On the Design, Selection Algorithm and Performance Analysis of Limited Feedback Transmit Beamforming

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***Abstract*—Multiple-input multiple-output (MIMO) systems achieve significant diversity and array gains by using transmit beamforming. When complete channel state information (CSI) is not available at the transmitter, a common set of beamformers (codebook) is used by both the transmitter and the receiver. For each channel realization, the best beamformer is selected at the receiver and its index is sent back to the transmitter via a limited feedback channel. In this paper, a codebook design method using the *genetic algorithm* is proposed, which reduces the design complexity and achieves large minimum-distance codebooks. Exploiting the specific structure of these beamformers, an *order and bound* algorithm is proposed to reduce the beamformer selection complexity at the receiver side. The exact bit error rate (BER) of the optimal beamforming in finite-series expression is used to facilitate the BER analysis of limited feedback beamforming. By employing a geometrical approach, an approximate BER of limited feedback beamforming is derived when the codebook size is relatively large (*high resolution analysis*). The simulation results show that the approximate BER is comparatively tight even for small size codebooks.**

***Index Terms*—Limited feedback beamforming, Grassmannian line packing, genetic algorithm, largest eigenvalue distribution, central Wishart matrix, high resolution analysis.**

I. Introduction

MULTIPLE-INPUT multiple-output (MIMO) wireless channels, created by deploying antenna arrays at both the transmitter and the receiver, promise high capacity and high quality wireless communication links [1], [2]. To fully exploit the benefits of MIMO channels, space-time modulation and receiver algorithms are required, providing a sensible performance and complexity tradeoff [3]. Popular MIMO techniques commonly assume the availability of channel state information (CSI) at the receiver, but not at the transmitter. However, in a slow fading environment, complete [4]–[6] or partial CSI [7]–[11] may be available at the transmitter. CSI at the transmitter may be exploited in two ways: antenna subset selection [12]– [14] and precoding. The optimum precoder matrix can be obtained based on the eigen structure of the channel matrix [4].

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In transmit beamforming, the optimal beamformer selects the subchannel corresponding to the largest singular value of the channel matrix by weighting the transmit signals with the corresponding eigenvector [15]. Transmit beamforming can achieve full diversity and array gain with a simple receiver structure. However, the bandwidth limits on the feedback channel [16] limits the availability of full CSI at the transmitter and therefore limited feedback beamforming techniques are of interest.

In limited feedback beamforming, the transmitter and receiver share a codebook of beamformers. Several methods have been proposed in the literature for codebook design, mainly based on vector quantization (VQ) and Lloyd algorithm [7], [17]–[19], or based on maximization of the minimum distance between each pair of beamformers in the codebook [8], [10], [11]. The simulation results show that the codebooks obtained by both methods perform identically in Rayleigh fading channels [20], [21]. The design complexity of these methods is large when the size of the codebook is large.

The authors in [10] map the design problem into the Grassmannian line packing (GLP) problem [22] and use the unitary structure presented in [23] for codebook design. This structure, originally proposed for differential unitary space-time modulation (DUSTM), consists of a diagonal matrix and a rectangular sub-matrix of the Discrete Fourier Transform (DFT) matrix. The diagonal terms are points on the unit circle in the complex plane where their angles are defined by integers that should be optimized offline. These angles are the only parameters that should be saved at the transmitter and receiver. Therefore, the implementation of the codebooks based on GLP needs small resources (memory), whereas in other previous design methods the whole codebook should be saved at the transmitter and receiver.

This paper uses the same unitary structure as [23], [10] and [11]. In these papers, the optimum rotation matrix is obtained via exhaustive search for small dimensions (number of antennas and/or codebook size) and random search for large dimensions. We propose to use the *genetic algorithm* [24] to find the optimum parameters. For this purpose, the design parameters are relaxed from positive integers [11] to positive real values. The simulation results show that the genetic codebooks not only achieve a larger minimum distance than those of [11], but also reduce the optimization complexity.

In limited feedback beamforming, the receiver selects the best beamformer for each realization of the channel by exhaus-

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tive search over the codebook. However, by exploiting the specific structure of the GLP codebooks, we present an *order and bound* algorithm to reduce the receiver's search complexity. In this algorithm, the beamformers of the codebook are ordered based on their vicinity to the optimal beamformer. The original beamformer selection, which is a maximization problem, is converted to a minimization problem, allowing the use of bounding techniques. Metric Bounding is the basic scheme used by minimization algorithms such as *sphere decoders* [25] to avoid unnecessary computations.

The performance of optimal transmit beamforming has been analyzed in the literature previously. In [26], the cumulative density function (cdf) and probability density function (pdf) of the largest singular value of the channel matrix, presented in the form of generalized hypergeometric functions, are used to calculate the *outage probability* in a system with optimal transmit beamforming. The expressions in [27] are in the form of infinite series of averaged BER over the distribution of the largest eigenvalue of central Wishart matrix, which itself is in the form of a hypergeometric function with matrix arguments. In general, hypergeometric functions are very slow converging functions, and therefore the final result presented in [27] is numerically hard to compute. This fact motivates another representation for the pdf of the largest eigenvalue of central Wishart matrix. We exploit the exact closed-form expression for the BER performance of the optimal transmit beamforming presented in [28] which is in finite series and therefore appropriate for the BER analysis of limited feedback beamforming.

The performance analysis of limited feedback beamforming is complicated for general MIMO systems. However, for multiple-input single-output (MISO) systems, the outage probability of transmit beamforming has been studied in [8], the symbol error rate with transmit correlation in [29] and a lower bound on the symbol error rate in [20]. In [30], a framework using high-resolution quantization theory is proposed for *distortion* analysis of MIMO systems with feedback where the distortion function is the capacity loss in a MISO system with limited feedback. Similarly, the authors in [21], use the SNR (signal-to-noise ratio) loss and outage capacity for distortion analysis of MIMO systems with limited feedback beamforming. In this paper, however, by assuming a large size codebook as [30] and [21] (high resolution analysis), we analyze the BER of limited feedback MIMO beamforming. By employing a geometrical approach, we derive an approximate BER of limited feedback MIMO beamforming in closed-form. The simulation results show that the approximate BER is comparatively tight even for small size codebooks.

