A Simple Exponential Integral Representation of the Generalized Marcum Q-Function $Q_M(a,b)$ for Real-Order M with Applications

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ABSTRACT — *This article derives a new exponential-type integral representation for the generalized M-th order Marcum Q-function, $Q_M(\alpha, \beta)$, when M is not necessarily an integer. Our new representation includes a well-known result due to Helstrom for the special case of positive integer M and an additional integral correction term that vanishes when M is an integer. The new form has both computational utility (numerous existing computational algorithms for $Q_M(\alpha, \beta)$ are limited to integer M) and analytical utility (e.g., performance analysis of selection diversity in bivariate Nakagami-m fading with arbitrary fading severity index, computation of the complementary cumulative distribution function of a noncentral chi-square random variable for both odd and even orders, unified analysis of correlated binary and quaternary modulations over generalized fading channels, development of a Markovian threshold model for packet errors in correlated Nakagami-m fading channels and so on). Our alternative representation for $Q_M(\alpha, \beta)$ also leads into a new, exact exponential integral formula for the cumulative distribution function (CDF) of signal-to-noise ratio (SNR) at the output of a dual-diversity selection combiner in correlated Nakagami-m fading (including the non-integer fading index case), which several researchers in the past have concluded as unobtainable. Simple yet tight upper and lower bounds for $Q_M(\alpha, \beta)$ (for any arbitrary real order $M \geq 0.5$) are also derived.*

I. INTRODUCTION

The canonical representation of the M-th order generalized Marcum Q-function is given by [1]

$$Q_M(\alpha, \beta) = \int_\beta^\infty x\left(\frac{x}{\alpha}\right)^{M-1} e^{-(x^2+\alpha^2)/2} I_{M-1}(\alpha x)\,dx, \qquad (1)$$

where α and β assume non-negative real values, and $I_M(.)$ is the M-th order modified Bessel function of the first kind. This quantity relates to the complementary CDF of a noncentral chi-square random variable (also known as generalized Rice distribution). If Y is the sum of squares of N statistically independent Gaussian random variates (RVs) with means m_i and variance σ^2, then the CDF is given by

$$F_Y(x) = 1 - Q_{N/2}\left(\frac{s}{\sigma}, \frac{\sqrt{x}}{\sigma}\right), \qquad (2)$$

where $s = \sqrt{\sum m_i^2}$ denotes the noncentrality parameter. Since the above CDF (2) corresponds to a quadratic summation of $N/2$ independent sampled matched filter receiver outputs for nonfading signal in Gaussian noise, it arises in numerous applications such as radar detection and performance evaluation of partially coherent, differentially coherent and noncoherent communications [2]-[4]. Thus reliable and efficient algorithms for computing $Q_M(\alpha, \beta)$ have been an active research subject for a considerable time. Simple algorithms based on recursive computation of modified Bessel functions are almost certain to fail for some parameter values (e.g., large values of M, α and β). Computer arithmetic overflow and underflow problem may also arise in such cases. These issues have been extensively studied by many authors, with series solutions, recursive methods, bounds, and saddle-point techniques (see [5]-[17] and references therein). However, all of the above algorithms are restricted to even values of N in (2) (or equivalently, the order M in (1) is a natural number).

If $Q_M(\alpha, \beta)$ is used to express error rates for noncoherent digital communication systems, the statistical average of $Q_M(a, bx)$, $Q_M(ax, b)$ or $Q_M(ax, bx)$ over the probability density function (PDF) of fading signal amplitude x, where a and b are modulation specific constants, is usually needed. Several integrals of the first two types have been studied by Nuttall [18]-[19]. The third form can be expediently handled by representing Q_M-function in a desirable exponential form [2] in analogy with Craig-type representation for the Gaussian Q-function [20]. For instance, Helstrom's integral representation for $Q_M(\alpha, \beta)$ with integer order M may be expressed in a compact form as [2, Appendix C]

$$Q_M(\alpha, \beta) = \begin{cases} H_M(\alpha, \beta) & \text{if } \alpha < \beta \\ \frac{1}{2} + H_M(\alpha, \alpha) & \text{if } \alpha = \beta \\ 1 + H_M(\alpha, \beta) & \text{if } \alpha > \beta \end{cases}, \qquad (3)$$

where $\zeta = \alpha/\beta > 0^+$, and

$$H_M(\alpha, \beta) = \frac{\zeta^{1-M}}{\pi}\int_0^\pi e^{-\frac{1}{2}[\alpha^2+\beta^2-2\alpha\beta\cos(\theta)]} \times \frac{\cos((M-1)\theta) - \zeta\cos(M\theta)}{1+\zeta^2-2\zeta\cos(\theta)}d\theta. \qquad (4)$$

Unlike (1), the alternative "exponential" integral representation (3) exhibits: (i) finite-range integral with upper and lower integration limits that are independent of α and β; (ii) the integrand whose non-exponential part is a function of α/β only;

