Numerical computation of the lognormal sum distribution

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***Abstract*—This paper develops highly accurate numerical techniques for evaluating the mgf/ chf of a single lognormal variable and for computing the lognormal sum cdf. Complex integration techniques based on the steepest-descent integration are thus developed for evaluating the lognormal mgf/ chf. The saddlepoint of the integrand is explicitly expressed using the Lambert-W function. The optimal steepest-descent contour passing through the saddlepoint is then derived. Even a simple mid-point-rule-based integration technique can be used along this contour to evaluate the mgf/ chf at extremely high precision.**

A highly efficient, extremely accurate numerical method is then developed for evaluating the cdf of sum of independent lognormal variables. The cdf is expanded as an alternating series, on which the Epsilon algorithm for convergence acceleration is applied. This reduces the computational load significantly.

I. Introduction

The lognormal distribution is widely used in various branches of science and engineering [1]–[3]. It is used in wireless communications, to model large-scale signal fading and co-channel interference for cellular mobile networks, ultra-wide band systems and others [4]–[10]. Closed-form analytical expressions for the lognormal characteristic function (chf) and the moment generating function (mgf); as well as the cumulative distribution function (cdf) of a sum of independent lognormal random variables (RVs) remain elusive. Wireless researchers have attacked this famous problem for several decades [11], [12].

The literature on sum of lognormals (SLN) can be broadly classified into two categories. The main feature of the first category is the approximation of the SLN distribution by yet another lognormal distribution, whose parameters are determined by moment/ cumulant matching techniques. Fenton [11] gives approximations based on the first and second moments, as well as the third and fourth-order moments; while Schleher [12] gives a Gram-Charlier series based on cumulant matching. Schwarz-Yeh's [13], Wilkinson's [13] and Farley's [13] approximations are also based on moment matching. Negative moment matching [14] and generalized moment matching [6], [7], [14] techniques are also found. Such methods have also been extended for correlated lognormal sums [15], [16].

The lognormal cdf is a straight line on the lognormal paper [11] whereas the actual SLN cdf is a convex curve. Therefore, aforementioned SLN approximations are not accurate over the entire range of the argument of the cdf [17], [18]. A second category of approximations has thus been devised to overcome this issue. They seek to represent the SLN distribution by distributions other than the lognormal distribution. Power lognormal and generalized lognormal distributions [19]–[21], Log-shifted Gamma distribution [22], [23], Type IV Pearson distribution [24], and least squares estimates [25], [26] are some of them.

This paper focuses on the development of efficient, highly accurate computational methods for lognormal mgf/ chf and the SLN cdf. The difficulty of providing highly accurate numerical evaluations of SLN probabilities has been understood for a long time and is the main motivation for development of the approximations mentioned above. Direct numerical methods, as opposed to approximations, have not been investigated much, except by Beaulieu and Xie [27]. They experimented with various direct integration techniques such as Simpson's rule, the trapezoidal rule, and Hermite polynomials and finally settled on the modified Clenshaw-Curtis rule.

The main computational difficulties of the lognormal chf are highly oscillatory nature and the slow decay rate of the integrand. To overcome these issues, we present a highly accurate numerical algorithm based on complex contour integration techniques [28]. Since the mgf/ chf can be viewed as a contour integral in the complex plane, Cauchy's theorem suggests that the contour can be deformed without affecting the integral. Exploiting this idea, Gubner [29] suggested integrating along a contour parallel to the X-axis. But his contour is not the best for numerical evaluation, because the best contour needs to have the two following properties: (1) the integrand along the contour does not oscillate at all; (2) the integrand along the contour has the fastest rate of decay and hence facilitates the use of simple numerical integration methods with high accuracy. These properties are satisfied only by the steepest-descent contour, a constant-phase contour passing through the saddle point of the integrand. We show that the saddle point can be explicitly expressed by the Lambert function $\mathcal{W}(x)$ [30] for the case of the lognormal mgf/ chf; derive this steepest-descent contour; and use a simple integration rule (namely the mid-point rule) to compute the mgf/ chf with extremely high precision.