II. System Model

Consider a narrow-band, flat fading communication system with N_t transmit and N_r receive antennas (MIMO(N_t, N_r)). In a beamforming scenario, the linear transformation between the transmit and receive antennas can be modeled as

$$\mathbf{x} = \sqrt{\rho}\,\mathbf{H}\mathbf{f}s + \mathbf{v} \tag{1}$$

where the vector $\mathbf{x} \in \mathcal{C}^{Nr}$ is the complex received vector, s is the transmitted signal, $\mathbf{f} \in \mathcal{C}^{N_t}$ is the beamformer

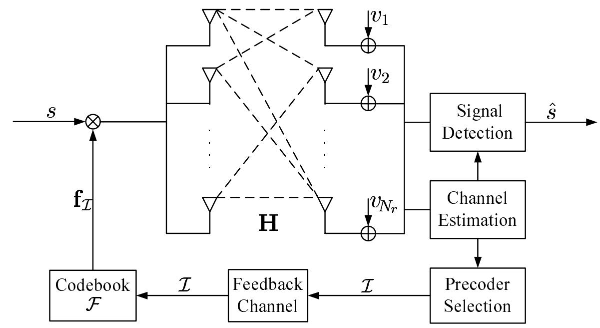

Fig. 1. Block diagram of a MIMO system with limited feedback transmit beamforming.

vector, $\mathbf{v} \in \mathcal{C}^{N_r}$ is the additive noise vector, $\mathbf{H} \in \mathcal{C}^{N_r \times N_t}$ is the channel matrix and ρ is the total transmit power at each signaling interval. Entries of $\mathbf{H}$ and $\mathbf{v}$ are independent and identically distributed (i.i.d.) complex Gaussian random variables with zero mean and unit variance, $\mathcal{CN}(0, 1)$. For each transmission, according to the input information, s is chosen from a signal constellation (e.g. PAM or QAM) with unit average energy. The transmit signal is weighted and parallelized by the beamformer $\mathbf{f}$ to be sent over N_t transmit antennas. To ensure that the transmit power on each signaling interval is ρ, the beamformer vector should satisfy the power constraint: $|\mathbf{f}^*\mathbf{f}| = 1$ where $(\cdot)^*$ and $|\cdot|$ denote the Hermitian (transpose conjugate) and absolute value, respectively.

To obtain the equivalent single-input single-output (SISO) system model of transmit beamforming, we combine the received signals by multiplying (1) with $\mathbf{w}^*$. The optimum combiner vector $\mathbf{w}$, which maximizes the received SNR for each transmitted symbol, is $\mathbf{w} = \mathbf{H}\mathbf{f}$ [3]. Therefore, the equivalent SISO model of transmit beamforming will be

$$\tilde{s} = \sqrt{\rho}\,\|\mathbf{H}\mathbf{f}\|^2\, s + z \tag{2}$$

where z is the additive white noise with $\mathcal{CN}\left(0, \|\mathbf{H}\mathbf{f}\|^2\right)$ distribution and $\|\cdot\|$ denotes the Frobenius norm. Based on (2), the received SNR (γ) for each transmitted symbol is $\gamma = \rho\,\|\mathbf{H}\mathbf{f}\|^2$. Considering the ordered singular value decomposition (SVD) of the channel matrix as $\mathbf{H} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^*$ where $\mathbf{U} \in \mathcal{U}^{N_r \times N_r}$ and $\mathbf{V} \in \mathcal{U}^{N_t \times N_t}$ are unitary matrices, i.e. $\mathcal{U}^*\mathcal{U} = \mathbf{I}$, and $\mathbf{\Sigma} \in \mathcal{R}_+^{N_r \times N_t}$ is a diagonal matrix with decreasing order, i.e. $\sigma_i \geqslant \sigma_{i+1}$, $i = 1, ..., m$, we have:

$$\begin{aligned}\|\mathbf{H}\mathbf{f}\|^2 &= \|\mathbf{\Sigma}\mathbf{V}^*\mathbf{f}\|^2 \\ &= \sigma_1^2|\mathbf{v}_1^*\mathbf{f}|^2 + \sum_{i=2}^{m}\sigma_i^2|\mathbf{v}_i^*\mathbf{f}|^2\end{aligned} \tag{3}$$

where $m = \min(N_t, N_r)$ and $\mathbf{v}_i$, $i = 1, ..., N_t$, is the ith right eigenvector of the channel matrix corresponding to the ith largest singular value. Clearly, the optimal beamformer (which maximizes the received SNR) is $\mathbf{f}_{opt} = \mathbf{v}_1$ and consequently $\gamma_{opt} = \rho\,\sigma_1^2$. The optimal beamforming is applicable in very slow fading environments or when the system uses time division duplex (TDD) technique for transmission where the transmitter and receiver use the same bandwidth.

In limited feedback systems where $\mathbf{f}_{opt}$ is not available at the transmitter, we use a pre-designed set (codebook) $\mathcal{F}$ of $L = 2^{N_b}$ beamformers (Fig. 1) where N_b is the number of feedback bits. For a given $\mathbf{H}$, the only feedback parameter is $\mathcal{I}$ which is the index of $\mathbf{f}_{\mathcal{I}} \in \mathcal{F}$ that maximizes $\|\mathbf{H}\mathbf{f}_k\|^2$, $\forall\, \mathbf{f}_k \in \mathcal{F}$.

This paper will investigate the following issues:

- How the codebook $\mathcal{F}$ should be designed?
- How the appropriate beamformer $\mathbf{f}_{\mathcal{I}}$ should be selected from $\mathcal{F}$ for each realization of the channel?
- How well a given codebook $\mathcal{F}$ with L beamformers will perform?

III. Codebook Design

In suboptimal beamforming, a vector $\mathbf{f}$ is used instead of the optimal $\mathbf{v}_1$ and the received SNR per symbol is related to $\|\mathbf{H}\mathbf{f}\|^2$. Thus, it is convenient to define the following *distortion* minimization problem [11] as a figure of merit for codebook $\mathcal{F}$:

$$\mathrm{E}_{\mathbf{H}}\left\{\min_{\mathbf{f}\in\mathcal{F}}\left(\|\mathbf{H}\mathbf{v}_1\|^2 - \|\mathbf{H}\mathbf{f}\|^2\right)\right\}. \quad (4)$$

It has been shown [11] that the minimization in (4) leads to the maximization of the *chordal distance* between any pairs of precoders in $\mathcal{F}$. The chordal distance is defined as

$$\begin{aligned} d_c(\mathbf{f}_i,\mathbf{f}_j) &= \frac{1}{\sqrt{2}}\|\mathbf{f}_i\mathbf{f}_i^* - \mathbf{f}_j\mathbf{f}_j^*\| \\ &= \sqrt{1-\lambda^2(\mathbf{f}_i^*\mathbf{f}_j)} \\ &= \sqrt{1-|\mathbf{f}_i^*\mathbf{f}_j|^2} \triangleq \sin(\theta_{ij}) \quad (5) \end{aligned}$$

where $0 \leqslant i \neq j < L$ and θ_{ij} denotes the angle between pair vectors $(\mathbf{f}_i, \mathbf{f}_j) \in \mathcal{F}$. The optimum codebook is the one with the maximum θ_{min} defined as

$$\begin{aligned} \theta_{min} &\triangleq \min_{\forall \mathbf{f}_i \neq \mathbf{f}_j \in \mathcal{F}} \arcsin\left(\sqrt{1-|\mathbf{f}_i^*\mathbf{f}_j|^2}\right) \quad (6) \\ &\leqslant \arcsin\left(\sqrt{\frac{L}{L-1}\times\frac{N_t-1}{N_t}}\right) \quad (7) \end{aligned}$$

where the inequality (7) is called the Welch bound [31] or Rankin bound [22] for $L \geqslant N_t$.

