Power Allocation in Wireless Relay Networks: A Geometric Programming-Based Approach

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***Abstract*—[1] In this paper, we consider an amplify-and-forward (AF) wireless relay system where multiple source nodes communicate with their corresponding destination nodes with the help of relay nodes. While each user[2] is assisted by one relay, one relay can assist many users. Conventionally, each relay node is assumed to equally distribute the available bandwidth and power resources to all sources for which it helps to relay information. Realizing the sub-optimality of this approach, in this paper, we present efficient power allocation schemes to i) maximize the minimum end-to-end signal-to-noise ratio among all users; ii) minimize the total transmit power over all sources; iii) maximize the system throughput. Our approach is based on geometric programming (GP), a well-studied class of nonlinear and nonconvex optimization. Since a GP problem is readily transformed into an equivalent convex optimization problem, optimal power allocation can be obtained efficiently. Numerical results demonstrate the effectiveness of our proposed approach.**

Index terms– **Power allocation, geometric programming, relay networks.**

I. Introduction

It has been shown that the operation efficiency and quality-of-service (QoS) of cellular and/or ad-hoc networks can be increased through the use of relay(s) [1], [2]. In such systems, the information from the source to the corresponding destination is transmitted via a direct-link and also forwarded via relays. A critical issue for improving the performance of wireless networks is efficient management of available radio resources. Particularly, resource allocation via power control is commonly used to ensure the performance and stability of the wireless network.

There have been numerous works that attempt to optimize the available communication resources, i.e., power and bandwidth to improve the system performance [7]-[10]. A single source-destination pair is typically considered in the aforementioned papers. In [7], for example, the authors derive closed-form expressions for the optimal and near-optimal relay transmission powers for the single relay and the multiple relays cases. Furthermore, the problem of minimizing the transmission power given that a target outage probability is achieved was tackled in [8]. In [9], the authors derive power allocation strategies for 3-node amplify-and-forward (AF) relaying system based on the knowledge of channel means. Given either channel state information (CSI) or channel statistics, two power allocation schemes to minimize the outage probability are presented in [10].

We note, however, that very few existing works have considered the 2-hop relay model with multiple users. The latter setup of multiple users is the more practical as compared to the previously considered configurations. Therefore, the above mentioned analysis is applicable only to a special case of the problem in hands, since each relay is usually delegated to assist more than one users, especially when the number of relays is usually (much) smaller than the number of users. An example of such scenario is the deployment of few relays in a cell at appropriate locations to assist mobile users operating in heavily scattering environment for uplink transmission. Resource allocation in a multi-user system usually has to take into account the fairness issue among users, their relative QoS requirements, channel quality and available resources. Mathematically, optimization of relay networks with multiple users is a difficult (if tractable) problem, especially for systems with large number of sources and relays.

In this paper, we develop efficient power allocation schemes for multi-user wireless relay systems. Particularly, we derive optimal power allocation schemes to i) maximize the minimum end-to-end signal-to-noise ratios (SNRs) among all users; ii) minimize the total transmit power of all sources; iii) maximize the system throughput. We show that the corresponding optimization problems can be formulated as *geometric programming* (GP) problems. Therefore, optimal power allocation can be obtained efficiently even for large-scale networks using convex optimization techniques. Note that GP has been successfully applied to solve the problem of power allocation in traditional cellular and ad hoc networks [5]–[6].

II. System Model

Consider a multi-user relaying model where a set of M source nodes S_i, $i \in \{1,...M\}$ wants to transmit data to their corresponding destination nodes D_i, $i \in \{1,...M\}$.[3] Moreover, L relay nodes, denoted by R_j, $j \in \{1,...,L\}$ are employed for forwarding the information from source to destination nodes. The conventional two-stage AF relaying

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[2]Hereafter, the term 'user' refers to a source-destination (S-D) pair or only the source node depending on the context.

