A novel complementary sequence and PMEPR reduction in OFDM

Nouvelle séquence complémentaire et réduction du PMEPR pour les signaux OFDM

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This paper introduces a novel sequence called the "β-complementary sequence" to encode orthogonal frequency-division multiplexing (OFDM) signals, with which one can substantially increase the code rate while maintaining a tight peak-to-mean envelope power ratio (PMEPR) of at most 2. Moreover, encoding schemes based on β-complementary sequences provide very good tradeoff performance between PMEPR and code rate. This observation follows from an intrinsic property of the β-complementary sequences considered. Furthermore, the distribution of code rate versus β is determined, and numerical results based on these properties are presented. Finally, comparisons with the well-known Golay complementary sequences and the generalized Golay complementary sequences (called "G_N-complementary sequences" in this paper) are made.

On présente dans cet article une nouvelle séquence dénommée "séquence β-complémentaire" pour le codage des signaux OFDM, qui permet d'accroître de manière substantielle le taux de code tout en jouissant d'une valeur PMEPR (rapport puissance de crête–puissance moyenne d'enveloppe) resserrée d'au plus 2. En plus, les mécanismes de codage basés sur les séquences β-complémentaires permettent de réaliser un bon compromis entre les performances PMEPR et le taux de code. Cette observation se base sur une propriété intrinsèque des séquences β-complémentaires examinées. En plus de cela, on détermine la distribution du taux de code par rapport à β et on présente les résultats numériques basés sur ces propriétés. Finalement, on compare le nouveau modèle aux séquences complémentaires Golay bien connues et à la séquence complémentaire Golay généralisée (dénommée "séquence complémentaire G_N" dans cet article).

I Introduction

In multi-carrier communications, orthogonal frequency-division multiplexing (OFDM) has been used for wireless large-area networks (LANs) by the committees for the international standards IEEE 802.11 and ETSI BRAN, since it provides great immunity to impulse noise and fast fades and eliminates the need for equalizers while enabling efficient hardware implementations using fast Fourier transform (FFT).

However, a major drawback of OFDM is the high peak-to-mean envelope power ratio (PMEPR) of the uncoded OFDM signal. Numbers of PMEPR reduction schemes based on the oversampled sequence have been proposed [1]–[6]. On the other hand, several coding schemes to reduce the PMEPR of the OFDM waveform have been studied in [7]–[12]. Some of them enjoy the large Euclidean distance and an efficient soft-decision decoding algorithm.

One idea introduced in [13] and developed in [14] is to use the Golay complementary sequences [15] to encode the OFDM signals with PMEPR of at most 2. Recently, [16] made further advances on this work and observed that the 2^h-ary Golay complementary sequences of length 2^m [15] can be obtained from certain second-order cosets of the classical first-order Reed-Muller code. As a result of this intrinsic observation, [7] was able to obtain, for a small number of carriers ($n \leq 32$), a range of binary, quaternary, and actuary OFDM codes with good error-correcting capabilities, efficient encoding and decoding, reasonable code rates, and controlled PMEPR of at most 2. Follow-up work in [17] investigated the tradeoffs between code rate and PMEPR by using the complementary set [18], which determined the codes with PMEPR of at most the exponential of 2 in the second-order Reed-Muller code. Since this tradeoff only slightly increases the code rate by relaxing the PMEPR, it remains an open problem to discover low-PMEPR error-correcting code constructions for a moderately large subcarrier.

In this paper, we introduce a novel sequence called the "β-complementary sequence" to encode the OFDM signal in a way that substantially increases the code rate. At the same time, such code also enjoys good performance in terms of the tradeoff between code rate and PMEPR. We first investigate the properties of β-complementary sequences and determine the distribution of code rate versus β. Then we make comparisons with the well-known Golay complementary sequences and the generalized Golay complementary sequences (called "G_N-complementary sequences" in this paper).

II β-complementary sequences and power control

Before proceeding further, let us introduce the OFDM signals, the PMEPR, and related concepts.

II.A Preliminaries

Let j be the imaginary unit, i.e., $j^2 = -1$. Then the n-subcarrier complex baseband OFDM signal can be represented as

$$s(t) = \sum_{k=-\infty}^{\infty} \sum_{\ell=0}^{n-1} c_{k,\ell} e^{j2\pi(f_c+\ell\Delta f)t} g[t - k(T+T_g)], \qquad (1)$$

where $0 \leq t < T$; $c_{k,\ell}$ is the data symbol for the ℓ-th subcarrier and the k-th OFDM symbol; the frequency separation between any two adjacent subcarriers is $\Delta f = 1/T$; and f_c is the carrying frequency, which is much larger than Δf ($f_c \gg \Delta f$). The unit rectangular pulse $g(t)$ is of duration $T + T_g$, where T_g is known as the guard

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interval. For the PMEPR problem, it is enough to consider only a single OFDM symbol, since there is almost no overlap between different OFDM symbols. In practice, filtering can cause some intersymbol interference, but this will be ignored in this discussion. The guard interval T_g is used to repeat parts of each OFDM signal and has no effect on PMEPR. For convenience of discussion, we may set T_g equal to 0 and $c_{0,\ell}$ equal to c_ℓ.

