A New Twist on the Generalized Marcum Q-Function $Q_M(a, b)$ with Fractional-Order M and Its Applications

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ABSTRACT — *A new exponential-type integral for the generalized M-th order Marcum Q-function $Q_M(\alpha, \beta)$ is obtained when M is not necessarily an integer. This new representation includes a classical formula due to Helstrom for the special case of positive integer order M and an additional integral correction term that vanishes when M assumes an integer value. The new form has both computational utility (numerous existing computational algorithms for $Q_M(\alpha, \beta)$ are limited to integer M) and analytical utility (e.g., performance evaluation of selection diversity receiver in correlated Nakagami-m fading with arbitrary fading severity index, unified analysis of binary and quaternary modulations over generalized fading channels, and development of a Markovian threshold model for block errors in correlated Nakagami-m fading channels). Tight upper and lower bounds for $Q_M(\alpha, \beta)$ that holds for any arbitrary real order $M \geq 0.5$ are also derived.*

I. MOTIVATION

The canonical representation of the M-th order generalized Marcum Q-function is given by [1]

$$Q_M(\alpha, \beta) = \int_{\beta}^{\infty} x\left(\frac{x}{\alpha}\right)^{M-1} e^{-(x^2+\alpha^2)/2} I_{M-1}(\alpha x)dx, \qquad (1)$$

where α and β assume non-negative real values, and $I_M(.)$ is the M-th order modified Bessel function of the first kind. This quantity relates to the complementary CDF of a noncentral chi-square random variable (also known as generalized Rice distribution). If Y is the sum of squares of N statistically independent Gaussian random variates (RVs) with means m_i and variance σ^2, then the CDF is given by

$$F_Y(x) = 1 - Q_{N/2}\left(\frac{s}{\sigma}, \frac{\sqrt{x}}{\sigma}\right), \qquad (2)$$

where $s = \sqrt{\sum m_i^2}$ denotes the noncentrality parameter. Since the above CDF (2) corresponds to a quadratic summation of $N/2$ independent sampled matched filter receiver outputs for nonfading signal in Gaussian noise, it arises in numerous applications such as radar detection and performance evaluation of partially coherent, differentially coherent and noncoherent communications [2]-[4]. Thus reliable and efficient algorithms for computing $Q_M(\alpha, \beta)$ have been an active research subject for a considerable time (see [5]-[9] and references therein). However, all of the above algorithms (with the exception of our recent study [9]) are restricted to even values of N in (2) (or equivalently, the order M in (1) is a natural number).

Moreover, in the performance evaluation of partially coherent, differentially coherent and non-coherent communication systems, the statistical average of $Q_M(a, bx)$, $Q_M(ax, b)$ or $Q_M(ax, bx)$ over the probability density function (PDF) of fading signal amplitude x, where a and b are modulation specific constants, is usually needed. Several integrals of the first two types have been studied by Nuttall [10]-[11]. The third form can be expediently handled by representing Q_M-function in a desirable exponential form in [2, Appendix C] analogy with Craig-type representation for the Gaussian Q-function [12]. Simon [13] also derives an equivalent sine integral representation when M is a positive integer. Incidentally, Proakis [3, pp. 885] provides a complex contour integral representation for $Q_M(\alpha, \beta)$. In [14], we show that both Helstrom's and Simon's representations are special cases of this contour integral. As readers can verify, the entire development (i.e., expanding the Q_M-function as an infinite series of Bessel functions, replacing these by their trigonometric integral representations and summing up the resulting series) in [2, Appendix C] and [13] presumes that M is a positive integer (because $I_v(x) \neq I_{-v}(x)$ if v is not an integer).