Given the mgf /chf, numerical integration is required once again to compute the SLN cdf. We develop a highly efficient and extremely accurate method for this purpose based on Longman [31]. He has noted that a highly oscillating infinite integral can be broken into a series of finite integrals, whose

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ranges of integration fall between the consecutive zeros of the integrand. Consequently, the integral is expressed as an alternating series whose individual terms are amenable to simple numerical methods. Using this approach, we derive the SLN cdf as an alternating series.

The Epsilon algorithm, which is more powerful than the Euler's transform, is then used for convergence acceleration [32], [33]. Convergence acceleration in this case yields remarkable computational efficiencies.

This paper is organized as follows. Section II presents the background needed to appreciate this work. Computation of the mgf is explored in section III. LNS problem is addressed next, in section IV. Numerical results follow in section V, highlighting the advantages proposed approach brings in.

Notation: The imaginary unit $\jmath = \sqrt{-1}$. Given a complex number z, the complex conjugate is z^*; the argument is $\arg(z)$; the magnitude is $|z|$; the real part is $\Re(z)$ and the imaginary part is $\Im(z)$. $\mathbb{E}[\cdot]$ denotes the statistical expectation. $f'(z)$ is the derivative of a function $f(z)$ with respect to z. $\mathbb{R}$ and $\mathbb{C}$ are respectively the sets of real and complex numbers.

II. The basics

If $X \sim N(\mu, \sigma^2)$ is a Gaussian RV with mean μ and variance σ^2, $Y = \mathrm{e}^X$ is a *lognormal* RV. Often, in wireless communications, X represents a signal power level, and, hence, the dB scale is used for the mean and variance. In this case, $\mu = \xi\mu_{\text{dB}}$, and $\sigma^2 = \xi^2\sigma^2_{\text{dB}}$, where $\xi = \frac{\ln(10)}{10} = 0.2303$.

The mgf of the lognormal distribution is given by

$$M(s) = \mathbb{E}\left[\mathrm{e}^{-sY}\right] = \int_{-\infty}^{\infty} \mathrm{e}^{-s\mathrm{e}^t}\, \frac{\mathrm{e}^{-\frac{1}{2}\left(\frac{t}{\sigma}\right)^2}}{\sqrt{2\pi\sigma^2}}\, \mathrm{d}t,\ \Re(s) \geq 0. \quad (1)$$

The mgf is analytic in the right half of the complex $s = \alpha + \jmath\omega$ plane, i.e. for all $\alpha \geq 0$. The special case $\alpha = 0$ yields the chf. Without loss of generality, only the case $\omega \leq 0$ is considered because $M(s^*) = M^*(s)$. Moreover, there is no loss of generality in assuming that X has zero mean, because $\mathbb{E}\left[\mathrm{e}^{-s\mathrm{e}^{X+\mu}}\right] = M(s\mathrm{e}^{\mu})$. Although the mgf is customarily defined as $\mathbb{E}\left[\mathrm{e}^{sY}\right]$, the mgf is defined here as the double-sided Laplace transform of the probability density function, using the negative exponent. This definition is preferred as it allows the LNS cdf to be computed through numerical Laplace inversion of corresponding mgf. The lognormal chf is given by $\phi(\omega) = \mathbb{E}\left[\mathrm{e}^{\jmath\omega Y}\right] = M(-\jmath\omega)$.

Consider a complex function $f(z) = u(x,y) + \jmath v(x,y)$ of the complex variable $z = x + \jmath y$, where $x, y, u(x,y), v(x,y) \in \mathbb{R}$. If $f(z)$ is differentiable, we have

$$f'(z) = \frac{\partial u(x,y)}{\partial x} + \jmath\frac{\partial v(x,y)}{\partial x} = \frac{\partial v(x,y)}{\partial y} - \jmath\frac{\partial u(x,y)}{\partial y}. \quad (2)$$