For an M-ary phase modulation OFDM, $c_\ell \in \xi^{\mathbb{Z}_M} = \{\xi^k : k \in \mathbb{Z}_M\}$, where $\xi = \exp(2\pi j/M)$ and $\mathbb{Z}_d = \{0, \ldots, M-1\}$. Let $\theta = 2\pi\Delta f t$. Suppose that $f_c = K\Delta f$ for some large $K \in \mathbb{Z}$, the integer set. Then, for a codeword $b = (b_0, \ldots, b_{n-1})$ with $b_\ell \in \xi^{\mathbb{Z}_M}$, the complex envelope (1) can be reduced to

$$s_b(\theta) = \sum_{\ell=K}^{K+n-1} b_\ell e^{j\ell\theta}, \quad \theta \in [0, 2\pi). \tag{2}$$

The instantaneous power of the complex envelope $s_b(\theta)$ is defined by

$$P_b(\theta) = |s_b(\theta)|^2. \tag{3}$$

Therefore the PMEPR of the codeword b is defined by

$$\mathrm{PMEPR}(b) = \frac{1}{n} \sup_{0 \le \theta < 2\pi} |s_b(\theta)|^2. \tag{4}$$

Obviously, in such a PMEPR problem, one can assume that $K = 0$.

A $\xi^{\mathbb{Z}_M}$-sequence a of length n is called a Golay complementary sequence [15] if there is a $\xi^{\mathbb{Z}_M}$-sequence b of length n such that

$$P_a(\theta) + P_b(\theta) = 2n. \tag{5}$$

It is easy to see that $\mathrm{PMEPR}(a) \le 2$ if a is a Golay complementary sequence. A generalization of the Golay complementary sequence is the complementary set [18]. A set of $\xi^{\mathbb{Z}_M}$-sequences $a^0, \ldots, a^{N-1}$ of length n is said to be a complementary set if

$$P_{a^0}(\theta) + \cdots + P_{a^{N-1}}(\theta) = Nn. \tag{6}$$

Any sequence in the complementary set is called a G_N- or $G_N(n)$-complementary sequence. Obviously, $\mathrm{PMEPR}(a) \le N$ if a is a G_N-complementary sequence, and any G_2-complementary sequence is a Golay complementary sequence. In this paper, we will further generalize the G_N-complementary sequences to substantially increase the code rate.

II.B β-complementary sequences

In a Golay complementary set, one actually uses the sequence $a^0 \in \xi^{\mathbb{Z}_M}$ only to encode the OFDM signals. Therefore the other sequences $a^1, \ldots, a^{N-1}$ can be any sequence in $\mathbb{C}^n$, the n-tuple complex number. This observation is one of the primary motivations for introducing the β-complementary sequence. Note that $s_b(\theta)$, defined in (2), and $P_b(\theta)$, defined in (3), are also well defined for any sequence $b \in \mathbb{C}^n$. We therefore give the following definition.

DEFINITION 1: *A $\xi^{\mathbb{Z}_M}$-sequence a of length n is said to be a β- or $\beta(n)$-complementary sequence for some $\beta \ge 1$ if there is a sequence $b \in \mathbb{C}^n$ such that*

$$P_a(\theta) + P_b(\theta) = \beta n.$$

It is easy to see that $\mathrm{PMEPR}(a) \le \beta$ if a is a β-complementary sequence. In the following, we will also show an intrinsic property of β-complementary sequences which signifies that any $\xi^{\mathbb{Z}_M}$-sequence a, where $\mathrm{PMEPR}(a) \le \beta$, is a β-complementary sequence. First we show some fundamental properties of β-complementary sequences.

II.C Some fundamental properties of β-complementary sequences

For a sequence $a = (a_0, \ldots, a_{n-1}) \in \mathbb{C}^n$, the aperiodic autocorrelation of a is defined as

$$R_a(\ell) = \begin{cases} \sum_{k=0}^{n-\ell-1} a_{k+\ell}\bar{a}_k, & 0 \le \ell < n, \\ 0, & \text{otherwise}, \end{cases}$$

where $\bar{a}_k$ is the complex conjugate of a_k. Immediately we can use the aperiodic autocorrelation to describe β-complementary sequences.