Finding a codebook (a pack) $\mathcal{F}$ of L vectors in N_t dimensional complex space with maximum possible θ_{min} (6), is called *Grassmannian line packing* problem in applied mathematics and information theory [10], [22]. For arbitrary L and N_t, line packings that achieve equality in (7) are often impossible to design. The most practical method for generating packings is to use the unitary matrix structure proposed in [23] for non-coherent space-time modulation. Similar to the codebook structure proposed in [23] that can be easily implemented and yields codebooks with large minimum distances, the authors in [10] construct the codebook $\mathcal{F}$ as follows

$$\mathcal{F} = \left\{\mathbf{f}_k \middle| \mathbf{f}_k = \frac{1}{\sqrt{N_t}}\left[e^{j\frac{2\pi}{L}ku_1}, e^{j\frac{2\pi}{L}ku_2}, \ldots, e^{j\frac{2\pi}{L}ku_{N_t}}\right]^T, k=0,\ldots,L-1\right\}$$

where $0 \leqslant u_i < L$ are the integer design parameters and should be optimized as follows

$$\mathbf{u} = \arg\max_{\{u_i\}}\min_{1\leqslant k<L} d_c(\mathbf{f}_0,\mathbf{f}_k)$$

where $\mathbf{u} = [u_1, u_2, \ldots, u_{N_t}]^T$.

In previous works [23], [11], the design parameters, $\{u_i\}$, are restricted to integers. Exhaustive computer search or random search for their optimum values is employed since analytical determination of the optimum appears impossible. Moreover, because the computational complexity increases exponentially with dimensions (N_t and L), it is impossible to find the optimum parameters for large dimensions with exhaustive search.

To overcome these problems, we propose to employ the genetic algorithm [24]. Although it does not guarantee the global optimality, we find that genetic solutions have larger θ_{min} than the optimum values from exhaustive search. This seemingly contradictory result is obtained by relaxing the design parameters to be real rather than integer numbers, i.e. the codebook $\mathcal{F}$ is constructed as follows

$$\mathcal{F} = \left\{\mathbf{f}_k \middle| \mathbf{f}_k = \frac{1}{\sqrt{N_t}}\left[e^{jk\alpha_1}, e^{jk\alpha_2}, \ldots, e^{jk\alpha_{N_t}}\right]^T, k=0,\ldots,L-1\right\} \quad (8)$$

where $0 \leqslant \alpha_i < 2\pi$ are the design parameters and should be optimized as follows

$$\boldsymbol{\alpha} = \arg\max_{\{\alpha_i\}}\min_{1\leqslant k<L} d_c(\mathbf{f}_0,\mathbf{f}_k) \quad (9)$$

where $\boldsymbol{\alpha} = [\alpha_1, \alpha_2, \ldots, \alpha_{N_t}]^T$. This relaxation increases the search space of the design parameters, thereby improving the chance to obtain codebooks with larger θ_{min}.

Genetic algorithm is an adaptive heuristic search algorithm based on the evolutionary ideas of natural selection and genetics. It represents an intelligent exploitation of a random search used to solve optimization problems. Although randomized, genetic algorithm is by no means random, instead it exploits historical information to direct the search into the region of better performance within the search space. Genetic algorithm simulates the survival of the fittest among individuals over consecutive generations for solving a problem. In our problem, fitness of a solution is determined by θ_{min} (6). Each generation consists of a population of bit (gene) strings that are analogous to the chromosome that we see in our DNA. Each individual represents a point in the search space and a possible solution. By assuming N_g bits (genes) for each α_i in (9), we define a string of $N_g \times N_t$ bits for each solution (individual in the population). The individuals in the population are first generated randomly, and then are made to go through a process of evolution. A fitness score (θ_{min}) is assigned to each solution representing the ability of an individual to *compete*. Individuals with higher fitness scores are selected as parents for the next generation with higher probability.

In a basic genetic algorithm, the next generation is composed of three types of children as follow:

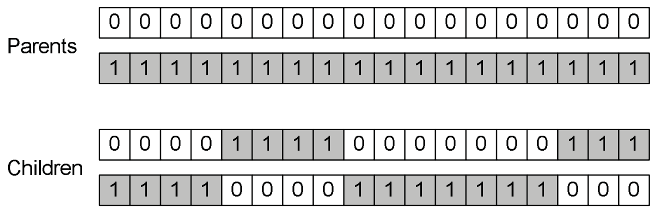

Fig. 2. Multi-point crossover.

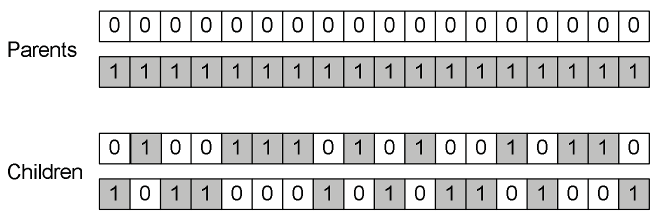

Fig. 3. Scattered crossover.

- *Elite Children*: Children in the current generation are selected for the next generation based on their fitness values. Since the selection rule here is probabilistic not deterministic, fitter solutions (larger θ_{min}) are typically more likely to be selected.
- *Crossover Children*: These children are created by combining pairs of parents in the current population. Generally, the crossover operation recombines selected solutions (parents) by swapping parts of them for producing divergent solutions to explore the search space. Many crossover techniques exist to produce a child of a pair parents. However, all of them are surprisingly simple to implement, involving random number generation and some partial string exchange. Fig. 2 and Fig. 3 illustrate two different techniques used in crossover generation. Scattered crossover is a popular technique used for crossover generation. This method first creates a random binary vector with the same size of parents. Then if the ith bit is 0, corresponding gene is selected from the first parent, otherwise it is selected from the second parent. Ultimately, all selected genes are combined to form the child.
- *Mutation Children*: The algorithm generates mutation children by randomly changing the bits of individual parent in the current population. This process can be done by adding a random vector from a Gaussian distribution to the parent. The aim of mutation in the algorithm is to avoid local optima by preventing the population from becoming too similar to each other, thus slowing or even stopping the evolution.

As a result, new mutated members along with new crossed over members and the rest of those selected from the previous population form the new generation. The genetic algorithm terminates when there is no improvement in the objective function for a specific number of successive iterations. Interested readers are referred to [24] and [32] for sophisticated genetic algorithms and more information about the algorithm such as robustness, marginal and large dimensional behavior, comparisons and benefits over other optimization techniques, etc.