Nevertheless, $Q_M(\alpha, \beta)$ with real order M arises in a few practical important cases: (i) the complementary CDF of noncentral chi-square RV with odd order N takes the form of $Q_{N/2}(\alpha, \beta)$; (ii) the PDF of SNR at the output of a dual-branch selection diversity combiner (SDC) in correlated Nakagami-m fading channels can be expressed in terms of $Q_m(\alpha, \beta)$, where $m \geq 0.5$ denotes the fading severity index; and (iii) the conditional bit error probability for a number of binary and quaternary modulation schemes can be expressed in terms of $Q_M(\alpha, \beta)$ [7]. Thus a natural question is to ask whether [2, Appendix C] can be generalized to handle both integer and non-integer values of M? In [9], we derived an exponential-type integral expression for $Q_M(\alpha, \beta)$ that is valid for any real $M > 0$, viz.,

$$Q_M(\alpha, \beta) = \begin{cases} H_M(\alpha, \beta) + C_M(\alpha, \beta) & \text{if } \alpha < \beta \\ \frac{1}{2} + H_M(\alpha, \alpha) + C_M(\alpha, \alpha) & \text{if } \alpha = \beta \\ 1 + H_M(\alpha, \beta) + C_M(\alpha, \beta) & \text{if } \alpha > \beta \end{cases}, \qquad (3)$$

where $\zeta = \alpha/\beta > 0^+$,

$$H_M(\alpha, \beta) = \frac{\zeta^{1-M}}{\pi}\int_0^{\pi} e^{-\frac{1}{2}[\alpha^2 + \beta^2 - 2\alpha\beta\cos(\theta)]} \times \frac{\cos((M-1)\theta) - \zeta\cos(M\theta)}{1 + \zeta^2 - 2\zeta\cos(\theta)}d\theta, \quad (4)$$

and the "correction" term $C_M(\alpha, \beta)$ is given by

$$C_M(\alpha, \beta) = \frac{\sin(M\pi)}{\pi}\int_0^1 \frac{(x/\zeta)^{M-1}}{\zeta + x} e^{-\frac{1}{2}\left(\alpha^2 + \beta^2 + \alpha\beta\left(x + \frac{1}{x}\right)\right)} dx. \quad (5)$$

Unlike (1), the alternative "exponential" integral representation (3) exhibits: (i) finite-range integral with upper and lower integration limits that are independent of α and β; (ii) the integrand whose non-exponential part is a function of α/β only; and (iii) the argument of the exponential term contains only α^2, β^2 and αβ. These features greatly facilitate the averaging problem of $Q_M(ax, bx)$ over generalized fading environments [4][7], and thus permits unified performance evaluation of partially coherent, differentially coherent and noncoherent digital communications.

Note that the term $C_M(\alpha, \beta)$ vanishes for integer M, and more importantly, it retains the desirable properties necessary for the fading averaging problems. However, (4) and (5) assume an indeterminate form when $\zeta = 0$ while $M \neq 1$, which explains the restriction on the region of validity of (3) stated above. However, (3) converge smoothly to $\Gamma(M, \beta^2/2)/\Gamma(M)$ as $\zeta \to 0$, where $\Gamma(a, z) = \int_z^{\infty} t^{a-1}e^{-t}dt$ denotes the complementary incomplete Gamma function. Therefore, (3) can still be used for numerical evaluation when $\alpha = 0$ by replacing it with a small value (e.g., $\alpha = 10^{-4}$).

II. EXPONENTIAL-TYPE INTEGRALS OF $Q_M(\alpha, \beta)$

It can be shown that (1) satisfies the recurrence relation

$$Q_M(\alpha, \beta) - Q_{M-1}(\alpha, \beta) = \left(\frac{\beta}{\alpha}\right)^{M-1} e^{-(\alpha^2 + \beta^2)/2} I_{M-1}(\alpha\beta), \quad (6)$$

by integrating (1) by parts with $u = (x/\alpha)^{M-1} I_{M-1}(\alpha x)$, $dv = xe^{-(x^2 + \alpha^2)/2}dx$, and using identity [15, eq. (8.486.5)]

$$\frac{\partial}{\partial x}\left[\left(\frac{x}{\alpha}\right)^{v} I_v(\alpha x)\right] = \alpha\left(\frac{x}{\alpha}\right)^{v} I_{v-1}(\alpha x), \quad (7)$$

which holds for arbitrary real v.