PROPOSITION 1: *A $\xi^{\mathbb{Z}_M}$-sequence is a $\beta(n)$-complementary sequence if and only if there is a sequence $b \in \mathbb{C}^n$ such that $R_b(0) = (\beta-1)n$ and $R_a(\ell) + R_b(\ell) = 0$ for $\ell = 1, \ldots, n-1$.*

PROOF: For any $b \in \mathbb{C}^n$,

$$P_b(\theta) = R_b(0) + \sum_{\ell=1}^{n-1} \left[R_b(\ell)e^{j\ell\theta} + \bar{R}_b(\ell)e^{-j\ell\theta} \right].$$

Denote the real and imaginary parts of $R_b(\theta)$ by $R_b^R(\theta)$ and $R_b^I(\theta)$ respectively. Then

$$P_b(\theta) = R_b(0) + 2\sum_{\ell=1}^{n-1} \left[R_b^R(\ell)\cos\ell\theta + R_b^I(\ell)\sin\ell\theta \right].$$

Since $\{1, \cos\theta, \ldots, \cos(n-1)\theta, \sin\theta, \ldots, \sin(n-1)\theta\}$ is an orthogonal system in the square integrable function space ($L^2[0, 2\pi]$), we conclude that the condition $P_a(\theta) + P_b(\theta) = \beta n$ is equivalent to the conditions $R_a(0) + R_b(0) = \beta n$ and $R_a^R(\ell) + R_b^R(\ell) = 0 = R_a^I(\ell) + R_b^I(\ell)$ for $\ell = 1, \ldots, n-1$. By noting that $R_a(0) = n$ when a is a $\xi^{\mathbb{Z}_M}$-sequence, we have the equivalent conditions $R_b(0) = (\beta-1)n$ and $R_a(\ell) + R_b(\ell) = 0$ for $\ell = 1, \ldots, n-1$. This completes the proof. □

The following property indicates how to produce new β-complementary sequences from a known β-complementary sequence.

PROPOSITION 2: *If $a = (a_0, \ldots, a_{n-1})$ is a β-complementary sequence, then for any $\zeta \in \xi^{\mathbb{Z}_M}$, $\zeta a = (\zeta a_0, \ldots, \zeta a_{n-1})$, $\bar{a}^r = (\bar{a}_{n-1}, \ldots, \bar{a}_0)$, $\bar{a} = (\bar{a}_0, \ldots, \bar{a}_{n-1})$, $a^r = (a_{n-1}, \ldots, a_0)$, and $a^\zeta = (a_0\zeta^0, \ldots, a_{n-1}\zeta^{n-1})$ are also β-complementary sequences.*

PROOF: Since

$$P_{\zeta a}(\theta) = \left| \sum_{\ell=0}^{n-1} \zeta a_\ell e^{i\ell\theta} \right|^2 = \left| \sum_{\ell=0}^{n-1} a_\ell e^{i\ell\theta} \right|^2 = P_a(\theta)$$

and

$$\begin{aligned} P_{\bar{a}^r}(\theta) &= \left| \sum_{\ell=0}^{n-1} \bar{a}_{n-\ell-1} e^{j\ell\theta} \right|^2 \\ &= \left| \sum_{\ell=0}^{n-1} \bar{a}_\ell e^{j(n-\ell-1)\theta} \right|^2 \\ &= \left| \sum_{\ell=0}^{n-1} \bar{a}_\ell e^{-j\ell\theta} \right|^2 \\ &= \left| \sum_{\ell=0}^{n-1} a_\ell e^{j\ell\theta} \right|^2 = P_a(\theta), \end{aligned}$$

it is easy to see that ζa and $\bar{a}^r$ are also β-complementary sequences by the definition of β-complementary sequences. On the other hand,

$$P_{\bar{a}}(\theta) = \left| \sum_{\ell=0}^{n-1} \bar{a}_\ell e^{j\ell\theta} \right|^2 = \left| \sum_{\ell=0}^{n-1} a_\ell e^{-j\ell\theta} \right|^2 = P_a(-\theta).$$

This shows that $P_{\bar{a}}(-\theta) + P_{\bar{b}}(-\theta) = 2n$ for some $b \in \mathbb{C}^n$ if a is a β-complementary sequence, and therefore $\bar{a}$ is also a β-complementary sequence. Immediately, this implies that $a^r = \overline{\bar{a}^r}$ is a β-complementary sequence. Suppose that $\zeta = e^{jd/M}$ for some

$d \in \mathbb{Z}_M$. Then

$$\begin{aligned} P_{a^\zeta}(\theta) &= \left|\sum_{\ell=0}^{n-1} \zeta^\ell a_\ell e^{j\ell\theta}\right|^2 \\ &= \left|\sum_{\ell=0}^{n-1} a_\ell e^{j\ell(\theta+d/M)}\right|^2 \\ &= P_a\left(\theta + \frac{d}{M}\right). \end{aligned}$$

Therefore $P_a(\theta + d/M) + P_b(\theta + d/M) = \beta n$ if a is a β-complementary sequence, and therefore a^ζ is also a β-complementary sequence. This completes the proof. □

Using Proposition 2, we can investigate the combination of β-complementary sequences.