TABLE I
θ_{min} (IN DEGREES) OF BEAMFORMERS OBTAINED BY GENETIC ALGORITHM AND EXHAUSTIVE SEARCH AND THEIR CORRESPONDING WELCH BOUND.

N_t	N_b	L	θ_{min}(exhaustive)	θ_{min}(genetic)	θ_{min}(Welch)
4	3	8	60.00	64.67	81.78
4	4	16	54.43	57.32	78.46
4	5	32	45.00	47.68	76.95
4	6	64	37.12	38.93	76.22
4	7	128	29.92	31.33	75.87
6	3	8	72.06	75.76	87.27
6	4	16	64.18	67.27	83.62
6	5	32	61.87	64.11	81.96
6	6	64	55.92	57.65	81.17
6	7	128	50.75	51.97	80.78

Table I shows the θ_{min} obtained by using exhaustive search and by the genetic algorithm. For comparison, the Welch bound for θ_{min} is also included. Our simulations show that using $N_g = 8$ bits for each α_i is enough to obtain the results in Table I. The genetic solutions, not only have a larger θ_{min} than those from exhaustive search, but also are obtained much faster than exhaustive search due to the computational complexity of exhaustive search in large dimensions (N_t and L). Genetic optimization can also be used for other applications such as *precoder design for multiplexing* and *code design for differential unitary space-time modulation* [33].

Although there are other methods proposed in the literature for codebook design, particularly based on vector quantization (VQ) and Lloyd algorithm [7], [17]–[19], the simulation results show that the codebooks obtained by other methods perform the same as GLP codebooks in the Rayleigh fading channels [20], [21], and the design complexity of all previous methods is large when the size of the codebook is high. But codebook design using the structure proposed in (8) and by exploiting the genetic algorithm has the following benefits:

- Genetic solutions have larger θ_{min}.
- Genetic algorithm reduces the design complexity effectively, especially in large dimensions.
- The only parameters that should be saved at the transmitter and receiver are $\{\alpha_1, \alpha_2, ..., \alpha_{N_t}\}$, and thus this method is easy to implement, while in other methods, the whole codebook should be saved at the transmitter and receiver.
- The structure proposed in (8) allows us to propose a faster algorithm than exhaustive search to reduce the beamformer selection complexity at the receiver side. This algorithm is presented in the next section.

IV. Beamformer Selection

For every realization of the channel matrix $\mathbf{H}$, the best beamformer $\mathbf{f}_{\mathcal{I}} \in \mathcal{F}$ is selected and only the index $\mathcal{I}$ is fed back to the transmitter. Since the codebook $\mathcal{F}$ is known to the transmitter, it uses $\mathbf{f}_{\mathcal{I}}$ to beamform the signals. Since the received SNR for each symbol is $\gamma = \rho\|\mathbf{H}\mathbf{f}_{\mathcal{I}}\|^2$, $\mathcal{I}$ must be selected from the following optimization problem:

$$\mathcal{I} = \arg\max_{0 \leqslant k < L} \|\mathbf{H}\mathbf{f}_k\|^2 \tag{10}$$

where $\mathbf{f}_k$ is defined in (8).

The maximization problem in (10) is simply an exhaustive search over all members of $\mathcal{F}$, which can be computationally complex for large dimensions (L, N_t and also N_r). Moreover, the beamformers in $\mathcal{F}$ should be stored at the receiver and transmitter which needs dedicated memory, specially for large dimensions. Therefore, it is expedient to find an intelligent algorithm to solve (10) efficiently with reasonable memory, particularly when the structure of the beamformers is known.

According to the Rayleigh-Ritz inequality [34] $\|\mathbf{Ha}\|^2 \leqslant \sigma^2_{max}(\mathbf{H})$ when $\|\mathbf{a}\|^2 = 1$, or equivalently $\sigma^2_{max}(\mathbf{H}) - \|\mathbf{Ha}\|^2 \geqslant 0$ when $\|\mathbf{a}\|^2 = 1$. Thus, the maximization problem (10) can be changed to the following minimization problem:

$$\begin{aligned}
\mathcal{I} &= \arg \min_{0 \leqslant k < L} \sigma^2_{max}(\mathbf{H}) - \|\mathbf{Hf}_k\|^2 \\
&= \arg \min_{0 \leqslant k < L} \sigma^2_{max}(\mathbf{H}) \mathbf{f}_k^* \mathbf{I}^* \mathbf{I} \mathbf{f}_k - \mathbf{f}_k^* \mathbf{H}^* \mathbf{H} \mathbf{f}_k \\
&= \arg \min_{0 \leqslant k < L} \mathbf{f}_k^* \mathbf{G} \mathbf{f}_k
\end{aligned}$$

where $\mathbf{I}$ is the identity matrix and $\mathbf{G} = \lambda_{max}(\mathbf{H}^*\mathbf{H})\mathbf{I} - \mathbf{H}^*\mathbf{H}$. Finally by Cholesky decomposition of $\mathbf{G}$ as $\mathbf{G} = \mathbf{R}^*\mathbf{R}$ where $\mathbf{R}$ is an upper triangular matrix, we have

$$\mathcal{I} = \arg \min_{0 \leqslant k < L} \|\mathbf{Rf}_k\|^2. \tag{11}$$

Due to the upper triangularity of $\mathbf{R}$ and by expanding $\|\mathbf{Rf}_k\|^2$ to its scaler form

$$\|\mathbf{Rf}_k\|^2 = \sum_{q=N_t}^{1} \left| \sum_{t=q}^{N_t} r_{q,t}\, f_{k,t} \right|^2, \tag{12}$$

it can be seen that (12) consists of an outer sum of non-negative real terms where the computational load of each term is increased when the index of the outer sum (q) is decreased. Thus, if we know that $\|\mathbf{Rf}_{\mathcal{I}}\|^2 \leqslant \mathcal{B}$ where $\mathcal{B}$ is a bound, we can compare the outer sum value in (12), index by index, to the bound $\mathcal{B}$ and if it is greater than the bound, the rest of the computations (for the given $\mathbf{f}_k$) are discarded. By this bounding technique, the computational complexity of (11) is reduced efficiently.

Clearly, the bound $\mathcal{B}$ plays a critical role on the complexity reduction. Initially, $\mathcal{B}$ is set to infinity for the first $\mathbf{f}_k$, i.e. $\mathbf{f}_0$, but for the rest of $\mathbf{f}_k$'s, $k = 1, 2, \ldots, L-1$, $\mathcal{B}$ is set to the minimum $\|\mathbf{Rf}_k\|^2$ obtained thus far during the algorithm. Consequently, it is expedient to run the proposed algorithm in a rational ordering for k (not simply k from 0 to $L-1$) so that the probability of obtaining as small $\mathcal{B}$ as possible in primary k's, is as high as possible.