As a result, $f(z)$ is **analytic** whenever the Cauchy-Riemann conditions

$$\frac{\partial u(x,y)}{\partial x} = \frac{\partial v(x,y)}{\partial y}, \quad \frac{\partial v(x,y)}{\partial x} = -\frac{\partial u(x,y)}{\partial y} \quad (3)$$

are satisfied. These conditions imply that the gradient of $u(x,y)$ and $v(x,y)$ are orthogonal; thus, the contour lines $u(x,y) = C_u$ are orthogonal to the contour lines $v(x,y) = C_v$ for all constants $C_u, C_v \in \Re$. If a function $f(z)$ is analytic at $z_0 = x_0 + \jmath y_0 \in \mathbb{C}$ and $f'(z_0) = 0$, then z_0 is a called a saddle point of $f(z)$. The point is named such, because the contours $u(x,y)$ and $v(x,y)$ have valleys and hills at this point causing the surface formed by the curves to bear in the neighborhood of z_0 the shape of a saddle. Any contour (path) that passes through this point and that satisfies $v(x,y) = v(x_0, y_0)$ is the **steepest descent** contour for $u(x,y)$.

We consider now the integrals of the form

$$I = \int_{\mathcal{C}} \mathrm{e}^{-\lambda f(z)}\, \mathrm{d}z, \quad (4)$$

where $\mathcal{C}$ is some contour lying completely in the region of analyticity of $f(z)$. Cauchy's theorem shows that deforming the contour does not change the value of the integral, provided that the contour stays in the region. Since $u(x,y)$ determines the magnitude of the integrand, and $v(x,y)$ its phase, the oscillations can be totally eliminated by choosing a contour along which $v(x,y)$ remains constant. The contributions along this contour are all in-phase (i.e., they add coherently). Moreover, this contour can be chosen to pass through the saddle point z_0 so that $u(x,y)$, and hence, the magnitude of the integrand decays rapidly. Steepest-descent contours are often used to derive asymptotic expansions as $\lambda \to \infty$ [34]. The idea of using steepest-descent contours for numerical evaluation, as opposed to asymptotic expansions, is not new. References [28], [35], [36] use this idea for numerical computations. However, the use of this idea to compute the lognormal mgf and chf appears to be new.

In sum, the steepest-descent contour is given by the constant-phase condition $\Im(f(z)) = v(x,y) = \Im(f(z_0))$; i.e.,

$$\mathcal{L} = \{x + \jmath y|,\ v(x,y) = v(x_0, y_0)\}. \quad (5)$$

Then the integral (4) becomes

$$I = \mathrm{e}^{-\jmath\lambda v(x_0,y_0)} \int_{\mathcal{L}} \mathrm{e}^{-\lambda u(x,y)}\, \mathrm{d}z. \quad (6)$$

Note the two main advantages:

- The oscillations in the integrand, which would have been caused by the $\jmath v(x,y)$ term in the exponent, are completely removed.
- As the integration moves away from the saddle point z_0 along the contour $\mathcal{L}$, the integrand has the fastest rate of decay on $\mathcal{L}$. It enormously facilitates the use of simple numerical integration methods with high accuracy.

III. Computation of the mgf

A. Saddle point

We can readily show that the integrand of (1) satisfies the Cauchy-Riemann conditions (3) and is analytic; thus, the integration path, which was originally the real axis in (1), can be deformed to a more general contour on the complex plane, and the mgf can be written as

$$M(s) = c\int_{\mathcal{L}} \mathrm{e}^{-\lambda\vartheta(z)}\, \mathrm{d}z, \quad (7)$$

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where $c = \frac{1}{\sqrt{2\pi\sigma^2}}$, $\lambda = 1/\sigma^2$, $\zeta = s\sigma^2$, $\vartheta(z) = \zeta e^z + \frac{z^2}{2}$, and $\mathcal{L}$ is a constant-phase contour to be determined later. Note that (7) reverts to (1), if $\mathcal{L}$ is taken to be the real axis. As mentioned in Section II the best contour for numerical purposes is the steepest-descent contour passing through the saddle point. The saddle point z_0 satisfies $\vartheta'(z_0) = \zeta e^{z_0} + z_0 = 0$. Conveniently, the saddle point is explicitly given by

$$z_0 = -\mathcal{W}(\zeta). \tag{8}$$

B. Steepest descent contour

Numerical evaluation of the mgf requires $\mathcal{L}$, as well as the derivative $\frac{dy}{dx}$ along $\mathcal{L}$ to be determined. Let $\vartheta(z_0) = C_0 + \jmath C_1$. The derivative $\frac{d}{dz}\vartheta(z) = \vartheta'(z) = \zeta e^z + z$ is given by

$$\begin{aligned}\vartheta'(z) &= \zeta e^{x+\jmath y} + (x + \jmath y)\\ &= |\zeta| e^x \cos(y+\theta) + x + \jmath\left\{|\zeta| e^x \sin(y+\theta) + y\right\},\end{aligned} \tag{9}$$

where $\theta = \arg\zeta$. By setting (9) to zero, the relationship between C_0 and C_1 can be determined.