[3]This includes the case of one destination node for all sources, for example, base station in a cellular network, or central processing unit in a sensor network.

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is assumed. We also assume *orthogonal transmission* using time division [10], [1], [2]. Each source S_i is assisted by one relay R_{S_i}.[4] The set of source nodes which use the relay R_j is denoted by $\mathcal{S}(R_j)$, i.e., $\mathcal{S}(R_j) = \{S_i \mid R_{S_i} = R_j\}$.

Let P_{S_i}, $P_{R_{S_i}}$ denote the power transmitted by source S_i and relay R_{S_i} corresponding to S_i-R_{S_i}-D_i link, respectively. Since unit duration time slots are assumed, P_{S_i} and $P_{R_{S_i}}$ correspond also to the average energies consumed by source S_i and relay R_{S_i}. For simplicity, we present the signal model for link S_i-R_{S_i}-D_i only. In the first time slot, source S_i transmits the signal x_i with unit energy to the relay R_{S_i}.[5] The received signal at relay R_{S_i} can be written as

$$r_{S_i R_{S_i}} = \sqrt{P_{S_i}} a_{S_i R_{S_i}} x_i + n_{R_{S_i}}$$

where $a_{S_i R_{S_i}}$ stands for the channel gain for link S_i-R_{S_i}, $n_{R_{S_i}}$ is the additive circularly symmetric white Gaussian noise (AWGN) at the relay R_{S_i} with variance $N_{R_{S_i}}$. The channel gain includes the effects of path loss, shadowing and fading. We assume that the relay R_{S_i} knows the CSI for link S_i-R_{S_i}. If the AF protocol is used, i.e., the signal received by a relay node is normalized and retransmits to the destination node D_i, the received signal at the destination node D_i can be expressed as

$$\begin{aligned} r_{D_i} &= \sqrt{P_{R_{S_i}}} a_{R_{S_i} D_i} \frac{r_{S_i R_{S_i}}}{\sqrt{E\{|r_{S_i R_{S_i}}|^2\}}} + n_{D_i} \\ &= \sqrt{\frac{P_{R_{S_i}} P_{S_i}}{P_{S_i} |a_{S_i R_{S_i}}|^2 + N_{R_{S_i}}}} a_{R_{S_i} D_i} a_{S_i R_{S_i}} x_i + \hat{x}_{D_i} \end{aligned}$$

where $E\{.\}$ denotes statistical expectation operator, $a_{R_{S_i} D_i}$ is the channel coefficient for link R_{S_i}-D_i, n_{D_i} is the AWGN at the destination node D_i with variance N_{D_i}, $\hat{x}_{D_i}$ is the modified AWGN noise at D_i with equivalent variance $N_{D_i} + \big(P_{R_{S_i}} |a_{R_{S_i} D_i}|^2 N_{R_{S_i}}\big) / \big(P_{S_i} |a_{S_i R_{S_i}}|^2 + N_{R_{S_i}}\big)$.

The equivalent end-to-end SNR of the virtual channel between the nodes S_i and D_i can be written as [10]

$$\begin{aligned} \gamma_i &= \frac{P_{R_{S_i}} P_{S_i} |a_{R_{S_i} D_i}|^2 |a_{S_i R_{S_i}}|^2}{P_{S_i} |a_{S_i R_{S_i}}|^2 N_{D_i} + P_{R_{S_i}} |a_{R_{S_i} D_i}|^2 N_{R_{S_i}} + N_{D_i} N_{R_{S_i}}} \\ &= \frac{P_{S_i} P_{R_{S_i}}}{\eta_i P_{S_i} + \alpha_i P_{R_{S_i}} + \beta_i} \end{aligned}$$

where

$$\eta_i = \frac{N_{D_i}}{a^2_{R_{S_i} D_i}}, \quad \alpha_i = \frac{N_{R_{S_i}}}{a^2_{S_i R_{S_i}}}, \quad \beta_i = \frac{N_{R_{S_i}} N_{D_i}}{a^2_{S_i R_{S_i}} a^2_{R_{S_j} D_i}}.$$