We resort to the geometry of N_t dimensional vectors in $\mathcal{F}$. We define $\mathbf{f}_{k_0}$ as the reference vector for any arbitrary $0 \leqslant k_0 < L$, and the reference set $\Theta = \{\theta_0, \theta_1, \ldots, \theta_{L-1}\}$ where $\theta_i \triangleq \arcsin\left(\sqrt{1 - |\mathbf{f}_{k_0}^* \mathbf{f}_i|^2}\right)$ is the angle between pair vectors $(\mathbf{f}_{k_0}, \mathbf{f}_i)$. For a realization of the channel matrix $\mathbf{H}$, we calculate $\theta_{\mathbf{H}} \triangleq \arcsin\left(\sqrt{1 - |\mathbf{f}_{k_0}^* \mathbf{v}_1|^2}\right)$ where $\mathbf{v}_1$ is the right eigenvector corresponding to the largest singular value of $\mathbf{H}$. Then an ordered set $\mathcal{K}$ of the angle indexes in Θ is constructed based on their vicinity to $\theta_{\mathbf{H}}$, or equivalently

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Algorithm 1: The order and bound algorithm

Data : $\mathbf{H}, \Theta, \mathbf{f}_0$
Result: $\mathcal{I}$
$[\,\mathbf{U}, \mathbf{\Sigma}, \mathbf{V}] \leftarrow \mathrm{svd}(\mathbf{H})\,;\ \lambda_{max} \leftarrow \max(\,\mathrm{diag}(\,\mathbf{\Sigma})\,)^2\,;$
$\mathbf{v}_1 \leftarrow \mathbf{V}(:,1)\,;\ \theta_{\mathbf{H}} \leftarrow \arcsin\left(\sqrt{1 - |\,\mathbf{f}_0^* \mathbf{v}_1\,|^2}\right)\,;$
$\mathcal{K} \leftarrow \mathrm{Index}\,\{\mathrm{Sort}\,\{\,|\Theta - \theta_{\mathbf{H}}|\,\}\}\,;$
$\mathbf{R} \leftarrow \mathrm{chol}(\lambda_{max}\mathbf{I} - \mathbf{H}^*\mathbf{H})\,;$
$L \leftarrow \mathrm{size}(\Theta)\,;\ N \leftarrow \mathrm{size}(\mathbf{R})\,;$
$\mathcal{B} \leftarrow \infty\,;\ d_0 = |\,\mathbf{R}(N,N)\,|^2;$
for $i = 1 : L$ **, do**
  $k \leftarrow \mathcal{K}(i)\,;\ \mathbf{f}_k \leftarrow \mathbf{f}_0^k\,;\ d \leftarrow d_0\,;$
  for $q = N-1 : -1 : 1$ **, do**
    $d \leftarrow d + |\,\mathbf{R}(q, q:N)\,\mathbf{f}_k(q:N)\,|^2\,;$
    if $d > \mathcal{B}$ **then**
      break;
    else
      if $q == 1$ **then**
        $\mathcal{B} \leftarrow d\,;$
        $\mathcal{I} \leftarrow k\,;$
      end
    end
  end
end

$$\mathcal{K} = \mathrm{Index}\left\{\mathrm{Sort}\left\{\,|\,\Theta - \theta_{\mathbf{H}}\,|\,\right\}\right\} \tag{13}$$

where $\mathrm{Index}\{|\theta_k - \theta_{\mathbf{H}}|\} = k$ and the operator $\mathrm{Sort}\{\cdot\}$ sorts its argument set from minimum to maximum. Thus, search for the minimum in (11) is re-ordered as follows:

$$\mathcal{I} = \arg \min_{k \in \mathcal{K}} \|\mathbf{Rf}_k\|^2 \tag{14}$$

In summary, for a given codebook, the angle set Θ is calculated and stored at the receiver. For a given channel matrix $\mathbf{H}$, $\theta_{\mathbf{H}}$ is computed and the ordered set $\mathcal{K}$ (13) is constructed. Then the minimization in (14) is executed with respect to the structure of $\|\mathbf{Rf}_k\|^2$ (12) and the bound $\mathcal{B}$. This algorithm is referred to as *order and bound* algorithm. Algorithm 1 presents the semi-code of the algorithm where $\mathbf{f}_0$ is adopted as the reference vector.

V. Performance Analysis of Optimal Transmit Beamforming

In order to perform the BER analysis of limited feedback beamforming, we first present the exact BER of optimal transmit beamforming. Although the performance of optimal transmit beamforming has been analyzed in the literature before, those expressions are infinite series and therefore are not suitable for our BER analysis of limited feedback beamforming presented in Section VI.

Since in optimal beamforming the right eigenvector corresponding to the largest singular value of the channel matrix ($\mathbf{v}_1$) is used for beamforming, by substituting (3) into (2), the system model for optimal beamforming becomes

$$\tilde{s} = \sqrt{\rho}\,\sigma_1^2\, s + z \tag{15}$$

where z is the additive white noise sample with $\mathcal{CN}\left(0, \sigma_1^2\right)$ distribution.

A. Exact BER expression for PAM and QAM

Assume the transmitted signal s in (15) is selected from a $I \times J$ rectangular QAM constellation with unit average energy and a Gray code mapping [35]. I and J denote the number of in-phase and quadrature amplitudes respectively. We define

$$P_{I|\sigma_1^2}(k) = \sum_{i=0}^{(1-2^{-k})I-1} \beta_I(k,i) Q\left(\sqrt{\eta(i)\sigma_1^2}\right) \tag{16}$$

where

$$\beta_I(k,i) = (-1)^{\lfloor \frac{i\,2^{k-1}}{I} \rfloor} \left(\frac{2}{I}\right)\left(2^{k-1} - \left\lfloor \frac{i\,2^{k-1}}{I} + \frac{1}{2}\right\rfloor\right), \tag{17}$$

$$Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^{\infty} e^{-\frac{u^2}{2}}\, du\,, \tag{18}$$

$$\eta(i) = \frac{6(2i+1)^2\rho}{I^2 + J^2 - 2} \tag{19}$$

and $\lfloor x \rfloor$ denotes the largest integer to x. Now the average BER of the $I \times J$ rectangular QAM conditioned on σ_1^2 is expressed as [36]

$$P_{b|\sigma_1^2} = \frac{1}{\log_2(I \cdot J)} \left(\sum_{k=1}^{\log_2 I} P_{I|\sigma_1^2}(k) + \sum_{l=1}^{\log_2 J} P_{J|\sigma_1^2}(l)\right). \tag{20}$$

Note that (20) reduces to the BER of BPSK for $I = 2$ and $J = 1$, I-array PAM for $J = 1$, and M-array square QAM for $I = J = \sqrt{M}$. Thus, the exact BER is obtained by averaging (20) over the distribution of σ_1^2. Interested readers may refer to [36] for more details on the derivation of (16) and (20).