Next iterating (6) in the backward and forward directions and noting that $Q_{-\infty}(\alpha, \beta) = 0$ and $Q_{\infty}(\alpha, \beta) = 1$, we obtain two formal Neumann series expansions for $Q_M(\alpha, \beta)$, viz.,

$$Q_M(\alpha, \beta) = 1 - e^{-(\alpha^2 + \beta^2)/2} \sum_{v = M}^{\infty} \left(\frac{\beta}{\alpha}\right)^{v} I_v(\alpha\beta), \quad (8)$$

$$Q_M(\alpha, \beta) = e^{-(\alpha^2 + \beta^2)/2} \sum_{v = 1-M}^{\infty} \left(\frac{\alpha}{\beta}\right)^{v} I_{-v}(\alpha\beta), \quad (9)$$

where the summation in (8) and (9) is in increments of one. Note that (8) is identical to [2, eq. (C-23)] but (9) differs from [2, eq. (C-24)] in that the sign of the order for $I_{-v}(\alpha\beta)$ is opposite from the latter. But $I_v(x) = I_{-v}(x)$ if v assumes an integer value. Notice also that (8) and (9) are valid regardless of the ratio $\zeta = \alpha/\beta$ although these two series may convergence at different rates depending on the value of ζ.

A. Case $\alpha > \beta$

In order to derive (3), we should use [15, eq. (8.431.5)]

$$I_v(z) = \frac{1}{\pi}\int_0^{\pi} e^{z\cos\theta}\cos(v\theta)d\theta - \frac{\sin(v\pi)}{\pi}\int_0^{\infty} e^{-z\cosh(t) - vt}dt, \quad (10)$$

in (8). Using [15, eqs. (8.485) and (8.432.1)], it can be shown that (10) is still valid when $v < 0$ while $|\arg z| < \pi/2$. Hence the restriction $\mathrm{Re}\ v > 0$ in [15, eq. (8.431.5)] is unnecessary, except for direct numerical evaluation of (10) itself. Note also that the second RHS term of (10) vanishes if v is an integer. Substituting (10) into (8), we obtain

$$Q_M(\alpha, \beta) = 1 - \frac{1}{\pi}e^{-(\alpha^2 + \beta^2)/2}\int_0^{\pi} e^{\alpha\beta\cos\theta}\left[\sum_{v = M}^{\infty}\left(\frac{\beta}{\alpha}\right)^{v}\cos(v\theta)\right]d\theta + \frac{1}{\pi}e^{-(\alpha^2 + \beta^2)/2}\int_0^{\infty} e^{-\alpha\beta\cosh(t)}\left[\sum_{v = M}^{\infty}\left(\frac{\beta}{\alpha}e^{-t}\right)^{v}\sin(v\pi)\right]dt. \quad (11)$$

Now summing the geometric progressions in (11), we get

$$Q_M(\alpha, \beta) = 1 - \frac{\alpha}{\pi}\left(\frac{\beta}{\alpha}\right)^{M} e^{-(\alpha^2 + \beta^2)/2}\left\{\int_0^{\pi} e^{\alpha\beta\cos\theta} \times \frac{[\alpha\cos(M\theta) - \beta\cos((M-1)\theta)]}{\alpha^2 - 2\alpha\beta\cos(\theta) + \beta^2}d\theta - \int_0^{\infty} e^{-\alpha\beta\cosh(t) - Mt} \times \frac{[\alpha\sin(M\pi) - \beta e^{-t}\sin((M-1)\pi)]}{\alpha^2 + 2\alpha\beta e^{-t} + \beta^2 e^{-2t}}dt\right\}, \quad (12)$$