It can be seen that for fixed $P_{R_{S_i}}$, γ_i is a concave increasing function of P_{S_i}. However, no matter how large P_{S_i} is, the maximum achievable γ_i can be shown to be equal to $P_{R_{S_i}}/\eta_i$. Vice versa, when P_{S_i} is fixed, γ_i is a concave increasing function of $P_{R_{S_i}}$ and the corresponding maximum achievable γ_i is P_{S_i}/α_i. Moreover, since γ_i is concave increasing on P_{S_i}, the incremental change in γ_i is smaller for large P_{S_i}, and, therefore, γ_i is monotone. Note that monotonicity is a useful property helping to provide some insights into optimization problems at optimality.

We assume a central unit (CU) to coordinate the power allocation at the sources and at the relays. For such purpose, the CU should have CSI for all the transmission links, i.e., source-relay and relay-destination links. The power allocation factors can be communicated to relays and sources via a secured channel. The sources and relays then adjust their transmit power accordingly. A slow fading environment is also assumed. The letter assumption corresponds to networks with stationary topology or low-mobility users.

III. Problem formulations

In this section, we propose power allocation schemes for multi-user wireless relay systems.

A. *Max-min SNR based allocation*

Power control in wireless networks often has to take into account the fairness among different users. In other words, the performance of the worst user(s), i.e., the user(s) with smallest end-to-end SNR, is often of concern to the network operator. The traditionally used maximum sum SNR power allocation is biased towards users that have the best channel quality and is unfair to the other links. Here we consider the max-min fair power allocation problem which aims at maximizing the minimum SNR over all users.[6] The problem can be mathematically posed as

$$\max_{P_{S_i},\, P_{R_{S_i}}} \quad \min_{i=1,\dots,M} \gamma_i(P_{S_i}, P_{R_{S_i}}) \tag{1a}$$

$$\text{subject to:} \quad \sum_{S_i \in \mathcal{S}(R_j)} P_{R_{S_i}} \le P^{\max}_{R_j}, \; j = 1, \dots, L \tag{1b}$$

$$\sum_{i=1}^{M} P_{S_i} \le P \tag{1c}$$

$$0 \le P_{S_i} \le P^{\max}_{S_i}, \; \forall S_i, \; i \in \{1, ...M\} \tag{1d}$$

where $P^{\max}_{R_j}$ is the total power available at the relay node R_j and P is the total power allocated to all sources. The right-hand side of (1b) is the total power that the relay R_j allocates to the users which it assists. This power is constrained to be less than the relay's total power. Similarly, constraints (1d) specify the peak power limit $P^{\max}_{S_i}$ for each source S_i.

It can be seen that the set of linear inequality constraints with positive variables in the optimization problem (1a)–(1d) is compact and nonempty. Hence, the problem (1a)–(1d) is always feasible. Moreover, since the objective function $\min_{i=1,\dots,M} \gamma_i$ is an increasing function of the allocated powers P_{S_i} and $P_{R_{S_i}}$, the inequality constraints (1b), (1c) must be met with equality at optimality. It can be observed that while the performance of user i depends only on the allocated powers P_{S_i} and $P_{R_{S_i}}$, the performance of all users interact

[4] The single relay assignment may be done during the connection setup phase, or by relay selection process [10].

[5] We consider the case in which the source-to-relay link is (much) stronger than the source-to-destination link, that is usual scenario in practice.

[6] In this way, the minimum data rate among users is also maximized since data rate is a monotonic increasing function of SNR.

with each other via shared and limited power resource at the relays and the sources. Therefore, proper power allocation among users is necessary to maximize a specific criterion on the system performance. Resource allocation in a multi-user network is not as simple as allocation of resources for each user individually, albeit orthogonal transmissions are assumed.