To describe the steepest-descent contour, we use $z = x + \jmath y$ and $\theta = \arg(s) = \tan^{-1}\left(\frac{\omega}{\alpha}\right)$ to expand the exponent as

$$\begin{aligned}\vartheta(z) &= \Re(\vartheta(z)) + \jmath\Im(\vartheta(z)) = (\alpha + \jmath\omega)\sigma^2 e^{x+\jmath y} + \frac{(x+\jmath y)^2}{2}\\ &= \left\{|\zeta| e^x \cos(y+\theta) + \left(\frac{x^2 - y^2}{2}\right)\right\}\\ &\qquad + \jmath\left\{|\zeta| e^x \sin(y+\theta) + xy\right\}.\end{aligned} \tag{10}$$

By utilizing the constant phase condition (5) in conjunction with (10), we obtain the steepest-descent contour as

$$\mathcal{L} = \{x + \jmath y \mid |\zeta| e^x \sin(y+\theta) + xy = C_1\}. \tag{11}$$

Since (x_0, y_0) satisfies (11), this contour $\mathcal{L}$ passes through the saddle point. It can be shown that $\mathcal{L}$ lies in the (x, y) range: $-\infty < x < \infty$ and $0 \le y \le -\theta$. Moreover, the derivative $\frac{dy}{dx}$ along the contour $\mathcal{L}$ can be computed from (11) to be

$$\frac{dy}{dx} = -\frac{|\zeta| e^x \sin(y+\theta) + y}{|\zeta| e^x \cos(y+\theta) + x}. \tag{12}$$

Although (11) and (12) supply what is needed for the problem at hand, two more points must be clarified here.

First, we note that the contour (11) is not given in closed-form solution: with y as a function of x. Thus, for a given x, y must be determined numerically. This determination is not particularly difficult because we know that $y \in [0, -\theta]$ for all x. Consequently, a simple bisection search can be used to determine y for each x. Since the derivative is explicitly known (12), the Newton-Raphson method, which has a higher quadratic convergence rate [37] can be used alternatively. We adopted a hybrid strategy that uses Newton-Raphson method only when $|x - x_0|$ is large enough for the derivative to be stable. It reverts to bisection method at the vicinity of x_0. We found the strategy to perform 100 times faster than the simple bisection method.

The second difficulty is that both the numerator and denominator of (12) approach zero at the saddle point (x_0, y_0) (This is the cause of the instability which prompted the hybrid method to be adopted.). This finding is not surprising because (9) shows that the derivative is the ratio between the real and imaginary parts of $\vartheta'(z)$. Thus, the derivative at the saddle point is given by a limiting process, which can be evaluated by using *L'Hospital's rule*. It can be shown that

$$\left.\frac{dy}{dx}\right|_{\substack{x=x_0\\y=y_0}} = \sqrt{\frac{(1-x_0)^2}{y_0^2} + 1} - \left(\frac{1-x_0}{y_0}\right). \tag{13}$$

C. Numerical evaluation

Now we are in a position to evaluate the mgf via the complex integral (6) and the contour (11). When integrating over the contour $\mathcal{L}$, either x or y can be taken as the independent variable. However, the upper limit of y is dependent on the argument of $M(s)$, whereas the limits of x are not. Moreover, when s is real, the limit of y is zero; i.e., the optimal contour is simply the real axis. For these reasons, we use x as the independent variable. Since $dz = dx + \jmath\, dy = (1 + \jmath\frac{dy}{dx})\, dx$, the mgf (11) is given by

$$\begin{aligned}M(s) = c e^{-\jmath\lambda C_1} \int_{-\infty}^{\infty} & e^{-\lambda\left(|\zeta| e^x \cos(y+\theta) + \frac{x^2 - y^2}{2}\right)}\\ & \left(1 - \jmath\frac{|\zeta| e^x \sin(y+\theta) + y}{|\zeta| e^x \cos(y+\theta) + x}\right) dx,\end{aligned} \tag{14}$$

where y is a function of x governed by (11).