for $\alpha > \beta$ using the identities [15, eqs. (1.447.1), (1.353.1), (1.447.2) and (1.353.3)], viz.,

$$\sum_{v = N}^{\infty} \Lambda^{v}\sin(v\theta) = \frac{\Lambda^{N}[\sin(N\theta) - \Lambda\sin((N-1)\theta)]}{1 - 2\Lambda\cos(\theta) + \Lambda^2}, \quad (13)$$

$$\sum_{v = N}^{\infty} \Lambda^{v}\cos(v\theta) = \frac{\Lambda^{N}[\cos(N\theta) - \Lambda\cos((N-1)\theta)]}{1 - 2\Lambda\cos(\theta) + \Lambda^2}. \quad (14)$$

Recognizing that $\sin((M-1)\pi) = -\sin(M\pi)$, the common factor $\alpha + \beta e^{-t}$ from the numerator and denominator of the third RHS term of (12) can be cancelled out. Finally, using a variable substitution $x = e^{-t}$, we obtain (3).

B. Case $\alpha < \beta$

Although the series (9) is formally correct, it is less useful for computational purposes. The reason is that the values of a modified Bessel function of non-integer order can be extremely large when the order is a large negative number (i.e., see the second integral term of (10)). The exponential integral that we derive next circumvents this difficulty, which in turn highlights the utility of (3) for computing $Q_M(\alpha, \beta)$ for non-integer order M while $\zeta < 1$. Substituting (10) into (9) and letting $\sin(-\nu\pi) = -\sin(\nu\pi)$ and $\cos(-\nu\theta) = \cos(\nu\theta)$ prior to summing the resulting geometric progressions using (13) and (14), we obtain

$$Q_M(\alpha, \beta) = e^{-(\alpha^2+\beta^2)/2}\frac{\zeta^{1-M}}{\pi}\left\{\int_0^\pi e^{\alpha\beta\cos\theta} \times \frac{[\cos((1-M)\theta) - \zeta\cos(-M\theta)]}{1 - 2\zeta\cos(\theta) + \zeta^2}d\theta + \int_0^\infty e^{-\alpha\beta\cosh(t) + (1-M)t} \times \frac{[\sin((1-M)\pi) - \zeta e^t\sin(-M\pi)]}{1 + 2\zeta e^t + \zeta^2 e^{2t}}dt\right\}, \quad (15)$$

for $\zeta = \alpha/\beta < 1$.

Once again, the common factor $1 + \zeta e^t$ from the numerator and denominator of the second RHS term of (15) can be cancelled out by noting that $\sin((M-1)\pi) = -\sin(M\pi)$. Next, using a variable substitution $x = e^{-t}$, we obtain (3).

C. Case $\alpha = \beta$

When $\zeta = \alpha/\beta = 1$, we must proceed by first combining (8) and (9), and then letting $\beta = \alpha$ so that

$$Q_M(\alpha, \beta) = \frac{1}{2} + \frac{1}{2}e^{-\alpha^2}\left[\sum_{\nu = 1-M}^{\infty} I_{-\nu}(\alpha^2) - \sum_{\nu = M}^{\infty} I_\nu(\alpha^2)\right]. \quad (16)$$

Substituting (10) into (16) and carrying out the geometric series summations, we arrive at

$$Q_M(\alpha, \alpha) = \frac{1}{2} + \frac{1}{2\pi}\int_0^\pi e^{-\alpha^2(1-\cos(\theta))}\frac{\sin((M-1/2)\theta)}{\sin(\theta/2)}d\theta + \frac{\sin(M\pi)}{\pi}\int_0^\infty \frac{e^{(1-M)t}}{1+e^t}e^{-\alpha^2(1+\cosh(t))}dt, \quad (17)$$

for $\alpha = \beta$ using $2\sin(x)\sin(y) = \cos(x-y) - \cos(x+y)$ and $1 - \cos(2x) = 2\sin^2(x)$. If M is a positive integer, then (17) simplifies into the familiar expression in [2, pp. 528] since the third RHS term of (17) vanishes in this case. Finally, we can show that (17) is equivalent to (3) using a variable substitution $x = e^{-t}$. In Section IV, we discuss two examples that highlight the computational/analytical utility of (3) for solving certain communication theory problems.