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and (iii) the argument of the exponential term contains only α^2, β^2 and $\alpha\beta$. These features greatly facilitate the averaging problem of $Q_M(ax, bx)$ over generalized fading environments [4][14], and thus permits unified performance evaluation of partially coherent, differentially coherent and noncoherent digital communications. Simon [21] also derives an equivalent sine integral representation when M is a positive integer. Incidentally, Proakis [3, pp. 885] provides a complex contour integral representation for $Q_M(\alpha, \beta)$. In [22]-[23], we show that both Helstrom's and Simon's representations are special cases of this contour integral. Note also that, Helstrom [2] derives (3) by expanding the Q_M-function as an infinite series of Bessel functions, replacing these by their trigonometric integral representations and summing up the resulting series. As readers can verify in [2, Appendix C] and [21], the entire development presumes that M is a positive integer (because $I_v(x) \neq I_{-v}(x)$ if v is not an integer). However, $Q_M(\alpha, \beta)$ with real order M arises in a few practical cases: (i) the complementary CDF of noncentral chi-square RV with odd order N takes the form of $Q_{N/2}(\alpha, \beta)$; (ii) the PDF of SNR at the output of a dual-branch selection diversity combiner (SDC) in correlated Nakagami-m fading channels can be expressed in terms of $Q_m(\alpha, \beta)$, where $m \geq 0.5$ denotes the fading severity index; and (iii) the conditional bit error probability for a number of binary and quaternary modulation schemes can be expressed in terms of $Q_M(\alpha, \beta)$ [14]. Hence a natural question is to ask whether (3) can be generalized to handle both integer and non-integer values of M? To the best of our knowledge, no such integral representation has been reported in the literature. In this article, we show that for any real M, Helstrom's representation (3) can be generalized as

$$Q_M(\alpha, \beta) = \begin{cases} H_M(\alpha, \beta) + C_M(\alpha, \beta) & \text{if } \alpha < \beta \\ \frac{1}{2} + H_M(\alpha, \alpha) + C_M(\alpha, \alpha) & \text{if } \alpha = \beta \\ 1 + H_M(\alpha, \beta) + C_M(\alpha, \beta) & \text{if } \alpha > \beta \end{cases}, \quad (5)$$

where $H_M(\alpha, \beta)$ is defined in (4), $\zeta = \alpha/\beta > 0^+$, and the additional "correction" term $C_M(\alpha, \beta)$ is given by

$$C_M(\alpha, \beta) = \frac{\sin(M\pi)}{\pi}\int_0^1 \frac{(x/\zeta)^{M-1}}{\zeta + x} e^{-\frac{1}{2}\left(\alpha^2 + \beta^2 + \alpha\beta\left(x + \frac{1}{x}\right)\right)} dx. \quad (6)$$

We observe that the term $C_M(\alpha, \beta)$ vanishes for integer M as anticipated, but more importantly, it retains the desirable properties necessary for the fading averaging problems. Note also that when $\zeta = 0$, (4) and (6) assume an indeterminate form while $M \neq 1$, which explains the restriction on the region of validity of (5) stated above. However, (5) converge smoothly to $\Gamma(M, \beta^2/2)/\Gamma(M)$ as $\zeta \to 0$, where $\Gamma(a, z) = \int_z^\infty t^{a-1} e^{-t} dt$ denotes the complementary incomplete Gamma function. Thus, (5) can still be used for numerical evaluation when $\alpha = 0$ by replacing it with a small value (e.g., $\alpha = 10^{-4}$). Section II outlines a sketch of the derivation leading to (5), while its applications are highlighted in Section III.

II. EXPONENTIAL INTEGRALS OF $Q_M(\alpha, \beta)$

In order to emphasize the derivation of our new formula (5), we first describe how the two Nuemann series for $Q_M(\alpha, \beta)$ are obtained. It is not difficult to show that (1) satisfies the recurrence relation

$$Q_M(\alpha, \beta) - Q_{M-1}(\alpha, \beta) = \left(\frac{\beta}{\alpha}\right)^{M-1} e^{-(\alpha^2+\beta^2)/2} I_{M-1}(\alpha\beta), \quad (7)$$

by integrating (1) by parts with $u = (x/\alpha)^{M-1} I_{M-1}(\alpha x)$, $dv = xe^{-(x^2+\alpha^2)/2}dx$, and using identity [24, eq. (8.486.5)]

$$\frac{\partial}{\partial x}\left[\left(\frac{x}{\alpha}\right)^v I_v(\alpha x)\right] = \alpha\left(\frac{x}{\alpha}\right)^v I_{v-1}(\alpha x), \quad (8)$$

which holds for arbitrary real v. Next iterating (7) in the backward and forward directions and noting that $Q_{-\infty}(\alpha, \beta) = 0$ and $Q_\infty(\alpha, \beta) = 1$, we obtain two formal Neumann series expansions for $Q_M(\alpha, \beta)$, viz.,

$$Q_M(\alpha, \beta) = 1 - e^{-(\alpha^2+\beta^2)/2} \sum_{v=M}^{\infty} \left(\frac{\beta}{\alpha}\right)^v I_v(\alpha\beta), \quad (9)$$

$$Q_M(\alpha, \beta) = e^{-(\alpha^2+\beta^2)/2} \sum_{v=1-M}^{\infty} \left(\frac{\alpha}{\beta}\right)^v I_{-v}(\alpha\beta), \quad (10)$$

where the summation in (9) and (10) is in increments of one. Note that (9) is identical to [2, eq. (C-23)] but (10) differs from [2, eq. (C-24)] in that the sign of the order for $I_{-v}(\alpha\beta)$ is opposite from the latter. But $I_v(x) = I_{-v}(x)$ if v assumes an integer value. Notice also that (9) and (10) are valid regardless of the ratio $\zeta = \alpha/\beta$ although these two series may convergence at different rates depending on the value of ζ.