B. Power minimization based allocation

In practically used wireless networks, one of the targets of power allocation is to prolong the lifetime of battery-powered devices since nodes with long lifetime help to ensure uninterrupted information exchange. Therefore, the minimization of the power consumption of the source nodes in relay systems is particularly important. The transmit power minimization problem subject to constraints on the end-to-end SNR for each user can be formulated as follows

$$\min_{P_{S_i},\, P_{R_{S_i}}} \quad \sum_{i=1}^{M} P_{S_i} \tag{2a}$$

$$\text{subject to:} \quad \gamma_i \geq \gamma_i^{\min},\ i=1,\ldots,M \tag{2b}$$

$$\text{The constraints (1b), (1d)} \tag{2c}$$

where $\gamma_i^{\min}$ is the threshold SNR for ith user.[7]

In the problem (2a)–(2c), the relays are assumed to be power-limited but energy-unlimited. This problem answers the question of how to exploit efficiently the available power resource at the relays to minimize the power consumption at the battery-powered nodes. It can be seen that under the assumption of energy-unlimited relays the optimization problem (2a)–(2c) is always feasible. It also can be observed that at optimality, the inequality constraints (2b), (1b) in (2c) must be met with equality. This is because γ_i is an increasing function of P_{S_i} and $P_{R_{S_i}}$. In order to minimize the objective function in (2a)–(2c), P_{S_i}, γ_i and $P_{R_{S_i}}$ must attain their minimum and maximum values, respectively. Finally, note that we have implicitly assumed in (2a)–(2c) that none of the source needs transmits more than $P_{S_i}^{\max}$ at optimality.

C. Throughput maximization based allocation

The data rate $\mathcal{R}_i$ of the ith S-D pair can be written as a function of γ_i as

$$\mathcal{R}_i = \frac{1}{T}\log_2(1+K\gamma_i) \approx \frac{1}{T}\log_2(K\gamma_i)$$

where T is the symbol period, $K = \frac{-\zeta_1}{\ln(\zeta_2 \mathrm{BER})}$, BER is the target bit error rate, and $\zeta_1,\ \zeta_2$ are constants dependent on the modulation scheme [3]. Note that we have approximated $1+K\gamma_i$ as $K\gamma_i$. For notational simplicity we also set $K=1$. Then, the aggregate throughput for the system can be written as [6]

$$R_{\text{sys}} = \sum_{i=1}^{M} \mathcal{R}_i \approx \frac{1}{T}\ log_2\Big[\prod_{i=1}^{M}\gamma_i\Big].$$

[7]We assume that the threshold $\gamma_i^{\min}$ is not larger than the maximum achievable SNR for user i as discussed above.

The power allocation problem to maximize the system throughput can be mathematically posed as

$$\max_{P_{S_i},\, P_{R_{S_i}}} \quad R_{\text{sys}} = \frac{1}{T}\log_2\Big[\prod_{i=1}^{M}\gamma_i\Big] \tag{3a}$$

$$\text{subject to:} \quad \text{The constraints (1b), (1c), (1d).} \tag{3b}$$

Therefore, in the high SNR region, maximizing network throughput can be approximately replaced by maximizing the product of SNRs.[8] At optimality, the inequality constraints (1b), (1c) in (3b) of the problem (3a)–(3b) must be met with equality.

IV. Power Allocation in Relay Networks via GP

GP is a well-investigated class of nonlinear, nonconvex optimization problems with attractive theoretical and computational properties [5], [6]. Since equivalent convex reformulation is possible for a GP problem, only global optimum exists. Moreover, the availability of large-scale software solvers makes GP more appealing for practical use.