III. TIGHT BOUNDS FOR $Q_M(\alpha, \beta)$

In this section, we first derive a new infinite series representation of $Q_M(\alpha, \beta)$ which holds for any real $M > 0$ and subsequently show that $Q_M(\alpha, \beta)$ is a monotonically increasing function of M when all other parameters are kept constant. Then we show that $Q_K(\alpha, \beta)$ can be evaluated in closed-form for the special case of half-odd positive integer orders (i.e., $K \in \{0.5, 1.5, 2.5, \ldots\}$). Besides facilitating efficient computation of $Q_M(\alpha, \beta)$ for this particular case, our result lends itself to the development of tight upper and lower bounds on $Q_M(\alpha, \beta)$ for any real order $M \geq 0.5$.

Substituting $I_\nu(z) = \sum_{k=0}^{\infty}\frac{(z/2)^{\nu+2k}}{k!\Gamma(\nu+k+1)}$ [15, eq. (8.445)] in (1), we obtain

$$Q_M(\alpha, \beta) = e^{-\alpha^2/2}\sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{\alpha^2}{2}\right)^k\frac{\Gamma(M+k, \beta^2/2)}{\Gamma(M+k)}. \quad (18)$$

To the best of our knowledge, (18) is new. When $\alpha = 0$, (18) simplifies into $Q_M(0, \beta) = \Gamma(M, \beta^2/2)/\Gamma(M)$ (since all the terms in the summations except $k = 0$ reduce to zero), while $Q_M(\alpha, 0) = 1$ (since $\Gamma(x, 0) = \Gamma(x)$ and $e^x = \sum_{k=0}^{\infty} x^k/k!$). Similarly if M is a positive integer, then $\Gamma(M, x)/\Gamma(M)$ reduces into a finite polynomial. Besides, we can also conclude that $Q_{\nu_1}(\alpha, \beta) \geq Q_{\nu_2}(\alpha, \beta)$ if $\nu_1 > \nu_2 > 0$ since the inequality $\Gamma(\nu_1, c)/\Gamma(\nu_1) \geq \Gamma(\nu_2, c)/\Gamma(\nu_2)$ holds for any arbitrary real $\nu_1 > \nu_2 > 0$ and constant c. This property will be exploited in the derivation of upper and lower bounds for $Q_M(\alpha, \beta)$.

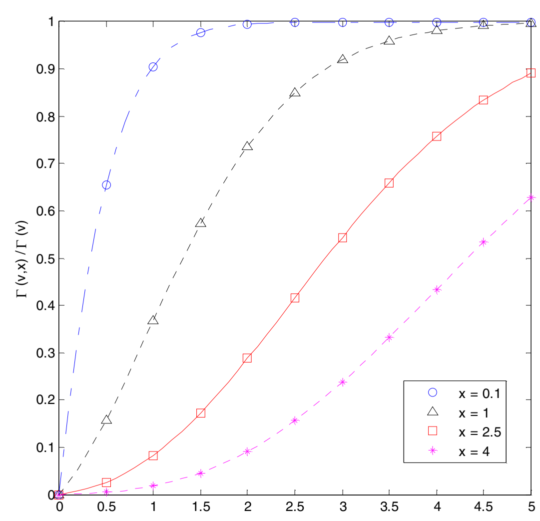

Fig. 1. $\Gamma(\nu, x)/\Gamma(\nu)$ plotted as a function of ν for several distinct values of x.