A. Case $\alpha > \beta$

Conventionally, (9) is used for numerical computation when $\alpha > \beta$ and for positive integer values of M only. We therefore need to prove (9) also holds for non-integer values of M as well. Our method of proof is to show that partial derivatives of both sides of (9) are equal for both integer and non-integer M values. Differentiating (1) with respect to α, we obtain

$$\begin{aligned} \frac{\partial}{\partial\alpha}Q_M(\alpha, \beta) &= \int_\beta^\infty \frac{\partial}{\partial\alpha}\left[(xe^{-(x^2+\alpha^2)/2})\left(\frac{x}{\alpha}\right)^{M-1} I_{M-1}(\alpha x)\right]dx \\ &= \alpha[Q_{M+1}(\alpha, \beta) - Q_M(\alpha, \beta)] \quad (11) \\ &= \frac{\beta^M}{\alpha^{M-1}} e^{-(\alpha^2+\beta^2)/2} I_M(\alpha\beta) \end{aligned}$$

with the aid of Leibnitz's differentiation rule [24, eq. (0.42)], identity [24, eq. (8.486.6)]

$$\frac{\partial}{\partial\alpha}\left[\left(\frac{\beta}{\alpha}\right)^{v} I_v(\alpha\beta)\right] = \alpha\left(\frac{\beta}{\alpha}\right)^{v+1} I_{v+1}(\alpha\beta)\,, \qquad (12)$$

and (7). We can also show that

$$\frac{\partial}{\partial\beta}Q_M(\alpha,\beta) = \frac{-\beta^M}{\alpha^{M-1}}e^{-(\alpha^2+\beta^2)/2} I_{M-1}(\alpha\beta)\,, \qquad (13)$$

which follows immediately from [24, eq. (0.411.2)] and the definition of (1) itself. Clearly, (11) and (13) hold for both integer and non-integer values of M since (8) and (12) hold regardless (i.e., valid for arbitrary real v). To check if the right-side (RHS) of (9) holds for arbitrary M, we differentiate (9) with respect to α and use (12) to arrive at

$$\frac{\partial}{\partial\alpha}RHS = -e^{-(\alpha^2+\beta^2)/2}\sum_{v=M}^{\infty}\alpha\left(\frac{\beta}{\alpha}\right)^{v+1} I_{v+1}(\alpha\beta) + \alpha e^{-(\alpha^2+\beta^2)/2}\sum_{v=M}^{\infty}\left(\frac{\beta}{\alpha}\right)^{v} I_v(\alpha\beta)\,, \qquad (14)$$

which reduces into (11). Similarly differentiating the RHS of (9) with respect to β and using (8), we obtain

$$\frac{\partial}{\partial\beta}RHS = -e^{-(\alpha^2+\beta^2)/2}\sum_{v=M}^{\infty}\alpha\left(\frac{\beta}{\alpha}\right)^{v} I_{v-1}(\alpha\beta) + \beta e^{-(\alpha^2+\beta^2)/2}\sum_{v=M}^{\infty}\left(\frac{\beta}{\alpha}\right)^{v} I_v(\alpha\beta)\,, \qquad (15)$$

which is identical to (13). Since the partial derivatives of both sides of (9) over α and β are identical, we conclude that the Nuemann series (9) holds for integer as well as non-integer values of M. Also note that, thus far we have not imposed any restrictions on whether $\alpha > \beta$, $\beta > \alpha$, or $\alpha = \beta$. Thus, (9) is valid regardless of the ratio $\zeta = \alpha/\beta$ (although its convergence may be slow when $\zeta < 1$).

In order to derive (5), we should use [24, eq. (8.431.5)]

$$I_v(z) = \frac{1}{\pi}\int_0^{\pi} e^{z\cos\theta}\cos(v\theta)d\theta - \frac{\sin(v\pi)}{\pi}\int_0^{\infty} e^{-z\cosh(t)-vt}dt\,, \quad (16)$$

in (9). Using [24, eqs. (8.485) and (8.432.1)], it can be shown that (16) is still valid when $v < 0$ while $|\arg z| < \pi/2$. Hence the restriction $\mathrm{Re}\ v > 0$ in [24, eq. (8.431.5)] is unnecessary, except for direct numerical evaluation of (16) itself. Note also that the second RHS term of (16) vanishes if v is an integer. Substituting (16) into (9), we obtain

$$Q_M(\alpha,\beta) = 1 - \frac{1}{\pi}e^{-(\alpha^2+\beta^2)/2}\int_0^{\pi} e^{\alpha\beta\cos\theta}\left[\sum_{v=M}^{\infty}\left(\frac{\beta}{\alpha}\right)^{v}\cos(v\theta)\right]d\theta + \frac{1}{\pi}e^{-(\alpha^2+\beta^2)/2}\int_0^{\infty} e^{-\alpha\beta\cosh(t)}\left[\sum_{v=M}^{\infty}\left(\frac{\beta}{\alpha}e^{-t}\right)^{v}\sin(v\pi)\right]dt\,. \quad (17)$$