Proposition 3: *If a $\xi^{\mathbb{Z}_m}$-sequence a is a $\beta(n)$-complementary sequence, then the concatenated sequence $a \vee a = (a_0, \ldots, a_{n-1} a_0, \ldots, a_{n-1})$ and the interleaved sequence $a \wedge a = (a_0, a_0, \ldots, a_{n-1}, a_{n-1})$ are $2\beta(2n)$-complementary sequences.*

Proof: There is a $b \in \mathbb{C}^n$ such that $R_b(0) = (\beta-1)n$ and $R_a(\ell) + R_b(\ell) = 0$ for $1 \le \ell \le n-1$, since, for $0 \le \ell \le n-1$,

$$R_{b\vee b}(\ell) = 2R_b(\ell) + R_b(n-\ell)$$

and, for $n \le \ell \le 2n-1$,

$$R_{b\vee b}(\ell) = R_b(\ell - n).$$

This completes the proof. □

II.D An intrinsic property of binary β-complementary sequences

If we restrict ourselves to binary phase-shift keying (BPSK), i.e., to the $\xi^{\mathbb{Z}_2}$-sequences, we find that the β-complementary sequence comprises all sequences with PMEPR of at most β. However, we cannot yet extend this result to M-ary phase-shift keying (MPSK).

Theorem 1: *A $\xi^{\mathbb{Z}_2}$-sequence a of length n is a β-complementary sequence if and only if $\mathrm{PMEPR}(a) \le \beta$.*

Proof: We need only to show that a is a β-complementary sequence if $\mathrm{PMEPR}(a) \le \beta$, since the necessity requirement has been demonstrated in Subsection II.B. We separate the proof into several steps.

1. Given $\mathrm{PMEPR}(a) \le \beta$, we have $P_a(\theta) \le \beta n$. Since $P_a(\theta) = n + 2\sum_{\ell=1}^{n-1} R_a(\ell)\cos \ell\theta$, we can write the non-negative trigonometric polynomial $\beta n - P_a(\theta) = p(\cos\theta)$, where p is a polynomial of degree $n-1$ with real coefficients. This polynomial can be factored as
$$p(c) = \alpha \prod_{k=1}^{n-1} (c - c_k),$$
where the zeros, c_k, of p appear either in complex duplets (c_k, $\bar{c}_k$), or in real singlets since $p = \bar{p}$. Meanwhile, we can also write
$$\beta n - P_a(\theta) = e^{jn\theta} P(e^{-j\theta}),$$
where P is a polynomial of degree $2(n-1)$. For $|z| = 1$, we have
$$\begin{aligned} P(z) &= z^n \alpha \prod_{k=1}^{n-1} \left(\frac{z+z^{-1}}{2} - c_k\right) \\ &= \alpha \prod_{k=1}^{n-1} \left(\frac{1}{2} - c_k z + \frac{1}{2} z^2\right). \end{aligned}$$
The two polynomials in the right- and left-hand sides also agree for all $z \in \mathbb{C}$ since a polynomial is an analytic function.
2. If c_k is real, then the zeros of $1/2 - c_k z + (1/2)z^2$ are $c_k \pm \sqrt{c_k^2 - 1}$. For $|c_k| \ge 1$, these are two real zeros (degenerate if $c_k = \pm 1$) of the form r_k, r_k^{-1}. For $|c_k| < 1$, the two zeros are complex conjugate and of absolute value 1, i.e., they are of the form $e^{j\alpha_k}$, $e^{-j\alpha_k}$. Since $|c_k| < 1$, such zeros correspond to "physical" zeros of $\beta n - P_a(\theta)$ (i.e., to values of θ for which $\beta n - P_a(\theta) = 0$). In order not to cause any contradiction with $\beta n - P_a(\theta) \ge 0$, these zeros must have even multiplicity.
3. If c_k is not real, we consider it together with $\bar{c}_k$. Then polynomial $(1/2 - c_k z + (1/2)z^2)(1/2 - \bar{c}_k z + (1/2)z^2)$ has four zeros, $c_k \pm \sqrt{c_k^2 - 1}$ and $\bar{c}_k \pm \sqrt{\bar{c}_k^2 - 1}$. One can easily check that the four zeros are all different and form a quadruplet z_k, z_k^{-1}, $\bar{z}_k$, $\bar{z}_k^{-1}$.
4. We therefore have
$$\begin{aligned} P(z) = R_a(n-1) &\cdot \left[\prod_{i=1}^{I} (z - z_i)\left(z - z_i^{-1}\right)(z - \bar{z}_i)\left(z - \bar{z}_i^{-1}\right)\right] \\ &\cdot \left[\prod_{k=1}^{K} \left(z - e^{j\alpha_k}\right)^2 \left(z - e^{-j\alpha_k}\right)^2\right] \\ &\cdot \left[\prod_{\ell=1}^{L} (z - r_\ell)\left(z - r_\ell^{-1}\right)\right], \end{aligned}$$
where we have regrouped the three kinds of zeros and $2I + 2K + L = n-1$.
5. For $z = e^{j\theta}$ on the unit circle, we have
$$\left|\left(e^{-i\theta} - z_0\right)\left(e^{-i\theta} - \bar{z}_0^{-1}\right)\right| = |z_0|^{-1}\left|e^{-i\theta} - z_0\right|^2.$$
Consequently,
$$\begin{aligned} \beta n - P_a(\theta) &= |\beta n - P_a(\theta)| = |P\left(e^{-j\theta}\right)| \\ &= \left[|R_a(n-1)| \prod_{i=1}^{I} |z_i|^{-2} \prod_{\ell=1}^{L} |r_\ell|^{-1}\right] \\ &\quad \cdot \left|\prod_{i=1}^{I} \left(e^{-j\theta} - z_i\right)\left(e^{-j\theta} - \bar{z}_i\right)\right|^2 \\ &\quad \cdot \left|\prod_{k=1}^{K} \left(e^{-j\theta} - e^{j\alpha_k}\right)\left(e^{-i\theta} - e^{-j\alpha_k}\right)\right|^2 \\ &\quad \cdot \left|\prod_{\ell=1}^{L} \left(e^{-j\theta} - r_\ell\right)\right|^2 \\ &= |s_b(\theta)|^2, \end{aligned}$$
where
$$\begin{aligned} s_b(\theta) = &\left[|R_a(n-1)| \prod_{i=1}^{I} |z_i|^{-2} \prod_{\ell=1}^{L} |r_\ell|^{-1}\right]^{1/2} \\ &\cdot \prod_{i=1}^{I} \left(e^{-j\theta} - z_i\right)\left(e^{-j\theta} - \bar{z}_i\right) \\ &\cdot \prod_{k=1}^{K} \left(e^{-j\theta} - e^{j\alpha_k}\right)\left(e^{-i\theta} - e^{-j\alpha_k}\right) \\ &\cdot \prod_{\ell=1}^{L} \left(e^{-j\theta} - r_\ell\right) \end{aligned}$$
is clearly a trigonometric polynomial of order $n-1$ with $b \in \mathbb{R}^n$. Then $P_a(\theta) + P_b(\theta) = \beta n$, that is, a is a β-complementary sequence. This completes the proof. □