B. Distribution of the largest singular value of $\mathbf{H}$

Assume $m = \min(N_t, N_r)$, $n = \max(N_t, N_r)$ and the Hermitian matrix $\mathbf{W} \in \mathcal{C}^{m\times m}$ defined as

$$\mathbf{W} = \begin{cases} \mathbf{H}^*\mathbf{H} & \text{if } N_t \leqslant N_r \\ \mathbf{H}\mathbf{H}^* & \text{if } N_t > N_r \end{cases}.$$

The nonzero singular values of $\mathbf{H}$ correspond to the eigenvalues of $\mathbf{W}$ by $\lambda_i = \sigma_i^2, i = 1, \ldots, m$. Therefore, $\lambda_1 = \lambda_{max}(\mathbf{W})$ and σ_1^2 have the same distribution. When $\mathbf{H}$ is a complex random matrix of i.i.d. elements with $\mathcal{CN}(0,1)$ distribution, $\mathbf{W}$ is a central complex Wishart matrix.

The distribution of λ_1 for the central complex Wishart matrix was originally calculated in general form by Khatri [37] in the form of hypergeometric functions with matrix arguments [38]. This result has been used in [26] to calculate the *outage probability* ($\Pr[\gamma = \rho\lambda_1 \leqslant \gamma_{th}]$) of a system with optimal transmit beamforming. The result has also been used in [27] for BER analysis of BPSK and BFSK modulations in optimal beamforming. Consequently, the final BER expressions presented in [27] are infinite series. Since the hypergeometric functions with matrix argument are numerically hard to compute [39], averaging (16) over this representation of the distribution of λ_1, makes the final results infinite series that are numerically hard to compute [27] and makes the BER analysis of limited feedback beamforming complicated.

On the other hand, it is easy to verify [40] that the pdf of λ_1 can also be represented by

$$f_{\lambda_1}(\lambda_1) = \sum_{t=1}^{m} e^{-t\lambda_1} G_t(\lambda_1) \tag{21}$$

where

$$G_t(\lambda_1) = \sum_{j=0}^{D_t} a_{t,j}\, \lambda_1^j \tag{22}$$

denotes the corresponding polynomial coefficient of $e^{-t\lambda_1}$, D_t is the degree of $G_t(\lambda_1)$ and the coefficients $a_{t,j}$ can be tabulated by simple integrations [28].

By averaging (16) over the pdf of λ_1 (21), we obtain

$$P_I(k) = \sum_{i=0}^{(1-2^{-k})I-1} \beta_I(k,i) \sum_{t=1}^{m}\sum_{j=0}^{D_t} \frac{a_{t,j}}{t^{j+1}}\, \phi\big(j+1, t, \eta(i)\big) \tag{23}$$

where

$$\phi(N,t,\alpha) = \Gamma(N)\left(\frac{1-\mu}{2}\right)^N \sum_{r=0}^{N-1} \binom{N-1+r}{r} \left(\frac{1+\mu}{2}\right)^r \tag{24}$$

and

$$\mu = \sqrt{\frac{\alpha}{2t+\alpha}}\,.$$

Consequently, the average BER of the $I \times J$ rectangular QAM signal transmitted through the system model in (15) is expressed as

$$P_b = \frac{1}{\log_2(I \cdot J)} \left(\sum_{k=1}^{\log_2 I} P_I(k) + \sum_{l=1}^{\log_2 J} P_J(l)\right). \tag{25}$$

VI. Performance Analysis of Limited Feedback Beamforming

The performance analysis of limited feedback beamforming has been studied in [29] for MISO systems with transmit antenna correlations. The analysis involves an $L-$tuple integration with infinite limits, which is computationally difficult even for small L (codebook size). On the other hand, a geometric approach has been proposed in [8] for the outage probability analysis and used in [20] for a lower bound on the symbol error rate of transmit beamforming, both for MISO systems. By using high-resolution quantization theory, the capacity loss in a MISO system with limited feedback has been derived in [30]. Similarly, the SNR loss and outage capacity of MIMO systems with limited feedback beamforming have been approximated in [21]. In this section, however, without distortion analysis, we analyze the BER of limited feedback beamforming for general MIMO systems.

Assume that codebook $\mathcal{F}$ with L beamformers is obtained by maximizing the minimum distance (6) where each vector in $\mathcal{F}$ represents a point on the N_t-dimensional complex unit hypersphere. For a large size codebook, the vector points are uniformly distributed over the surface of the unit hypersphere. On the other hand, the optimal beamformer $\mathbf{v}_1$ uniformly rotates on the unit hypersphere when channel gains are i.i.d distributed. Therefore, for a large size codebook we have:

$$\lim_{L\to\infty} |\mathbf{v}_1^* \mathbf{f}_{\mathcal{I}}| = 1 \quad , \quad \lim_{L\to\infty} |\mathbf{v}_i^* \mathbf{f}_{\mathcal{I}}| = 0\,,\; i = 2, \ldots, N_t \tag{26}$$

where $\mathbf{f}_{\mathcal{I}}$ is selected from (10) and $\mathbf{v}_i$ is the right eigenvector corresponding to the ith largest singular value of the channel. Considering (26) and the fact that $\sigma_1 \geqslant \sigma_2 \geqslant \cdots \geqslant \sigma_m$, we assume the following approximation to simplify the analysis:

$$\sum_{i=2}^{m} \sigma_i^2 |\mathbf{v}_i^* \mathbf{f}_{\mathcal{I}}|^2 \approx 0 \tag{27}$$

With this approximation, from (2) and (3) the equivalent channel model for limited feedback transmit beamforming will be

$$\tilde{s} = \sqrt{\rho}\, \lambda_1 (1 - X)\, s + z \tag{28}$$

where

$$X \triangleq 1 - |\mathbf{v}_1^* \mathbf{f}_{\mathcal{I}}|^2 = d_c^2(\mathbf{v}_1, \mathbf{f}_{\mathcal{I}})\,, \tag{29}$$

$z \sim \mathcal{CN}(0, \lambda_1(1 - X))$ and $\lambda_1 = \lambda_{max}(\mathbf{W})$. Assuming the transmitted signal s in (28) is selected from a $I \times J$ rectangular QAM constellation with unit average energy and a Gray code mapping, the average BER of the system conditioned on λ_1 and X is expressed as

$$P_{b|\lambda_1, X} = \frac{1}{\log_2(I \cdot J)} \left(\sum_{k=1}^{\log_2 I} P_{I|\lambda_1, X}(k) + \sum_{l=1}^{\log_2 J} P_{J|\lambda_1, X}(l) \right) \tag{30}$$

where

$$P_{I|\lambda_1, X}(k) = \sum_{i=0}^{(1-2^{-k})I-1} \beta_I(k, i)\, Q\Big(\sqrt{\eta(i) \lambda_1 (1 - X)}\Big)\,, \tag{31}$$

and $\beta_I(k, i)$ and $\eta(i)$ are defined in (17) and (19).