Before we evaluate (14), note that the main contribution to the integral occurs near the saddle point $x = x_0$. Thus, we substitute $\sqrt{\frac{\lambda}{2}}(x - x_0) = t$ and obtain

$$M(s) = \frac{e^{-\jmath\lambda C_1}}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-t^2} q(t)\, dt, \tag{15}$$

where

$$\begin{aligned}q(t) = e^{-\lambda\left(|\zeta| e^{\tilde{\sigma}t + x_0} \cos(y+\theta) + \frac{x_0^2 + 2\tilde{\sigma}x_0 t - y^2}{2}\right)}&\\ \left(1 - \jmath\frac{|\zeta| e^{\tilde{\sigma}t + x_0} \sin(y+\theta) + y}{|\zeta| e^{\tilde{\sigma}t + x_0} \cos(y+\theta) + \tilde{\sigma}t + x_0}\right)&,\end{aligned} \tag{16}$$

$\tilde{\sigma} = \sqrt{2}\sigma$, and y is given by (11) with $x = \tilde{\sigma}t + x_0$. The mgf can now be evaluated by using a simple mid-point rule as follows:

$$M(s) = \frac{h e^{-\jmath\lambda C_1}}{\sqrt{\pi}} \sum_{k=-\infty}^{\infty} e^{-k^2 h^2} q(kh + h/2) + E_h, \tag{17}$$

where the error term E_h rapidly decays with decreasing h. This computation requires three simple steps:

1) Calculate the saddle point z_0 by using (8).
2) Solve $y = y_k$ in (11) for each $x = x_k = \tilde{\sigma}kh + x_0$.
3) Sum (17) to the required precision.

This computational method has extremely high accuracy. For example, the LNS cdf can be calculated for 15-digit accuracy (i.e., an error less than 10^{-15}) even in a finite precision computational environment like MATLAB. See details in Section V.C.

Nevertheless, the above approach requires the numerical computation of the steepest-descent contour $\mathcal{L}$ (11). Numerical evaluation can be greatly improved if $\mathcal{L}$ can be replaced with a contour expressible in closed-form (i.e., as points $(x, y(x))$, $y(x)$ in closed-form). Closed-from 'atan' and 'tanh' contours that closely approximate $\mathcal{L}$ and achieve similar precision are investigated in [38].

IV. Lognormal sum cdf

This section develops a new method to compute the SLN cdf. Conventionally, the mgf or chf is integrated over a vertical contour (i.e., the Bromwich contour), which passes the real axis at $x = \alpha_0$, where $0 \leq \alpha_0 < \infty$. Accuracy may be improved by a simple search to find the best value of α_0. Various numerical integration techniques can be used to integrate along the Bromwich contour. Nevertheless, the integrand can be oscillatory and decay rather slowly (particularly if the number of summands is small), and the accuracy becomes poor as a result. Proposed method overcomes these problems.

A. Conventional approaches for the LNS cdf

It is required to compute the distribution of the lognormal sum

$$Y = \sum_{k=1}^{K} \mathrm{e}^{X_k}, \tag{18}$$

where $X_k \sim N(\mu, \sigma^2)$ are independent and identically distributed (i.i.d.). The non-identically distributed, but independent, case can also be treated, but is omitted for brevity. The cdf is given by

$$F(y) = \Pr[Y \leq y]. \tag{19}$$

Suppose the mgf of Y is $M_Y(s) = [M(s)]^K$, where $M(s)$ is the mgf of X_k, $k \in \{1, \ldots, n\}$. It is easy to show that

$$F(y) = \frac{1}{2\jmath\pi} \int_{\alpha_0 - \jmath\infty}^{\alpha_0 + \jmath\infty} \frac{M_Y(s)\mathrm{e}^{sy}}{s}\,\mathrm{d}s = \frac{1}{2\jmath\pi} \int_{\alpha_0 - \jmath\infty}^{\alpha_0 + \jmath\infty} \mathrm{e}^{\varsigma(s)}\,\mathrm{d}s, \tag{20}$$