This result immediately implies that a G_N-complementary sequence is a special β-complementary sequence.

COROLLARY 1: *A $\xi^{\mathbb{Z}_2}$-sequence is a β-complementary sequence with $\beta = N$ if a is a G_N-complementary sequence.*

PROOF: It is easy to see that $\text{PMEPR}(a) \leq N$ if a is a G_N-complementary sequence. Then Theorem 1 immediately shows that a is a β-complementary sequence with $\beta = N$. □

The argument used in the proof of Theorem 1 also implies the following results about the combination of β-complementary sequences.

COROLLARY 2: *If a $\xi^{\mathbb{Z}_2}$-sequence a is a $\beta(n)$-complementary sequence, then the concatenated sequence $a \vee a = (a_0, \ldots, a_{n-1}, a_0, \ldots, a_{n-1})$ and the interleaved sequence $a \wedge a = (a_0, a_0, \ldots, a_{n-1}, a_{n-1})$ are $4\beta(2n)$-complementary sequences.*

PROOF: There is a sequence $b \in \mathbb{C}^n$ such that

$$P_a(\theta) + P_b(\theta) = \beta n.$$

Then

$$P_{a\vee a}(\theta) + P_{b\vee b}(\theta) = |1 + z^n|^2(P_a(\theta) + P_b(\theta)) = \beta n|1 + e^{jn\theta}|^2.$$

Let $p(\theta) = 4\beta n - \beta n|1 + e^{jn\theta}|^2 + P_{b\vee b}(\theta)$. Then $P(\theta)$ is a nonnegative trigonometric polynomial of degree $4n - 2$. By the argument in the proof of Theorem 1, we conclude that there exists a $b' \in \mathbb{C}^{2n}$ such that $P_{b'}(\theta) = p(\theta)$, that is

$$P_{a\vee a}(\theta) + P_{b'}(\theta) = 4\beta n.$$

Similarly, we can prove that the interleaving sequence $a \wedge a$ is a $4\beta(2n)$-complementary sequence. This completes the proof. □