From [15, eq. (8.467)], we know that the modified Bessel function $I_{\pm(n+0.5)}(z)$ can be expressed in closed form as

$$I_{\pm(n+0.5)}(z) = \frac{1}{\sqrt{\pi}}\sum_{k=0}^{n}\frac{(n+k)!}{k!(n-k)!}\left[\frac{(-1)^k e^z \mp (-1)^n e^{-z}}{(2z)^{k+0.5}}\right], \quad (19)$$

where $n \in \{0, 1, 2, 3, \ldots\}$ is a non-negative integer. Now substituting $I_{n+0.5}(z)$ into (1), we get

$$Q_{n+\frac{3}{2}}(\alpha, \beta) = \frac{1}{\sqrt{2\pi}} \sum_{k=0}^{n} \frac{1}{\alpha^{n+k+1}} \left[(-1)^k \int_\beta^\infty x^{n-k+1} e^{-\frac{1}{2}(x-\alpha)^2} dx \right.$$

$$\left. - (-1)^n \int_\beta^\infty x^{n-k+1} e^{-\frac{1}{2}(x+\alpha)^2} dx \right] \frac{(n+k)!}{k!(n-k)!}. \quad (20)$$

Thus a closed-form formula for $Q_{n+1.5}(\alpha, \beta)$ may be derived by applying appropriate variable substitutions, expanding the resulting terms of the form $(x \pm \alpha)^{n-k+1}$ binomially, and using identity [15, eq. (3.351.2)]. However, a more concise closed-form expression for $Q_{n+1.5}(\alpha, \beta)$ can be attained with the aid of the recursion relation (6), namely

$$Q_{n+\frac{3}{2}}(\alpha, \beta) = Q_{\frac{1}{2}}(\alpha, \beta) + e^{-\frac{1}{2}(\alpha^2+\beta^2)} \sum_{i=1}^{n+1} \left(\frac{\beta}{\alpha}\right)^{i-\frac{1}{2}} I_{i-\frac{1}{2}}(\alpha\beta). \quad (21)$$

It is also important to recognize that $Q_{1/2}(\alpha, \beta)$ in (21) can be evaluated in closed-form (as shown in (22)) by letting $I_{-1/2}(z) = (e^z + e^{-z})/\sqrt{2\pi z}$ in (1) and carrying-out a few routine algebraic manipulations, viz.,

$$\begin{aligned} Q_{1/2}(\alpha, \beta) &= \int_\beta^\infty \sqrt{\alpha x}\, e^{-(x^2+\alpha^2)/2} I_{-1/2}(\alpha x) dx \\ &= \frac{1}{\sqrt{2\pi}} \int_\beta^\infty e^{-(x-\alpha)^2/2} dx + \frac{1}{\sqrt{2\pi}} \int_\beta^\infty e^{-(x+\alpha)^2/2} dx \\ &= Q(\beta - \alpha) + Q(\beta + \alpha) \end{aligned} \quad (22)$$

where $Q(x) = \int_x^\infty e^{-t^2/2} dt / \sqrt{2\pi} = \frac{1}{2}\mathrm{erfc}(x/\sqrt{2})$ denotes the Gaussian probability integral and $Q(-x) = 1 - Q(x)$. Finally, substituting (22) and (19) into (21), we get the desired expression for $Q_K(\alpha, \beta)$ ($K \in \{0.5, 1.5, 2.5, \ldots\}$) in closed-form:

$$Q_K(\alpha, \beta) = Q(\beta - \alpha) + Q(\beta + \alpha) + \frac{1}{\beta\sqrt{2\pi}} \sum_{i=1}^{K-0.5} \left(\frac{\beta}{\alpha}\right)^i$$

$$\times \sum_{k=0}^{i-1} \frac{(i+k-1)!}{k!(i-k-1)!} \left[\frac{(-1)^k e^{-(\alpha-\beta)^2/2} + (-1)^i e^{-(\alpha+\beta)^2/2}}{(2\alpha\beta)^k} \right]. \quad (23)$$