Now summing the geometric progressions in (17), we get

$$Q_M(\alpha,\beta) = 1 - \frac{\alpha}{\pi}\left(\frac{\beta}{\alpha}\right)^{M} e^{-(\alpha^2+\beta^2)/2}\left\{\int_0^{\pi} e^{\alpha\beta\cos\theta} \times \frac{[\alpha\cos(M\theta) - \beta\cos((M-1)\theta)]}{\alpha^2 - 2\alpha\beta\cos(\theta) + \beta^2}d\theta - \int_0^{\infty} e^{-\alpha\beta\cosh(t) - Mt} \times \frac{[\alpha\sin(M\pi) - \beta e^{-t}\sin((M-1)\pi)]}{\alpha^2 + 2\alpha\beta e^{-t} + \beta^2 e^{-2t}}dt\right\}, \qquad (18)$$

for $\alpha > \beta$ using the identities [24, eqs. (1.447.1), (1.353.1), (1.447.2) and (1.353.3)], viz.,

$$\sum_{v=N}^{\infty}\Lambda^{v}\sin(v\theta) = \frac{\Lambda^N[\sin(N\theta) - \Lambda\sin((N-1)\theta)]}{1 - 2\Lambda\cos(\theta) + \Lambda^2}\,, \qquad (19)$$

$$\sum_{v=N}^{\infty}\Lambda^{v}\cos(v\theta) = \frac{\Lambda^N[\cos(N\theta) - \Lambda\cos((N-1)\theta)]}{1 - 2\Lambda\cos(\theta) + \Lambda^2}\,. \qquad (20)$$

Since $\sin((M-1)\pi) = -\sin(M\pi)$, the common factor $\alpha + \beta e^{-t}$ from the numerator and denominator of the third RHS term of (18) can be cancelled out. Finally, using a variable substitution $x = e^{-t}$, we obtain (5). This completes the proof of (5) for $\alpha > \beta$.

B. Case $\alpha < \beta$

As before (i.e., similar to the development of (14) and (15)), we can show that (10) holds for both integer and non-integer values of M by considering the partial derivatives $\frac{\partial}{\partial\alpha}Q_M(\alpha,\beta)$ and $\frac{\partial}{\partial\beta}Q_M(\alpha,\beta)$ of both sides of (10) with the aid of (8) and (11)-(13) (details omitted here for brevity). If M is a positive integer, then (10) reduces to the well-known Nuemann series expansion [2, eq. (C-24)] since $I_k(x) = I_{-k}(x)$ when k is a natural number.

Although the series (10) is formally correct, it is less useful for computational purposes. The reason is that the values of a modified Bessel function of non-integer order can be extremely large when the order is a large negative number (i.e., see the second integral term of (16)). The exponential integral that we derive next circumvents this difficulty, which in turn highlights the utility of (5) for computing $Q_M(\alpha,\beta)$ for non-integer order M while $\zeta < 1$. Substituting (16) into (10) and letting $\sin(-v\pi) = -\sin(v\pi)$ and $\cos(-v\theta) = \cos(v\theta)$ prior to summing the resulting geometric progressions using (19) and (20), we obtain

$$Q_M(\alpha,\beta) = e^{-(\alpha^2+\beta^2)/2}\frac{\zeta^{1-M}}{\pi}\left\{\int_0^{\pi} e^{\alpha\beta\cos\theta} \times \frac{[\cos((1-M)\theta) - \zeta\cos(-M\theta)]}{1 - 2\zeta\cos(\theta) + \zeta^2}d\theta + \int_0^{\infty} e^{-\alpha\beta\cosh(t) + (1-M)t} \times \frac{[\sin((1-M)\pi) - \zeta e^{t}\sin(-M\pi)]}{1 + 2\zeta e^{t} + \zeta^2 e^{2t}}dt\right\}, \qquad (21)$$

for $\zeta = \alpha/\beta < 1$.

Once again, the common factor $1 + \zeta e^t$ from the numerator and denominator of the second RHS term of (21) can be cancelled out by noting that $\sin((M-1)\pi) = -\sin(M\pi)$. Next, using a variable substitution $x = e^{-t}$, we obtain (5).

C. Case $\alpha = \beta$

When $\zeta = \alpha/\beta = 1$, we must proceed by first combining (9) and (10), and then letting $\beta = \alpha$ so that

$$Q_M(\alpha, \beta) = \frac{1}{2} + \frac{1}{2}e^{-\alpha^2}\left[\sum_{v=1-M}^{\infty} I_{-v}(\alpha^2) - \sum_{v=M}^{\infty} I_v(\alpha^2)\right]. \quad (22)$$

Substituting (16) into (22) and carrying out the geometric series summations, we arrive at

$$Q_M(\alpha, \alpha) = \frac{1}{2} + \frac{1}{2\pi}\int_0^{\pi} e^{-\alpha^2(1-\cos(\theta))}\frac{\sin((M-1/2)\theta)}{\sin(\theta/2)}d\theta + \frac{\sin(M\pi)}{\pi}\int_0^{\infty}\frac{e^{(1-M)t}}{1+e^t}e^{-\alpha^2(1+\cosh(t))}dt, \quad (23)$$

for $\alpha = \beta$ using $2\sin(x)\sin(y) = \cos(x-y) - \cos(x+y)$ and $1 - \cos(2x) = 2\sin^2(x)$. If M is a positive integer, then (23) simplifies into the familiar expression in [2, pp. 528] since the third RHS term of (23) vanishes in this case. Finally, we can show that (23) is equivalent to (5) using a variable substitution $x = e^{-t}$. In the next section, we provide several examples that highlight the computational and analytical utility of (5) in a few applications (i.e., communication theory problems).