III The distribution of code rate versus β

By Theorem 1, for a β-complementary sequence, $\text{PMEPR} = \beta$. Then we can determine the distribution of code rate versus β by the distribution of the PMEPR of $\xi^{\mathbb{Z}_2}$-sequences. Using the assumption that the envelope $s_n(t)$ is asymptotically complex Gaussian, [19] and [20] determined good approximations of the distribution of the PMEPR by relying on the work of [21]. However, this assumption lacks rigorous proof. Let

$$\begin{cases} x_n(t) = \dfrac{1}{\sqrt{n}} \sum_{\ell=0}^{n-1} \left(A_\ell^R \cos \omega_\ell t - A_\ell^I \sin \omega_\ell t\right), \\ y_n(t) = \dfrac{1}{\sqrt{n}} \sum_{\ell=0}^{n-1} \left(A_\ell^R \cos \omega_\ell t + A_\ell^I \sin \omega_\ell t\right), \end{cases} \tag{7}$$

where A_ℓ^R and A_ℓ^I are, respectively, the real and imaginary parts of A_ℓ, and $\omega_\ell = 2\pi\ell/n$. Then one can write the complex envelope $s_n(t)$ as

$$s_n(t) = x_n(t) + jy_n(t). \tag{8}$$

Recently, the work done in [22] and [23], using Lindberg's central limit theorem for triangular arrays [24], rigorously proved that either $x_n(t)$ or $y_n(t)$ converges to a stationary Gaussian process on any closed interval in $\mathbb{R}$, in the sense of distribution (weakly). Therefore, $s_n(t)$ weakly converges to a complex Gaussian process. Consequently, using the modern extremal theory [25], the authors of [22] and [23] arrived at a very good approximation of the cumulative distribution function (CDF) of the PMEPR.

Unfortunately, the proof presented in [22] and [23] is not correct for binary OFDM signals since it requires that $x_n(t)$ and $y_n(t)$ both be stationary, which constraint, however, does not hold for a binary OFDM signal. In this section, we add a random phase to a binary OFDM signal and apply the results in [22] and [23] to the newly derived binary OFDM signal with random phase. With this technique, we establish a rigorous proof for the claim that the newly derived binary OFDM signal with random phase weakly converges to a stationary complex Gaussian process on any closed interval in $\mathbb{R}$. Then we employ the modern extremal theory for χ^2-process to estimate the CDF of the PMEPR for the original binary OFDM signals.

III.A Addition of random phase to the binary OFDM signals

For a binary OFDM signal $s_n(t)$, we have $M = 2$, $A_\ell^R = A_\ell$, and $A_\ell^I = 0$. Therefore the associated real part $x_n(t)$ and imaginary part $y_n(t)$ degenerate to

$$\begin{cases} x_n(t) = \dfrac{1}{\sqrt{n}} \sum_{\ell=0}^{n-1} A_\ell \cos \omega_\ell t, \\ y_n(t) = \dfrac{1}{\sqrt{n}} \sum_{\ell=0}^{n-1} A_\ell \cos \omega_\ell t, \end{cases} \tag{9}$$

neither of which is stationary. To make $x_n(t)$ and $y_n(t)$ stationary, we add a random phase to the OFDM signal $s_n(t)$.

Let θ be a random variable of uniform distribution, which equally takes values in the interval $[0, 2\pi]$. Consider the binary OFDM signal with random phase θ,

$$\tilde{s}_n(t) = \frac{1}{\sqrt{n}} \sum_{\ell=0}^{n-1} A_\ell e^{j(2\pi\ell t/n+\theta)}, \quad t \in [0, n]. \tag{10}$$

Since a random phase does not change the PMEPR of an OFDM signal, it is easy to see that

$$\text{PMEPR} = \max_{t\in[0,n]} |\tilde{s}_n(t)|^2.$$

Let $\tilde{A}_\ell = A_\ell e^{j\theta}$ and

$$\begin{cases} \tilde{x}_n(t) = \dfrac{1}{\sqrt{n}} \sum_{\ell=0}^{n-1} \left(\tilde{A}_\ell^R \cos \omega_\ell t + \tilde{A}_\ell^I \sin \omega_\ell t\right), \\ \tilde{y}_n(t) = \dfrac{1}{\sqrt{n}} \sum_{\ell=0}^{n-1} \left(\tilde{A}_\ell^R \cos \omega_\ell t + \tilde{A}_\ell^I \sin \omega_\ell t\right), \end{cases} \tag{11}$$

where $\tilde{A}_\ell^R$ and $\tilde{A}_\ell^I$ are, respectively, the real and imaginary parts of $\tilde{A}_\ell$, i.e., $\tilde{A}_\ell^R = A_\ell \cos\theta$ and $\tilde{A}_\ell^I = A_\ell \sin\theta$. Then we can rewrite $\tilde{s}_n(t)$ as

$$\tilde{s}_n(t) = \tilde{x}_n(t) + j\tilde{y}_n(t).$$

For the newly defined signal $\tilde{s}_n(t)$ with random phase θ, verify the constraints of Theorem 2 in [22]–[23]:

$$\begin{cases} E\left(\tilde{A}_\ell^R\right) = E\left(\tilde{A}_\ell^I\right) = 0, \\ E\left(\tilde{A}_\ell^R \tilde{A}_\ell^I\right) = 0, \\ E\left[\left(\tilde{A}_\ell^R\right)^2\right] = E\left[\left(\tilde{A}_\ell^I\right)^2\right] = \dfrac{1}{2}. \end{cases}$$