Since $Q_M(\alpha, \beta)$ is a monotonically increasing function with respect to its order M when both α and β are fixed, we may also employ (23) to derive new tight upper and lower bounds for $Q_M(\alpha, \beta)$ (which is valid for any real $M \geq 0.5$) as

$$Q_{\lfloor M \rfloor}(\alpha, \beta) \leq Q_M(\alpha, \beta) \leq Q_{\lceil M \rceil}(\alpha, \beta), \quad (24)$$

where $\lceil . \rceil$ denotes the smallest half-odd integer that is greater than its argument, while $\lfloor . \rfloor$ corresponds to the largest half-odd integer that is less than its argument. For instance, $Q_{0.5}(\alpha, \beta) \leq Q_M(\alpha, \beta) \leq Q_{1.5}(\alpha, \beta)$ if $0.5 < M < 1.5$, and so on. Besides, we also obtain a good closed-form approximation for the generalized Marcum Q-function as

$$Q_M(\alpha, \beta) \approx (1 - \delta) Q_{\lfloor M \rfloor}(\alpha, \beta) + \delta Q_{\lceil M \rceil}(\alpha, \beta), \quad (25)$$

where $\delta = M - \lfloor M \rfloor$.

IV. APPLICATIONS

A. Analysis of SDC in Correlated Nakagami-m Fading

The instantaneous SNR at the SDC receiver output, $\gamma_{SC} = max(\gamma_1, \gamma_2)$, has the PDF [14]

$$f_{\gamma_{SC}}(\gamma) = \sum_{i=1, k \neq i}^{2} \frac{\gamma^{m-1}}{\Gamma(m)} \left(\frac{m}{\Omega_i}\right)^m e^{-m\gamma/\Omega_i} \times \left[1 - Q_m\left(\sqrt{\frac{2\rho m \gamma}{(1-\rho)\Omega_i}}, \sqrt{\frac{2m\gamma}{(1-\rho)\Omega_k}} \right) \right], \quad (26)$$

where Ω_i ($i = 1, 2$) denote the mean branch SNRs, ρ is the correlation coefficient between the two signal envelopes, and $\Gamma(.)$ corresponds to the Euler Gamma function. Since $Q_M(\alpha, \beta)$ as represented in (3) depends on the relative values of its arguments, it is convenient to first define an auxiliary function $U(x) = 1$ if $x < 1$, $U(x) = 0$ if $x > 1$ and $U(1) = 0.5$, and then re-write (3) as

$$1 - Q_M(\alpha, \beta) = U(\alpha/\beta) - H_M(\alpha, \beta) - C_M(\alpha, \beta), \quad (27)$$

so that we do not have to consider its use in (26) separately for the different regions of its arguments. Substituting (27) into (26) and using (4) and (5), we obtain

$$f_{\gamma_{SC}}(\gamma) = \sum_{i=1, k \neq i}^{2} \frac{\gamma^{m-1}}{\Gamma(m)} \left(\frac{m}{\Omega_i}\right)^m e^{-m\gamma/\Omega_i} \left[U\left(\frac{\rho\Omega_k}{\Omega_i}\right) \right.$$

$$- \frac{\mu^m}{\pi} \int_0^\pi e^{\frac{-\rho m \gamma}{(1-\rho)\Omega_i}[1 + \mu^2 - 2\mu\cos(\theta)]} \frac{\mu\cos((m-1)\theta) - \cos(m\theta)}{1 + \mu^2 - 2\mu\cos(\theta)} d\theta$$

$$\left. - \frac{\mu^m \sin(m\pi)}{\pi} \int_0^1 \frac{x^{m-1}}{1 + \mu x} e^{\frac{-\rho m \gamma}{(1-\rho)\Omega_i}\left(1 + \mu^2 + \mu\left(x + \frac{1}{x}\right)\right)} dx \right], \quad (28)$$

where $\mu = \sqrt{\Omega_i/(\rho\Omega_k)}$ and $0^+ < \rho < 1$. Eq. (28) generalizes [16, eqs. (7) and (9)] since it holds for both integer and non-integer $m \geq 0.5$ values. If we can find an alternative representation of the conditional bit or symbol error probability (CEP) $P_s(\gamma)$ in a "desirable" exponential-form [4][7], then (28) and identity (29) [15, eq. (3.381.4)] can be utilized to unify the ASER performance analysis of a broad class of digital modulation schemes:

$$\int_0^\infty e^{-(s+a)x} x^{v-1} dx = \Gamma(v)/(s+a)^v, \ \mathrm{Re}\ v > 0 \quad (29)$$