III. APPLICATIONS

A. Algorithm for Calculating $Q_M(\alpha, \beta)$ *for Real Order M*

We have already pointed out in Section II.B that the Nuemann series expansion (10) (in conjunction with (16)) is not very useful for computing $Q_M(\alpha, \beta)$ when $\zeta = \alpha/\beta < 1$ because the values of $I_v(x)$ for non-integer order v can be extremely large when v is a large negative number. Moreover, the "marcumq" routine in MATLAB (i.e., based on extended Parl's algorithm [11]) is only valid for positive integer order M. But our new formula (5) overcomes the above-mentioned limitations. For instance, unlike most prior work [5]-[12] which are restricted to even values of N in the computation of of $Q_{N/2}(\alpha, \beta)$, (5) may be used for evaluating the CDF of noncentral chi-square distribution for both even and odd degrees of freedom (see (2)). Furthermore, in Section IV we have derived a new closed-form expression for computing $Q_K(\alpha, \beta)$ for the specific case of half-odd positive integer orders $K \in \{1/2, 3/2, 5/2, \ldots\}$.

B. Analysis of SDC in Bivariate Nakagami-m Fading

Let us now consider the evaluation of outage probability and the average symbol error rate (ASER) performance of a two-branch SDC receiver with correlated inputs over a Nakagami-m fading environment. Unlike [22] and [25], we shall not restrict the fading severity index to only positive integers but it assume any real $m \ge 0.5$. The instantaneous SNR at the SDC receiver output, $\gamma_{SC} = max(\gamma_1, \gamma_2)$, has the PDF [22]

$$f_{\gamma_{SC}}(\gamma) = \sum_{i=1, k\ne i}^{2} \frac{\gamma^{m-1}}{\Gamma(m)}\left(\frac{m}{\Omega_i}\right)^m e^{-m\gamma/\Omega_i} \times\left[1 - Q_m\left(\sqrt{\frac{2\rho m\gamma}{(1-\rho)\Omega_i}}, \sqrt{\frac{2m\gamma}{(1-\rho)\Omega_k}}\right)\right], \quad (24)$$

where Ω_i $(i = 1, 2)$ denote the mean branch SNRs, ρ is the correlation coefficient between the two signal envelopes, and $\Gamma(.)$ corresponds to the Euler Gamma function.

Since $Q_M(\alpha, \beta)$ as represented in (5) depends on the relative values of its arguments, it is convenient to first define an auxiliary function $U(x) = 1$ if $x < 1$, $U(x) = 0$ if $x > 1$ and $U(1) = 0.5$, and then re-write (5) as

$$1 - Q_M(\alpha, \beta) = U(\alpha/\beta) - H_M(\alpha, \beta) - C_M(\alpha, \beta), \quad (25)$$

so that we do not have to consider its use in (24) separately for the different regions of its arguments. Substituting (25) into (24) and using (4) and (6), we obtain

$$f_{\gamma_{SC}}(\gamma) = \sum_{i=1, k\ne i}^{2} \frac{\gamma^{m-1}}{\Gamma(m)}\left(\frac{m}{\Omega_i}\right)^m e^{-m\gamma/\Omega_i}\left[U\left(\frac{\rho\Omega_k}{\Omega_i}\right) - \frac{\mu^m}{\pi}\int_0^{\pi} e^{\frac{-\rho m\gamma}{(1-\rho)\Omega_i}[1+\mu^2-2\mu\cos(\theta)]}\frac{\mu\cos((m-1)\theta) - \cos(m\theta)}{1+\mu^2-2\mu\cos(\theta)}d\theta - \frac{\mu^m\sin(m\pi)}{\pi}\int_0^1\frac{x^{m-1}}{1+\mu x}e^{\frac{-\rho m\gamma}{(1-\rho)\Omega_i}\left(1+\mu^2+\mu\left(x+\frac{1}{x}\right)\right)}dx\right], \quad (26)$$

where $\mu = \sqrt{\Omega_i/(\rho\Omega_k)}$ and $0^+ < \rho < 1$. Eq. (26) generalizes [25, eqs. (7) and (9)] since it holds for both integer and non-integer $m \ge 0.5$ values. If we can find an alternative representation of the conditional bit or symbol error probability (CEP) $P_s(\gamma)$ in a "desirable" exponential-form [4][14], then (26) and identity (27) [24, eq. (3.381.4)] can be utilized to unify the ASER performance analysis of a broad class of digital modulation schemes:

$$\int_0^{\infty} e^{-(s+a)x}x^{v-1}dx = \Gamma(v)/(s+a)^v, \text{Re } v > 0 \quad (27)$$

In fact, the resulting expressions are more general yet more concise than those derived in [25] even for the specific case of positive integer fading severity index m!

It is also interesting to note that the PDF (26) simplifies into a simple exponential-type integral when $m = 1$, and thus it can be readily used in conjunction with [26, Table II] (i.e., the Laplace transform of the CEPs) to derive simple-to-evaluate ASER expressions for a broad range of digital modulation schemes with a dual-diversity SDC receiver over both independent and correlated Rayleigh fading channels.