By Theorem 2 in [22]–[23], we conclude that $\tilde{x}_n(t)$ and $\tilde{y}_n(t)$ weakly converge to the stationary Gaussian processes $\tilde{x}(t)$ and $\tilde{y}(t)$, respectively, on any closed interval in $\mathbb{R}$ as $n \to \infty$. Meanwhile, we have

$$E[\tilde{x}(s)\tilde{x}(t)] = E[\tilde{y}(s)\tilde{y}(t)] = \frac{\text{sinc}(2(t-s))}{2}$$

and

$$E[\tilde{x}(s)\tilde{y}(t)] = \frac{\sin^2(\pi(t-s))}{2\pi(t-s)}.$$

Consequently, $\tilde{s}_n(t)$ converges to a stationary complex Gaussian process $\tilde{s}(t) = \tilde{x}(t) + j\tilde{y}(t)$ on any closed interval in $\mathbb{R}$ as $n \to \infty$. This fixes the difficulty in the proof for binary OFDM signals in [22]

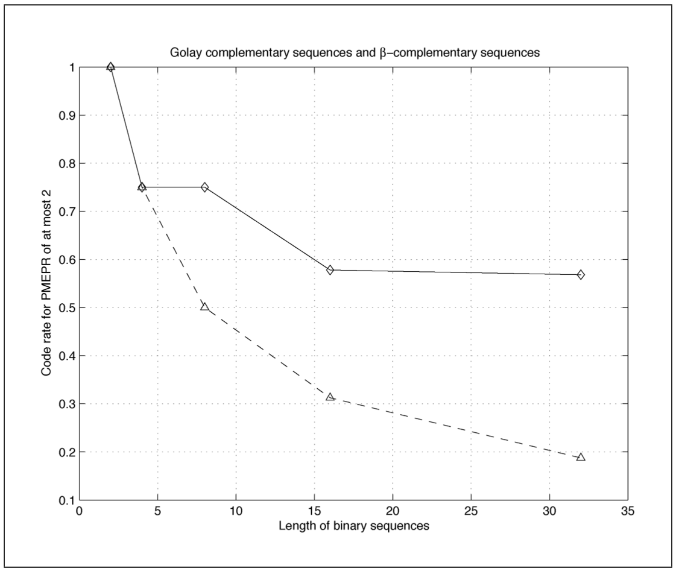

Figure 1: *The diamond-solid line is the code rate versus the length of the β-complementary sequences for PMEPR of at most 2. The triangle-dashed line is the code rate versus length of the Golay complementary sequences for PMEPR of at most 2.*

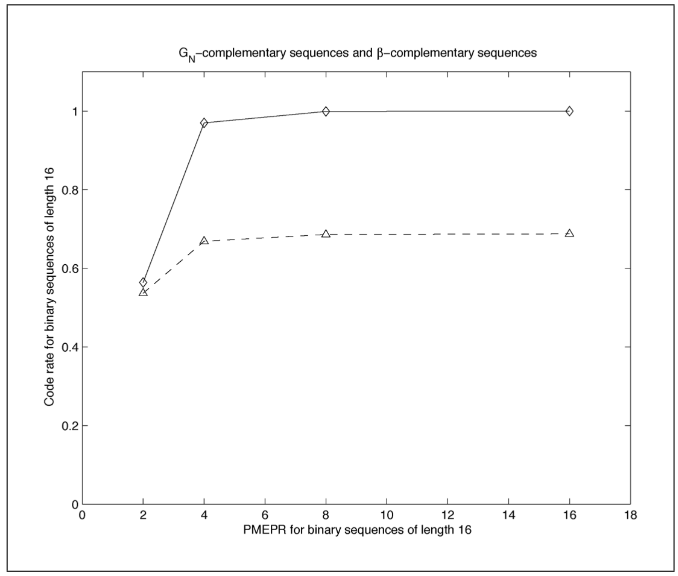

Figure 2: *The diamond-solid line is the code rate versus the PMEPR of the $\beta(16)$-complementary sequences. The triangle-dashed line is the code rate versus the PMEPR of the $G_N(16)$-complementary sequences with $N = \beta$.*

and [23] and establishes a rigorous proof for the claim that the newly derived binary OFDM signal $\tilde{s}_n(t)$ with random phase θ, rather than the original binary OFDM signal $s_n(t)$, is asymptotically Gaussian.