A. Max-min SNR based allocation

Introducing a new variable t, the optimization problem (1a)–(1c) can be equivalently rewritten as

$$\min_{P_{S_i},\, P_{R_{S_i}},\, t\geq 0} \quad \frac{1}{t} \tag{4a}$$

$$\text{subject to:} \quad \frac{P_{S_i}P_{R_{S_i}}}{\eta_i P_{S_i} + \alpha_i P_{R_{S_i}} + \beta_i} \geq t, i=1,\ldots,M \tag{4b}$$

$$\text{The constraints (1b), (1c), (1d).} \tag{4c}$$

The objective function in the problem (4a)–(4c) is a monomial function. Moreover, the constraints in (4b) can be easily converted into posynomial constraints. The constraints (1b), (1c), (1d) are linear with respect to the power variables, and thus, are posynomial constraints. Therefore, the optimization problem (4a)–(4c) is a GP problem.

B. Power minimization based allocation

In this case, the objective function is clearly a posynomial function. The constraints can also be written as posynomial ones. Therefore, the power minimization based allocation is a GP problem.

C. Throughput maximization based allocation

A simple manipulation of the optimization problem (3a)–(3b) gives

$$\min_{P_{S_i},\, P_{R_{S_i}}} \quad \frac{1}{\prod_{i=1}^{M}\gamma_i} \tag{5a}$$

$$\text{subject to:} \quad \text{The constraints (1b), (1c), (1d).} \tag{5b}$$

The objective function in this problem can be shown to be a posynomial. Moreover, the constraint can also be easily converted into posynomial constraints as for the previous

[8]Note, however, that in the low SNR region, the approximation of $1+\gamma_i$ by γ_i does not hold satisfactorily, and therefore, will not lead to accurate results.

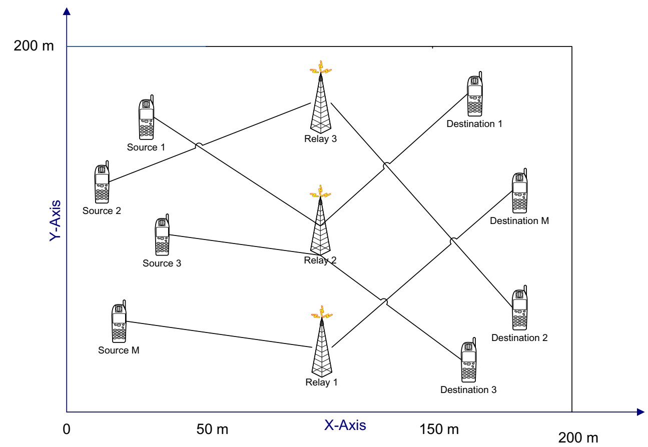

Fig. 1. A wireless multi-user relay system.

problems. Therefore, the optimization problem (5a)–(5b) also belongs to the GP class.

We have shown that the three aforementioned power allocation schemes can be reformulated as GP problems. The proposed optimization problems with distinct features of relaying model are mathematically similar to the ones in [6] developed for conventional cellular networks. However, the numerator and denominator of the SNR expression for each user considered in [6] are linear functions of the power variables which is not the case in our work.

V. Simulation Results

Consider a wireless relay network shown in Fig. 1 with ten users and three relays distributed in a two-dimensional region $200m \times 200m$. The relays are fixed at coordinates (100,50), (100,100), and (100,150). The ten source nodes and their corresponding destination nodes are deployed randomly in the area inside the box area $[(0,0),(50,200)]$ and $[(150,0),(200,200)]$ respectively. In our simulation, each source is assisted by a random (and then fixed) relay. We assume that there is no microscopic fading and the gain for each transmission link is computed using the path loss model as $a = 1/d$ where d is the Euclidean distance between two transmission ends. The noise power is assumed to be equal to $N_0 = -50$ dB. Although each relay node may assist different number of users, which are assumed to have the same maximum power level $P_{R_j}^{\max}$. Similarly, all users are also assumed to have the equal minimum SNR thresholds $\gamma^{\min}$. The software package for solving convex programs, which can be downloaded at [11], is used in our simulations.