where $\varsigma(s) = sy + \log(M_Y(s)) - \log(s)$, and $\alpha_0 > 0$ is a more or less free parameter in the range $0 \leq \alpha_0 < \infty$. The cdf is now given by the integral

$$F(y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{M_Y(\alpha_0 + \jmath\omega)\mathrm{e}^{(\alpha_0 + \jmath\omega)y}}{\alpha_0 + \jmath\omega}\,\mathrm{d}\omega, \tag{21}$$

which can be evaluated by using, for example, the Gauss-Chebysev quadrature rule. Another method is to choose $\alpha_0 = 0$ and use traditional numerical techniques (e.g., the trapezoidal rule, Simpson's rule, Clenshaw-Curtis rule) [27].

B. New cdf algorithm for a LNS

The conventional numerical solution of (21) is not accurate as y gets larger. In this case, α_0, which is a function of y, tends to zero, and the integration is then performed along a vertical line that approaches the imaginary axis. Thus, it makes sense to set $\alpha_0 = 0$. Deforming the contour is not possible because there is no analytic continuation of the lognormal mgf to the left half-plane.

Therefore, we develop a different method than that using the simple numerical integration of (21). Longman [31] developed a powerful technique for the numerical evaluation of integrals with an oscillating integrand. His key idea is to break down the infinite integral into a series of finite integrals, which are integrals over the intervals between consecutive zeros of the integrand.

Consider the integral in (21). We set $\alpha_0 = 0$ and note that $F(y) = 0$ for $y < 0$. Therefore, (21) can be simplified as

$$F(y) = \frac{2}{\pi} \int_0^{\infty} \frac{\Re\left(\phi_Y(\omega)\right)\sin(\omega y)}{\omega}\,\mathrm{d}\omega,$$

where $\phi_Y(\omega)$ is the chf of Y. Substitute $t = \omega y$, and the cdf is then given by

$$F(y) = \frac{2}{\pi} \sum_{k=0}^{\infty} \int_{k\pi}^{(k+1)\pi} \frac{\Re\left(\phi_Y\left(t/y\right)\right)}{t} \sin t\,\mathrm{d}t. \tag{22}$$

Considering the periodic nulls induced by the factor $\sin t$, this is further simplified as

$$F(y) = \sum_{k=0}^{\infty} a_k (-1)^k, \tag{23}$$

where

$$a_k = \frac{2}{\pi} \int_0^{\pi} \frac{\Re\left(\phi_Y\left((k\pi + t)/y\right)\right)}{t + k\pi} \sin t\,\mathrm{d}t. \tag{24}$$

The truncation error (e.g., at the n-th term) for the series (23) is upper bounded by the absolute value $|a_{n+1}|$ of the first omitted term, making it straightforward to truncate the series to meet any arbitrary precision goal.

The numerical evaluation of the k-th term a_k is quite easy because the integrand of (24) is smooth and well-behaved. Even the simple adaptive quadrature techniques available in, say, MATLAB, can reach an accuracy of 15 significant digits. However, the precision can get severely affected if a large number of a_k's needs to be added for (23). Fortunately, this issue can be avoided through convergence acceleration algorithms, whose purpose is to transform a given series to another series that converges with a fewer number of terms to the same limit as that of the original series.

Consider the partial sums $s_n = \sum_{k=0}^{n} (-1)^k a_k$ for $n \in \{0, 1, \ldots, N-1\}$. The objective is to estimate the limit s_∞ by using as few partial sums as possible. The Epsilon algorithm developed by Shanks [32] and Wynn [33] is a powerful convergence acceleration technique suitable for this purpose. The algorithm generates an array ϵ with $\epsilon_0(s_n) = s_n$; $\epsilon_{-1}(s_n) = 0$; and

$$\epsilon_{r+1}(s_n) = \epsilon_{r-1}(s_{n-1}) + \frac{1}{\epsilon_r(s_{n+1}) - \epsilon_r(s_n)}, \tag{25}$$

for $r \in \{1, 2, \ldots\}$. The accelerated approximations of $s_\infty = F(y)$ are obtained from the columns of the epsilon array.