In order to evaluate the outage probability P_{out} of a dual-diversity SDC receiver with correlated inputs, we need the CDF of its combiner output SNR. Integrating (26), we can show that the desired CDF is given by

$$F_{\gamma_{SC}}(x^*) = \int_0^{x^*} f_{\gamma_{SC}}(\gamma)d\gamma = \sum_{i=1, k\neq i}^{2} \left\{ U\left(\frac{\rho\Omega_k}{\Omega_i}\right)\gamma\left(m, \frac{mx^*}{\Omega_i}\right)/\Gamma(m) \right.$$

$$- \frac{\mu^m}{\pi}\int_0^{\pi} \frac{\gamma\left(m, \frac{mx^*}{\Omega_i}[1+[\rho/(1-\rho)](1+\mu^2-2\mu\cos(\theta))]\right)}{\Gamma(m)(1+[\rho/(1-\rho)][1+\mu^2-2\mu\cos(\theta)])^m}$$

$$\times \frac{\mu\cos((m-1)\theta)-\cos(m\theta)}{1+\mu^2-2\mu\cos(\theta)}d\theta - \frac{\mu^m\sin(m\pi)}{\pi}\int_0^1 \frac{t^{m-1}}{1+\mu t}$$

$$\left. \times \frac{\gamma\left(m, \frac{mx^*}{\Omega_i}(1+[\rho/(1-\rho)][1+\mu^2+2\mu(t+1/t)])\right)}{\Gamma(m)(1+[\rho/(1-\rho)][1+\mu^2+2\mu(t+1/t)])^m}dt \right\}, \quad (28)$$

where $\gamma(a, b) = \int_0^x e^{-t}t^{a-1}dt$ denotes the incomplete Gamma function, $P_{out} = F_{\gamma_{SC}}(x^*)$, and the SNR threshold x^* for a specified modulation scheme is obtained by solving $P_s(x) = P_s{}^*$, where $P_s{}^*$ is the prescribed target symbol error rate in an AWGN channel. To the best of our knowledge, (28) is a new result which can also admit fractional values of m, and thus can be viewed as a generalization of [25, eq. (16)]. Notice that for positive integer m, the second integral term vanishes while the incomplete Gamma function reduces into a finite polynomial [24, eq. (8.352.1)].

Using [27, eq. (3)], it is straight-forward to derive an infinite series solution for the desired CDF as

$$F_{\gamma_{SC}}(x^*) = \frac{(1-\rho)^m}{\Gamma(m)}\sum_{k=0}^{\infty}\frac{\rho^k}{k!\Gamma(m+k)}\prod_{i=1}^{2}\gamma\left(m+k, \frac{mx^*}{\Omega_i(1-\rho)}\right) \quad (29)$$

While the above series is very easy to use and can be used as an alternative to (28), it has a shortcoming, namely its convergence is rather poor for large ρ values. This fact has been described in detail in [28] which motivated the development of a single integral expression in the same form as (28) for the special case of Rayleigh fading [25, eq. (2)]. We have also evaluated (28) numerically and compared its computational speed and accuracy with the direct evaluation of the double integral representation of the joint PDF [22, eq. (10)], infinite series representation (29), and direct evaluation approach of the integral of (24). We noticed that (28) is clearly superior to all other methods especially for large x, m, and ρ values.

It is also worth mentioning that (28) may be used for ASER analysis (using a "CDF approach" [29]) aside from its utility in outage probability calculations, viz.,

$$\overline{P_s} = \int_0^{\infty} P_s(\gamma) f_{\gamma_{SC}}(\gamma)d\gamma = -\int_0^{\infty}\left[\frac{\partial}{\partial\gamma}P_s(\gamma)\bigg|_{\gamma=x}\right]F_{\gamma_{SC}}(x)dx. \quad (30)$$

Since the first order derivative of CEP with respect to SNR can be expressed in closed-form for a wide range of modulation schemes, we can express the ASER in terms of only one-dimensional integrals with finite integration limits. For instance, the ASER of coherent binary phase-shift-keying that employs a dual-diversity SDC receiver with closely-spaced antennas in a Nakagami-m channel is given by

$$\overline{P_s} = \frac{1}{2\sqrt{\pi}}\int_0^{\infty} x^{-1/2}e^{-x}F_{\gamma_{SC}}(x)dx = \frac{\Gamma(m+0.5)}{\sqrt{2}m\Gamma(m)}\sum_{i=1, k\neq i}^{2}\left(\frac{m}{\Omega_i}\right)^m$$

$$\times\left\{ U\left(\frac{\rho\Omega_k}{\Omega_i}\right){}_2F_1\left(m, m+\frac{1}{2}; m+1; -m/\Omega_i\right)\right.$$

$$-\frac{\mu^m}{\pi}\int_0^{\pi}{}_2F_1\left(m, m+\frac{1}{2}; m+1; \frac{1+\rho(\mu^2-2\mu\cos(\theta))}{-(1-\rho)(\Omega_i/m)}\right)$$

$$\times\frac{\mu\cos((m-1)\theta)-\cos(m\theta)}{1+\mu^2-2\mu\cos(\theta)}d\theta - \frac{\mu^m\sin(m\pi)}{\pi}\int_0^1\frac{t^{m-1}}{1+\mu t}$$

$$\left.\times{}_2F_1\left(m, m+\frac{1}{2}; m+1; \frac{1+\rho(\mu^2+2\mu(t+1/t))}{-(1-\rho)(\Omega_i/m)}\right)dt\right\}, \quad (31)$$

using identities [24, eqs. (6.455.2) and (9.131.1)]. This result can be readily extended to other digital modulation schemes (see [29] for further details).