Figs. 2 and 3 show the minimum rate among all users and the network throughput, i.e., the sum of users' rates, when the maximum power levels of the relays $P_{R_j}^{\max}$ and sources P are varied. The performance of the equal power allocation (EPA) scheme is also plotted for comparison. According to the EPA scheme the power is allocated equally among all sources, i.e., $P_{S_i} = P/10,\ \forall S_i$ and each relay distributes power equally among all users which it assists. For $P = 50$ (see Fig. 2), the optimal power allocation (OPA) scheme achieves about

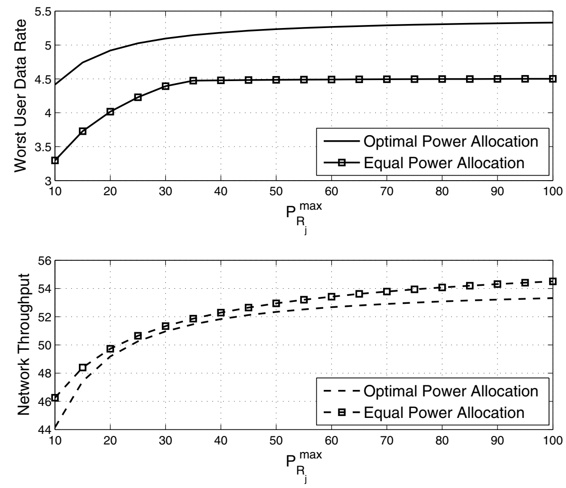

Fig. 2. Data rate versus $P_{R_j}^{\max}$, $P = 50$.

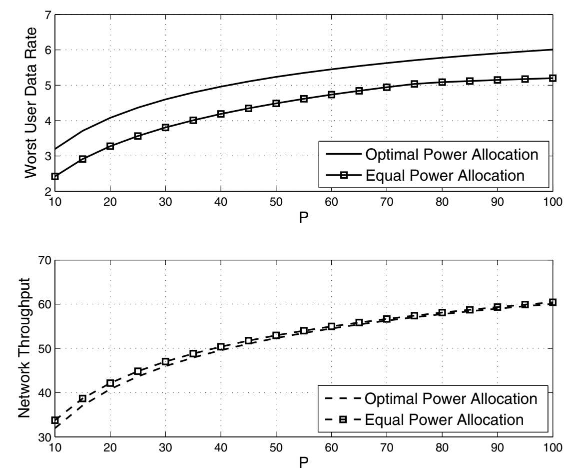

Fig. 3. Data rate versus P, $P_{R_j}^{\max} = 50$.

0.8 bits performance improvement over the EPA scheme for the worst user data rate. The performance improvement of both schemes is higher when $P_{R_j}^{\max}$ is small (less than 30). The EPA scheme provides a slight performance improvement for the worst user(s) for $P_{R_j}^{\max} \geq 35$. However, the OPA scheme is able to take advantage from larger $P_{R_j}^{\max}$. This demonstrates the effectiveness of OPA scheme in general and our proposed approach in particular. In Fig. 3, we fix $P_{R_j}^{\max} = 50$. It can be seen that the OPA scheme also outperforms the EPA scheme. The improvement is about 0.8 bits and increases when P increases. In both scenarios, there is a loss in the network throughput since our objective is to improve the performance of the worst user(s). This confirms the well-known fact that achieving max-min fairness among users usually results in performance loss for the whole system.