In sum, the computation of $F(y)$ is given by the following algorithm, which proves to be extremely accurate (Section V).

(1) eps = precision level
(2) begin $s_0 = a_0;\ k = 1;$
(3) $\quad a_k;\ s_k = s_{k-1} + (-1)^k a_k$
(4) $\quad$ if $|a_k| \leq$ eps then stop
(5) $\quad\quad$ else compute ϵ array using (25)
(6) $\quad\quad$ if ϵ is convergent then stop
(7) $\quad\quad\quad$ else $k = k + 1$; goto 3

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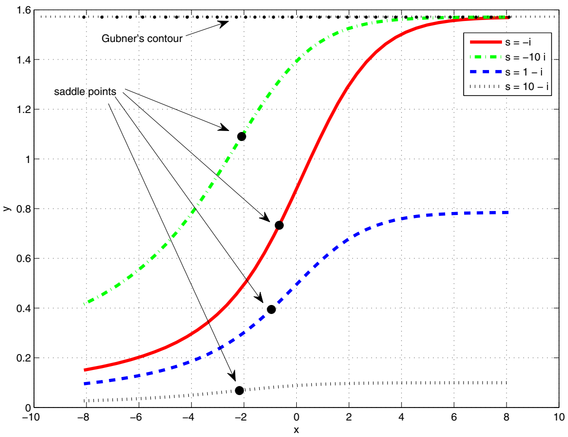

Fig. 1. Constant-phase steepest-decent contours for evaluating the lognormal mgf, for mean $\mu = 0\,\mathrm{dB}$ and standard deviation $\sigma = 6\,\mathrm{dB}$.

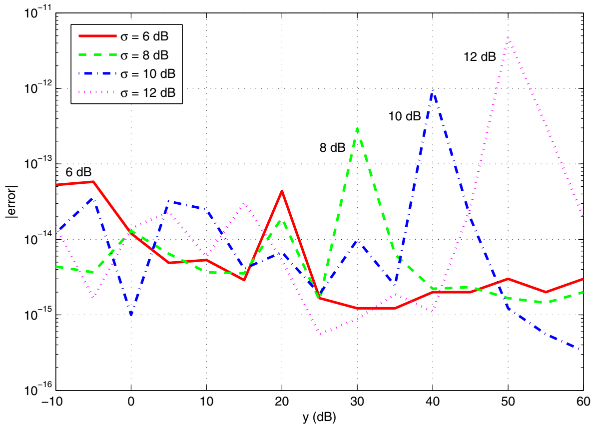

Fig. 2. Absolute Error between (23) and (26) for a single lognormal RV with mean 0 dB and standard deviation σ.

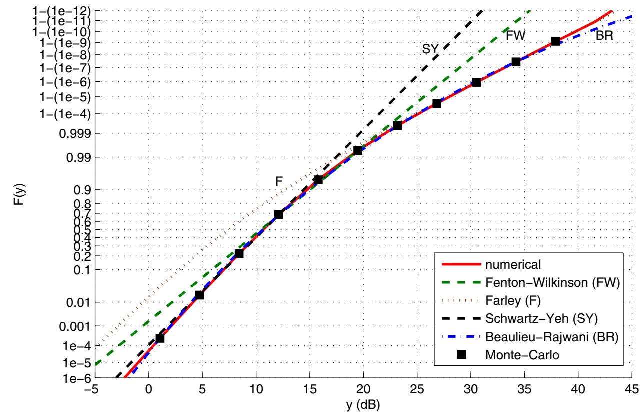

Fig. 3. The cdf of the sum of 6 i.i.d. lognormal RVs with mean 0 dB. and standard deviation 6 dB

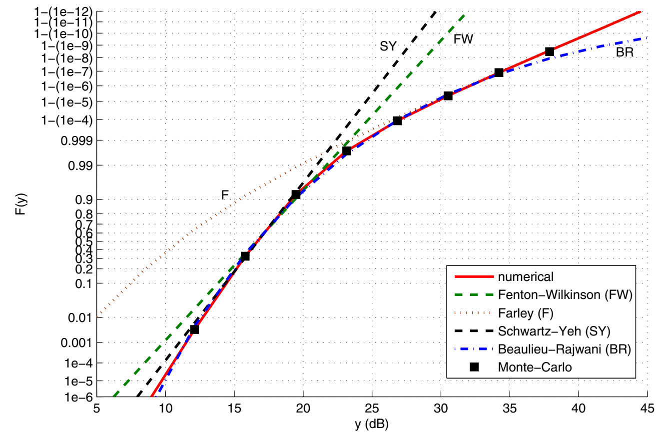

Fig. 4. The cdf of the sum of 20 i.i.d. lognormal RVs with mean 0 dB. and standard deviation 6 dB