C. Unified Analysis of Binary and Quaternary Modulations

From (1), we can readily observe that $Q_M(\alpha, 0) = 1$ while $Q_M(0, \beta) = \frac{\Gamma(M, \beta^2/2)}{\Gamma(M)}$ since $\lim_{\alpha\to 0}\frac{I_{M-1}(\alpha x)}{\alpha^{M-1}} = \frac{(x/2)^{M-1}}{\Gamma(M)}$, where $\Gamma(a, z) = \int_z^{\infty} t^{a-1}e^{-t}dt = \Gamma(a) - \gamma(a, z)$ is the complementary incomplete Gamma function. Hence the conditional bit error probability (i.e, error performance in an AWGN channel) for several binary and quaternary modulations with coherent, incoherent or differentially coherent detection schemes can be expressed as [14]

$$P_b(\gamma) = \frac{1}{2}[1 + Q_M(a\sqrt{\gamma}, b\sqrt{\gamma}) - Q_M(b\sqrt{\gamma}, a\sqrt{\gamma})], \quad (32)$$

where the modulation-specific coefficients a and b are summarized in Table 1.

Table 1. Modulation specific parameters for a generic CEP formula.

	M	a	b
Nonorthogonal incoherent BFSK	1	$\sqrt{\frac{1-\sqrt{1-\lvert\lambda\rvert^2}}{2}}$	$\sqrt{\frac{1+\sqrt{1-\lvert\lambda\rvert^2}}{2}}$
Gray-coded differential QPSK	1	$\sqrt{2-\sqrt{2}}$	$\sqrt{2+\sqrt{2}}$
Orthogonal incoherent BFSK	1	0	1
Antipodal DPSK	1	0	$\sqrt{2}$
Antipodal CPSK	1/2	0	$\sqrt{2}$
Orthogonal coherent BFSK	1/2	0	1
Correlated coherent BPSK	1/2	0	$\sqrt{2}\lvert\lambda\rvert$

Note: $0 \le \lvert\lambda\rvert \le 1$ is the magnitude of cross-correlation between two signals.

Next, using (5) and (25), (32) may be re-stated as

$$P_b(\gamma) = [H_M(a\sqrt{\gamma}, b\sqrt{\gamma}) - H_M(b\sqrt{\gamma}, a\sqrt{\gamma})]/2$$
$$+ [C_M(a\sqrt{\gamma}, b\sqrt{\gamma}) - C_M(b\sqrt{\gamma}, a\sqrt{\gamma})]/2, \quad (33)$$

which facilitates the averaging problem in different stochastic fading channel models (via moment generating function method). However, $a = 0$ in Table 1 should be replaced with a very small non-zero constant (e.g., $a = 10^{-4}$) when $M \neq 1$ for numerical computation using (33).

IV. TIGHT BOUNDS FOR $Q_M(\alpha, \beta)$

In this section, we first show that $Q_K(\alpha, \beta)$ can be evaluated in closed-form for the special case of half-odd integer orders ($K \in \{0.5, 1.5, 2.5, \ldots\}$). Aside from facilitating efficient computation of $Q_M(\alpha, \beta)$ for this particular case, our result lends itself to the development of tight upper and lower bounds on $Q_M(\alpha, \beta)$ for any real order $M \geq 0.5$.

From [24, eq. (8.467)], we know that the modified Bessel function $I_{\pm(n+0.5)}(z)$ can be expressed in closed form as

$$I_{\pm(n+0.5)}(z) = \frac{1}{\sqrt{\pi}} \sum_{k=0}^{n} \frac{(n+k)!}{k!(n-k)!}\left[\frac{(-1)^k e^z \mp (-1)^n e^{-z}}{(2z)^{k+0.5}}\right], \quad (34)$$

where $n \in \{0, 1, 2, 3, \ldots\}$ is a non-negative integer. Now substituting $I_{n+0.5}(z)$ into (1), we get

$$Q_{n+\frac{3}{2}}(\alpha, \beta) = \frac{1}{\sqrt{2\pi}} \sum_{k=0}^{n} \frac{1}{\alpha^{n+k+1}}\left[(-1)^k \int_{\beta}^{\infty} x^{n-k+1} e^{-\frac{1}{2}(x-\alpha)^2} dx - (-1)^n \int_{\beta}^{\infty} x^{n-k+1} e^{-\frac{1}{2}(x+\alpha)^2} dx\right] \frac{(n+k)!}{k!(n-k)!}. \quad (35)$$

Thus a closed-form formula for $Q_{n+1.5}(\alpha, \beta)$ may be derived by applying appropriate variable substitutions, expanding the resulting terms of the form $(x \pm \alpha)^{n-k+1}$ binomially, and using identity [24, eq. (3.351.2)]. However, a more concise closed-form expression for $Q_{n+1.5}(\alpha, \beta)$ can be attained with the aid of the recursion relation (7), namely

$$Q_{n+\frac{3}{2}}(\alpha, \beta) = Q_{\frac{1}{2}}(\alpha, \beta) + e^{-\frac{1}{2}(\alpha^2+\beta^2)} \sum_{i=1}^{n+1} \left(\frac{\beta}{\alpha}\right)^{i-\frac{1}{2}} I_{i-\frac{1}{2}}(\alpha\beta). \quad (36)$$