Fig. 4 displays the total power consumed by source nodes in two scenarios. The first scenario is to attain a minimum

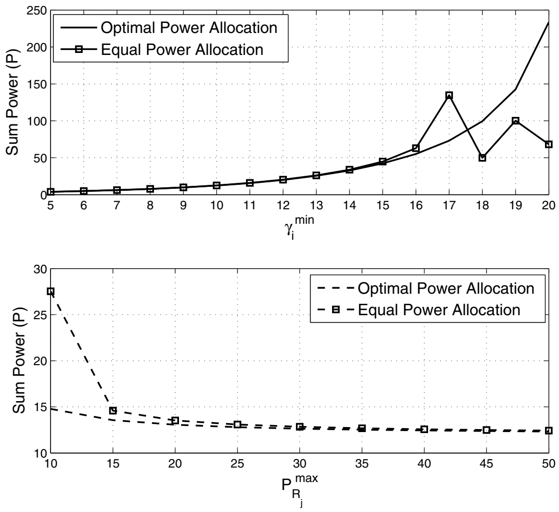

Fig. 4. Total power of sources.

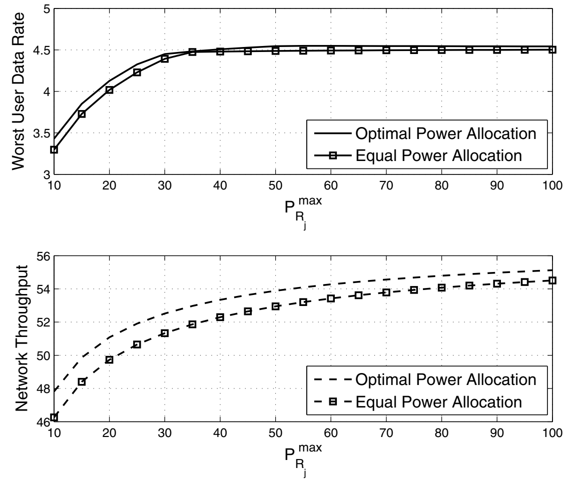

Fig. 5. Network throughput versus $P_{R_j}^{\max}$, $P = 50$.

SNR $\gamma^{\min}$ with fixed $P_{R_j}^{\max} = 50$, while in the second scenario $P_{R_j}^{\max}$ is varied when $\gamma^{\min} = 10$ dB is fixed. It can be seen that, in the first scenario, the OPA scheme allocates less required power than that of the EPA scheme when $\gamma^{\min} \leq 17$ dB. Moreover, when $\gamma^{\min} \geq 18$ dB, EPA scheme can not find a feasible power allocation (in fact, negative power allocation), represented by weird part in the EPA curve. It is because the threshold $\gamma^{\min} \geq 18$ dB exceeds the maximum value of γ_i for some users. It can be seen that by proper power distribution at the relays, the OPA scheme can find power allocation to achieve larger target SNR $\gamma^{\min}$. This further demonstrates the advantage of our proposed approach over the EPA scheme. For the second scenario, the OPA scheme requires less sum power than that of the EPA scheme, especially when $P_{R_j}^{\max}$ is small. It can be observed that as there is more available power $P_{R_j}^{\max}$, less sum power is required to achieve a target SNR.

In the last example, the OPA is used to maximize the sum users' throughput. Fig. 5 shows the performance of our proposed approach versus $P_{R_j}^{\max}$ when $P = 50$. The OPA scheme outperforms the EPA for all values of $P_{R_j}^{\max}$. It is noticeable that the OPA scheme achieves better performance in terms of both worst user data rate and network throughput. Comparing with the results in Figs. 2 and 3, we can see the tradeoff between achieving fairness and the sum throughput.

VI. Conclusion

In this paper, optimal power allocation schemes have been proposed for wireless relay networks. AF relaying model is assumed where each of the source nodes communicates with its corresponding destination node with the help of one relay node. The proposed approach is based on GP. Although GP is nonconvex, it allows for an equivalent convex reformulation which provides an efficient method for obtaining optimal solution. In particular, we presented power allocation schemes to i) maximize the minimum end-to-end SNR for all source-destination pairs; ii) minimize the total transmit power over all sources; iii) maximize the system throughput. Simulation results demonstrate the effectiveness of the proposed approachs over the EPA scheme.

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