In fact, the resulting expressions are more general yet more concise than those derived in [16] even for the specific case of positive integer fading severity index *m*.

B. Markovian Threshold Model for Block Error Rates

To study the effects of correlation properties of the fading mobile radio channels on data link layer protocols, it is plausible to consider a first-order Markov process (whose transition probabilities are a function of channel characteristics) to model the success/failure of transmission of data blocks. Here we wish to extend the analysis presented in [17] (for a Rayleigh channel) to a bivariate Nakagami-m channel.

In this model, the received signal power threshold x^* for making the block success/failure decision is given by the reciprocal of the fade margin F (i.e., $x^* = 1/F$) while the state transition probabilities of the channel transition probability matrix $\mathbf{C} = \begin{bmatrix} p & 1-p \\ 1-q & q \end{bmatrix}$ are defined as

$$p = \frac{F_{\gamma_{n-1},\gamma_n}(x^*, x^*)}{F_{\gamma_{n-1}}(x^*)}, \quad (30)$$

$$q = \frac{1 - F_{\gamma_{n-1}}(x^*) - F_{\gamma_n}(x^*) + F_{\gamma_{n-1},\gamma_n}(x^*, x^*)}{1 - F_{\gamma_{n-1}}(x^*)}, \quad (31)$$

where $F_{\gamma_n}(.)$ and $F_{\gamma_{n-1},\gamma_n}(.,.)$ denote the CDF and the joint CDF of sampled normalized fading signal powers $\gamma_n = \gamma(nT)$ and T corresponds to the duration of a single data block. Thus the steady-state probabilities for the block success state and the block failure state can be computed as $\pi_g = \frac{1-p}{2-p-q}$ and $\pi_b = \frac{1-q}{2-p-q}$ respectively. For the bivariate Nakagami-m fading channel, we can show that

$$F_{\gamma_{n-1},\gamma_n}(x^*, x^*) = \frac{2}{\Gamma(m)}\left\{ U(\rho)\gamma(m, mx^*) \right.$$
$$- \frac{\mu^m}{\pi}\int_0^{\pi}\frac{\gamma\left(m, mx^*[1 + [\rho/(1-\rho)](1 + \mu^2 - 2\mu\cos(\theta))]\right)}{(1 + [\rho/(1-\rho)][1 + \mu^2 - 2\mu\cos(\theta)])^m}$$
$$\times \frac{\mu\cos((m-1)\theta) - \cos(m\theta)}{1 + \mu^2 - 2\mu\cos(\theta)}d\theta - \frac{\mu^m\sin(m\pi)}{\pi}\int_0^1\frac{t^{m-1}}{1+\mu t}$$
$$\left. \times \frac{\gamma\left(m, mx^*\left(1 + \frac{\rho}{1-\rho}[1 + \mu^2 + 2\mu(t + 1/t)]\right)\right)}{\left(1 + \frac{\rho}{1-\rho}[1 + \mu^2 + 2\mu(t + 1/t)]\right)^m}dt \right\}, \quad (32)$$

where $\gamma(a, b) = \int_0^x e^{-t}t^{a-1}dt$ denotes the incomplete Gamma function and the correlation coefficient ρ can be related to the normalized Doppler bandwidth $f_D T$ as in [17]. The average block error probability is given by $\frac{1-q}{2-p-q}$ while $(1-p)^{-1}$ represents the average length of burst errors.

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