It is also important to recognize that $Q_{1/2}(\alpha, \beta)$ in (36) can be evaluated in closed-form (as shown in (37)) by letting $I_{-1/2}(z) = (e^z + e^{-z})/\sqrt{2\pi z}$ in (1) and carrying-out a few routine algebraic manipulations, viz.,

$$\begin{aligned} Q_{1/2}(\alpha, \beta) &= \int_{\beta}^{\infty} \sqrt{\alpha x} e^{-(x^2+\alpha^2)/2} I_{-1/2}(\alpha x) dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{\beta}^{\infty} e^{-(x-\alpha)^2/2} dx + \frac{1}{\sqrt{2\pi}} \int_{\beta}^{\infty} e^{-(x+\alpha)^2/2} dx \\ &= Q(\beta - \alpha) + Q(\beta + \alpha) \end{aligned} \quad ,(37)$$

where $Q(x) = \int_{x}^{\infty} e^{-t^2/2} dt/\sqrt{2\pi} = \frac{1}{2}\mathrm{erfc}(x/\sqrt{2})$ denotes the Gaussian probability integral and $Q(-x) = 1 - Q(x)$. Finally, substituting (37) and (34) into (36), we get the desired expression for $Q_K(\alpha, \beta)$ ($K \in \{0.5, 1.5, 2.5, \ldots\}$) in closed-form:

$$Q_K(\alpha, \beta) = Q(\beta - \alpha) + Q(\beta + \alpha) + \sum_{i=1}^{K-0.5} \left(\frac{\beta}{\alpha}\right)^i \sum_{k=0}^{i-1} \frac{(i+k-1)!}{k!(i-k-1)!} \times \frac{1}{\beta\sqrt{2\pi}}\left[\frac{(-1)^k e^{-(\alpha-\beta)^2/2} + (-1)^i e^{-(\alpha+\beta)^2/2}}{(2\alpha\beta)^k}\right]. \quad (38)$$

Moreover, if we can show that $Q_M(\alpha, \beta)$ is a monotonically increasing function with respect to its order M when both α and β are fixed, then we may also utilize (38) to derive new tight upper and lower bounds for $Q_M(\alpha, \beta)$ (which is valid for any real $M \geq 0.5$) as

$$Q_{\lfloor M \rfloor}(\alpha, \beta) \leq Q_M(\alpha, \beta) \leq Q_{\lceil M \rceil}(\alpha, \beta), \quad (39)$$

where $\lceil . \rceil$ denotes the smallest half-odd integer value that is greater than its argument, while $\lfloor . \rfloor$ corresponds to the largest half-odd integer value that is less than its argument. For instance, $Q_{0.5}(\alpha, \beta) \leq Q_M(\alpha, \beta) \leq Q_{1.5}(\alpha, \beta)$ if $0.5 < M < 1.5$, and so on. To rigorously prove that $Q_M(\alpha, \beta)$ is a monotonically increasing function of M when all other parameters are kept constant, we substitute the power series representation $I_v(z) = \sum_{k=0}^{\infty} \frac{(z/2)^{v+2k}}{k!\Gamma(v+k+1)}$ [24, eq. (8.445)] in (1) to arrive at a new alternative infinite series representation for $Q_M(\alpha, \beta)$ which holds for any real $M > 0$, viz.,

$$Q_M(\alpha, \beta) = e^{-\alpha^2/2} \sum_{k=0}^{\infty} \frac{1}{k!}\left(\frac{\alpha^2}{2}\right)^k \frac{\Gamma(M+k, \beta^2/2)}{\Gamma(M+k)}. \quad (40)$$

Since the inequality $\Gamma(v_1, c)/\Gamma(v_1) \geq \Gamma(v_2, c)/\Gamma(v_2)$ holds for any arbitrary real $v_1 > v_2 > 0$ and constant c, we can therefore also conclude that $Q_{v_1}(\alpha, \beta) \geq Q_{v_2}(\alpha, \beta)$ if $v_1 > v_2 > 0$. This completes our proof.

V. CONCLUSION

A new exponential-type integral (with finite integration limits) for $Q_M(\alpha, \beta)$ has been derived that is valid for arbitrary real *M* orders. It includes a well-known result due to Helstrom for the special case of positive integer *M* and an additional "correction term" that vanishes when *M* is an integer. The new form has both computational and analytical utility, and facilitates the averaging problem in various multipath fading environments and/or modulation formats. It also lends itself to the development of simple-to-evaluate expression for the outage probability of dual-diversity SDC with correlated Nakagami-m fading (for arbitrary fading severity index), which provides a significant speed-up factor compared to the direct evaluation of a double integral of the joint PDF as well as without losing precision/accuracy especially at high values of correlation coefficient (e.g., $\rho > 0.8$), which is an inherent problem with the infinite series method. Simple yet tight upper and lower bounds on $Q_M(\alpha, \beta)$ for any real order $M \geq 0.5$ were also derived.

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