V. Numerical results

A. Computation of the mgf

In Fig. 1, the optimal steepest-decent contour given in (11) is plotted for a lognormal RV having mean 0 dB and standard deviation 6 dB. We use a hybrid algorithm that switches between the simple bisection algorithm and the Newton-Raphson method [37] to compute the contours (x, y) in Fig. 1. Following trends can be noted.

- The optimal contour for evaluating $M(s)$ approaches the real axis when $\Re(s)$ is much larger than $\Im(s)$.
- On the other hand, when $\Im(s)$ is much larger, the optimal contour has the shape of an inverse tangent function.
- The optimal contour always lies in the horizontal strip $0 \leq y \leq |\arg(s)|$, where $|\arg(s)| = \frac{\pi}{2}$ for the chf.

Interestingly, Gubner [29] suggested the contour $\mathcal{L} = \{z = x + \jmath\pi/2, -\infty < x < \infty\}$ for numerical evaluation of the chf. A part of the the optimal contour approaches Gubner's contour as $\Re[s] \to 0$ and $x \to \infty$. (Numerical results for mgf/chf are found in [38]). Nevertheless, the level of accuracy of the proposed scheme for evaluating the mgf should be evident from the following results given for the SLN problem.

B. Verification of accuracy

Since the SLN cdf has no closed-form, we need an indirect way to estimate the overall accuracy. Fortunately, for a single lognormal RV e^Z, where Z is $N(\mu, \sigma^2)$, the cdf is given by

$$F_Z(y) = P[\mathrm{e}^Z < y] = \Phi\left(\frac{\log(y) - \mu}{\sigma}\right), \tag{26}$$

where $\Phi(x)$ is the cdf of the standard normal distribution (i.e., $N(0,1)$). We calculate the cdf of a single lognormal RV by using (26) and the proposed algorithm (23). The error is plotted in Fig. 2 for σ from 6 to 12 dB. The absolute error is always less than 10^{-11} and is less than 10^{-13} in most cases. This level of accuracy is, in fact, fairly close to the best accuracy that Matlab can provide. Indeed, if our algorithm is implemented in an arbitrary-precision environment, even higher precision levels can be expected. Note that this case is the most difficult test for any SLN cdf calculation algorithm because the chf $\phi(\omega)$ of a single lognormal RV decays slowly with ω. When the number of summands K increases, the chf decays much more rapidly, and the cdf calculation becomes easier.

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C. Lognormal sum cdf

Figures 3 and 4 present the LNS cdf results obtained respectively for 6 and 20 i.i.d. RVs having mean 0 dB and standard deviation 6 dB. Corresponding results provided by Fenton-Wilkinson, Farley, Schwartz-Yeh and Beaulieu-Rajwani approximations; as well as Monte-Carlo simulation (averaging over 10^{10} sample points) are provided for comparison. Note that our computed results match the simulation results extremely well. Only the BR method comes even close. Without acceleration, the number of terms needed for (23) ranges from 2 to 25000000. It reduces below 25 with convergence acceleration in use. Detailed results on computational saving are found in [38]. It also analyzes the cases with non-identical lognormal variables and those with different standard deviations.

VI. Conclusion

This paper developed an efficient, highly-accurate numerical method for evaluating the lognormal mgf/ chf. The computational challenges posed by the highly oscillatory nature and slow decay rate of the integrand are overcome by evaluating the integrand over a constant-phase, steepest-descent complex contour that we derived. This approach allows even a simple mid-point-rule-based integration technique to yield extremely high precision. We also developed, a highly efficient and extremely accurate numerical method for the SLN cdf, deriving it as an alternating series and using the Epsilon algorithm for convergence acceleration.

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