Since eigenvalues and eigenvectors of the central Wishart matrix are independent [40], therefore λ_1 and X are independent for i.i.d. channel matrix. Thus, we first average $P_{I|\lambda_1, X}(k)$ over λ_1 using the distribution of λ_1 presented in (21) to obtain

$$P_{I|X}(k) = \sum_{i=0}^{(1-2^{-k})I-1} \beta_I(k, i) \times \sum_{t=1}^{m} \sum_{j=0}^{D_t} \frac{a_{t,j}}{t^{j+1}}\; \phi\big(j + 1, t, (1 - X)\eta(i)\big) \tag{32}$$

where $\phi\big(j + 1, t, (1 - X)\eta(i)\big)$, defined in (24), should be averaged over the distribution of X.

To obtain the distribution of X, we use the geometrical method presented in [8]. For each vector $\mathbf{f}_k \in \mathcal{F}$, a spherical cap is defined on the surface of the hypersphere $S_k(x) = \big\{\mathbf{v}_1 \mid d_c^2(\mathbf{v}_1, \mathbf{f}_k) \leqslant x\big\}$ where $0 \leqslant x \leqslant 1$. By defining $A\{S_k(x)\}$ as the area of the cap $S_k(x)$, it is shown [8] that

$$A\{S_k(x)\} = \frac{2\pi^{N_t} x^{N_t - 1}}{(N_t - 1)!}. \tag{33}$$

Equation (33) shows that the surface of the unit hypersphere grows exponentially with N_t. Therefore, when N_t is increased, L should be increased accordingly so that the limits in (26) and the approximation in (27) hold tightly.

According to the definition of X in (29), we have

$$F_X(x) = \Pr\Big\{ \big[d_c^2(\mathbf{v}_1, \mathbf{f}_0) \leqslant x\big] \text{ or } \big[d_c^2(\mathbf{v}_1, \mathbf{f}_1) \leqslant x\big] \text{ or } \cdots \text{ or } \big[d_c^2(\mathbf{v}_1, \mathbf{f}_{L-1}) \leqslant x\big] \Big\}$$

where $F_X(x)$ denotes the cdf of X. When the channel matrix entries are i.i.d., the optimal vector $\mathbf{v}_1$ is uniformly distributed on the surface of the unit hypersphere. Therefore, we have

$$F_X(x) = \frac{A\left\{\cup_{k=0}^{L-1} S_k(x)\right\}}{A\left\{S_k(1)\right\}}$$

where $\cup_{k=0}^{L-1} S_k(x)$ denotes the union of the regions $\{S_k(x)\}$, $k = 0, \ldots, L - 1$. Using (33) and the fact that

$$A\left\{\cup_{k=0}^{L-1} S_k(x)\right\} \leqslant \sum_{k=0}^{L-1} A\left\{S_k(x)\right\},$$

we obtain

$$F_X(x) \leqslant \frac{\sum_{k=0}^{L-1} A\left\{S_k(x)\right\}}{A\left\{S_k(1)\right\}} = L\, x^{N_t - 1}. \tag{34}$$

Finally, by taking into account that $F_X(x) \leqslant 1$, the following approximate cdf and pdf for X can be defined:

$$F_X(x) \approx \begin{cases} L\, x^{N_t - 1}, & 0 \leqslant x \leqslant X_0 \\ 1, & x > X_0 \end{cases} \tag{35}$$

$$f_X(x) = L\,(N_t - 1)\, x^{N_t - 2}, \quad 0 \leqslant x \leqslant X_0 \tag{36}$$

where $X_0 = \left(\frac{1}{L}\right)^{\frac{1}{N_t - 1}}$.

Now by using the pdf of X (36), we can calculate the average of $P_{I|X}(k)$ over the distribution of X. By defining

$$\phi_L(N, t, \alpha) = \Gamma(N) \sum_{r=0}^{N-1} \binom{N - 1 + r}{r} \frac{L(N_t - 1)}{2^{N+r}} \times \int_0^{X_0} x^{N_t - 2} \big[1 + \mu(x)\big]^r \big[1 - \mu(x)\big]^N dx \tag{37}$$

where

$$\mu(x) = \sqrt{\frac{\alpha(1 - x)}{2t + \alpha(1 - x)}}\,,$$

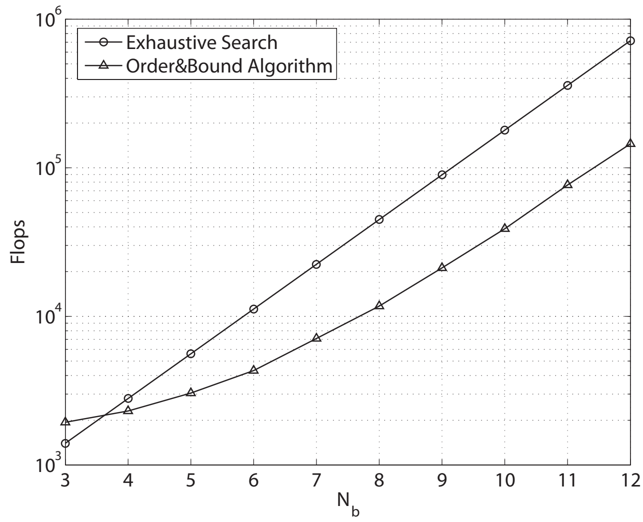

Fig. 4. Average flops of beamformer selection with exhaustive search and order and bound algorithm for MIMO(4,5) versus codebook size.

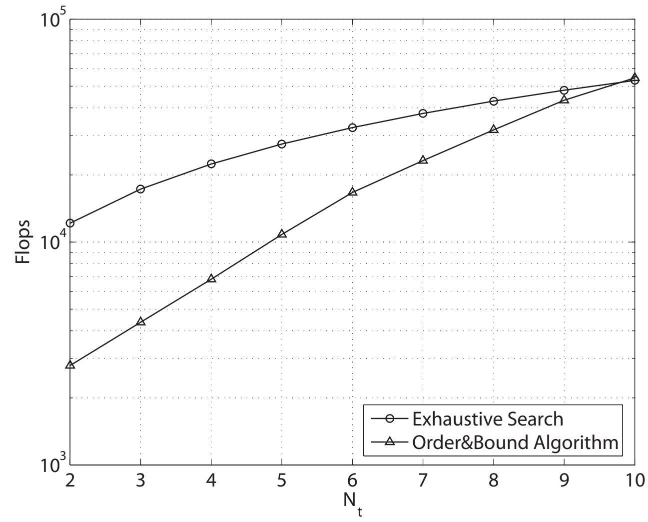

Fig. 5. Average flops of beamformer selection with exhaustive search and order and bound algorithm for MIMO with $N_r = 5$ receive antennas and $N_b = 7$ bits of feedback versus number of transmit antennas.

we conclude that

$$P_I(k, L) = \sum_{i=0}^{(1-2^{-k})I-1} \beta_I(k, i) \sum_{t=1}^{m} \sum_{j=0}^{D_t} \frac{a_{t,j}}{t^{j+1}} \phi_L\big(j+1, t, \eta(i)\big) \tag{38}$$

and consequently the approximate BER of a MIMO system with L beamformers is expressed as

$$P_b(L) \approx \frac{1}{\log_2(I \cdot J)} \left(\sum_{k=1}^{\log_2 I} P_I(k, L) + \sum_{l=1}^{\log_2 J} P_J(l, L) \right). \tag{39}$$

The accuracy of this approximate BER is verified in the next section.

