for a simple calculation at the critical point to replace the true covariance matrix by an approximate quantity. Furthermore, to hold the statistical efficiency, the nominal angle estimate from such an array covariance approximation should be equivalent to that given by the exact ML estimation. Due to both concerns, i.e., the computational complexity and the statistical efficiency, the motivation of large-sample approximation thus stems from the consistency available from a sample covariance matrix. The estimate in this way would probably be equivalent in large sample to that provided by the exact likelihood.

Some notations are involved, as follows: $\imath = \sqrt{-1}$ is the unit imaginary number. $\triangleq$ denotes the equality by definition. $(\cdot)^{\mathsf{T}}$ is the transpose. $(\cdot)^{\mathsf{H}}$ is the Hermitian transpose. $\delta_{\cdot,\cdot}$ is the Kronecker delta function. $\mathcal{E}\langle\cdot\rangle$ denotes the statistical expectation operator. $f(\delta_\phi|0;\sigma_\phi^2)$ denotes the conditional probability density function (pdf) for the random deviation δ_ϕ given the *a priori* knowledge of zero mean and the angular spread σ_ϕ. $\sqcap_x[a_1, a_2]$ is a unit function defined by $\sqcap_x[a_1, a_2] = 0;\ x < a_1, 1;\ a_1 \le x \le a_2, 0;\ x > a_2$. $\mathcal{F}(\cdot)$ is the

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