VII. Simulation Results

We first verify the computational performance of the order and bound algorithm, presented in Section IV for beamformer selection at the receiver side. Fig. 4 shows the average flops (floating point operations) of beamformer selection with exhaustive search and with the order and bound algorithm. By exploiting the proposed ordering and bounding method, the beamformer selection complexity is reduced 68% for $N_b = 7$ and 80% for $N_b = 12$, in a MIMO system with $N_t = 4$ transmit antennas and $N_r = 5$ receive antennas. The flops of the order and bound algorithm includes the flops consumed by the singular value decomposition and Cholesky decomposition. Since the complexity order of both decompositions is $\mathcal{O}(N_t^3)$, for small size codebooks, specifically $N_b = 3$ in Fig. 4, the order and bound algorithm is more complex than the exhaustive search.

The complexity of exhaustive search mostly depends on the number of receive antennas, N_r, and codebook size, L, while the complexity of the order and bound algorithm mostly depends on the number of transmit antennas, N_t, and codebook size, L, and almost independent of the number of receive antennas, N_r. Therefore, due to the dependency of the order and bound algorithm on SVD and Cholesky decomposition with $\mathcal{O}(N_t^3)$ order of complexity, when the number of transmit antennas is higher than the number of receive antennas, the complexity of proposed algorithm exceeds the complexity of exhaustive search (Fig. 5). Clearly, this issue depends on the codebook size since the sensitivity of exhaustive search to codebook size is significant.

Next, the approximate BER expression of limited feedback transmit beamforming (39) is evaluated. The approximate BER of MIMO(4,2) and MIMO(4,3) systems with $L = 8, 32$ and 128 beamformers are illustrated for 4-QAM and 16-QAM in Fig. 6 and Fig. 7 (On the figures, SNR represents ρ in (1)). For comparison, the exact simulated BER and the BER of the optimal transmit beamforming are included. GLP Codebooks have been used for the simulations. Although our approximate analysis was for large size codebooks, the simulation results show that the approximate BER expression in (39) is satisfactory tight even for small size codebooks.

For small size codebooks, the approximation made in (27) is not tight and consequently the effective SNR used in the equivalent system model (28) is less than the actual SNR used by the system. Therefore, we expect the approximate BER curves for small L to be an upper bound for the curves obtained by simulations. On the other hand, for large size codebooks, the approximation in (27) is tight enough, but since we used an upper bound approximation for $F_X(x)$, going from (34) to (35), the effective SNR used in the equivalent system model (28) is larger than the actual SNR used by the system. Therefore, we expect the approximate BER curves for large L to be a lower bound for the curves obtained by simulations. These behaviors are clearly observable in Fig. 6 and Fig. 7.

VIII. Conclusions

This paper develops the genetic GLP beamformer codebooks. The design parameters are relaxed from positive integers to positive real values (angles). The simulation results show that the genetic GLP codebooks achieve a larger minimum distance than those of [10] and reduce the optimization

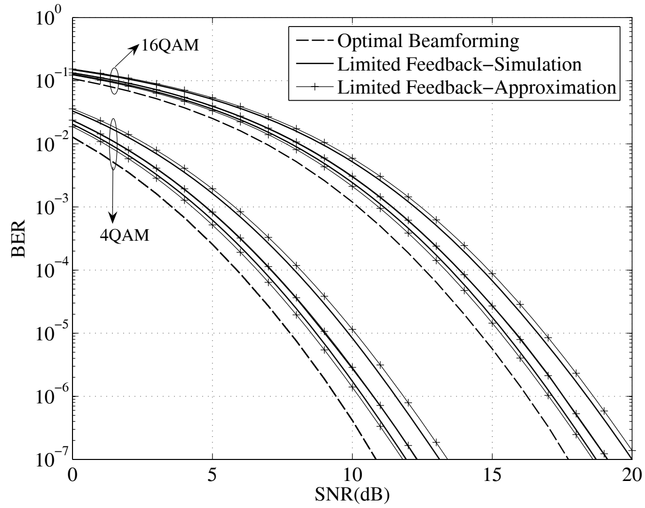

Fig. 6. BER of MIMO(4,2) system obtained from simulation and closed-form approximation for 4-QAM and 16-QAM signalings. Pair curves from left to right are for $N_b = 7, 5$ and 3 feedback bits, respectively.

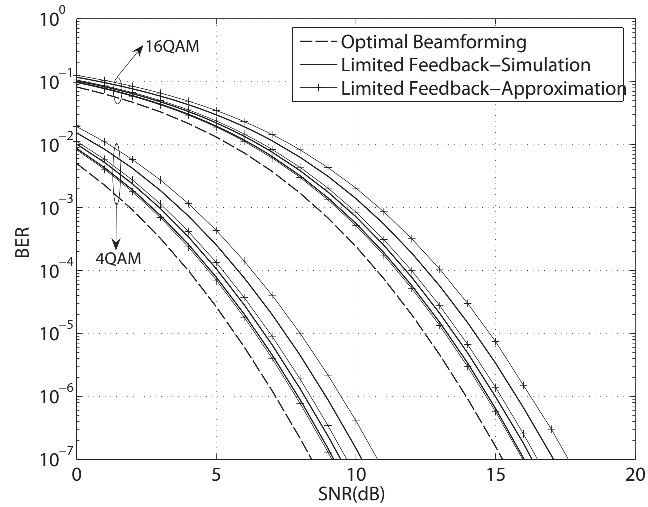

Fig. 7. BER of MIMO(4,3) system obtained from simulation and closed-form approximation for 4-QAM and 16-QAM signalings. Pair curves from left to right are for $N_b = 7, 5$ and 3 feedback bits, respectively.

complexity. By exploiting the specific structure of the GLP codebooks, the order and bound algorithm has been proposed to reduce the beamformer selection complexity at the receiver. This algorithm uses bounding techniques and avoids unnecessary computations.

By employing the singular value distribution of the channel matrix, the exact closed-form expression for the BER performance of the optimal beamforming was presented in finite summations for PAM and QAM constellations. The resulting expression was used to simplify the BER analysis of limited feedback transmit beamforming. By assuming a large size codebook (high resolution analysis) and employing a geometrical approach, an approximate BER performance for limited feedback beamforming was derived. The simulation results show that the approximate BER is satisfactorily tight even for small size codebooks.

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