III.B The distribution of code rate versus β

In the last subsection, we rigorously showed that the newly derived signal $\tilde{s}_n(t)$ with random phase weakly converges to a stationary complex Gaussian process $\tilde{s}(t)$ on any closed interval in $\mathbb{R}$ as $n \to \infty$. We will use this result and the extremal theory for χ^2-process to estimate the CDF of the PMEPR for binary OFDM signals in this section.

Before proceeding further, we should modify the derived binary OFDM signal $\tilde{s}_n(t)$ with random phase to a symmetric binary OFDM signal by setting $\omega_\ell = (2\pi/n)(\ell - (n-1)/2)$. By abusing the symbols in the following, we can still use $\tilde{s}_n(t) = \tilde{x}_n(t) + j\tilde{y}_n(t)$ to represent the modified symmetric binary OFDM signal with random phase. Then it is trivial to verify that $\tilde{s}_n(t)$ weakly converges to $\tilde{s}(t)$ on any closed interval in $\mathbb{R}$ by an argument similar to the one in the last section. As an advantage of such modification, we have

$$E[\tilde{x}(s)\tilde{y}(t)] = 0.$$

By the definition of χ^2-process,[1] the normalized instantaneous power $2|\tilde{s}_n(t)|^2 = 2\tilde{x}_n^2(t) + 2\tilde{y}_n^2(t)$ weakly converges to a χ^2-process on any closed interval in $\mathbb{R}$ as $n \to \infty$. Hence, for n sufficiently large, we have

$$\mathrm{PMEPR} = \max_{t\in[0,n]} |\tilde{s}_n(t)|^2 \approx \frac{1}{2} \max_{t\in[0,n]} \chi^2(t). \tag{12}$$

Using the extremal theory for χ^2-process [25] and referring to the argument in [22] and [23], we have, as $n \to \infty$,

$$P(a_n(\max_{t\in[0,n]} \chi(t) - b_n) \le \lambda) \to \exp(-e^{-\lambda}), \tag{13}$$

where $a_n = 1/2$ and $b_n = 2\log n + \log\log n - \log(\lambda/\pi)$. Combining (12) and (13), for n sufficiently large, we obtain the CDF of the PMEPR for binary OFDM signals as

$$P(\mathrm{PMEPR} \le \lambda) \approx \exp\left\{-e^{-\lambda} n \sqrt{\frac{\pi}{3}\log n}\right\}. \tag{14}$$

[1] For the stationary Gaussian processes $G_1(t), \ldots, G_d(t)$ ($d \ge 2$) of mean 0 and variance 1, the process $\chi(t) = \sum_{i=1}^{d} G_i^2(t)$ is called χ^2-process (with d degrees of freedom) if $G_1(t), \ldots, G_d(t)$ are mutually independent.

IV Numerical results

Consider the simplest $\xi^{\mathbb{Z}_2}$-sequences, i.e., the binary sequences. Exhaustive computer searching shows that there are, respectively, 4, 8, 64, 608, and 149 184 available codewords for $n = 2, 4, 8, 16, 32$. Therefore the corresponding code rates are, respectively, 1, 0.75, 0.75, 0.578, and 0.58 for $n = 2, 4, 8, 16, 32$. The distribution of code rate versus the length of binary β-complementary sequences is plotted in Fig. 1 by the diamond-solid line. In Fig. 1, we also plot the distribution of code rate versus the binary Golay complementary sequences with the triangle-dashed line. From [7], the code rates of binary Golay complementary sequences are $[\lfloor \log_2(m!) \rfloor + m + 1]/2^m$, where $m = \log_2 n$. Therefore the code rates of binary Golay complementary sequences are, respectively, 1, 0.75, 0.5, 0.3125, and 0.1875 for $n = 2, 4, 8, 16, 32$. From Fig. 1, both lines decrease as the number of subcarriers increases. However, the code rate of binary Golay sequences decreases very fast, while that of β-complementary sequences persists, relatively speaking. This implies that the β-complementary sequences provide a high code rate for a moderately large subcarrier in OFDM. Meanwhile, we find that the diamond-solid line lies above the triangle-dashed line, and that the difference is more than 0.38 for $n = 32$, which means that there are many more codes with PMEPR of at most 2 available beyond the Golay complementary sequences. Therefore it is reasonable to expect that we can determine an efficient encoding method using these codewords. This is another primary motivation for introducing the β-complementary sequences.

On the other hand, we can consider the tradeoff between the PMEPR and code rate as in [17]. We focus on the binary sequences of length $n = 16$. The distributions of code rate versus PMEPR for β-complementary sequences and versus G_N-complementary sequences ($N = \beta$) are plotted in Fig. 2. When one relaxes the PMEPR, the code rate of G_N-complementary sequences slowly increases, while that of β-complementary sequences increases very fast. For example, when the PMEPR is relaxed from 2 to 4, the code rate of G_N-complementary sequences increases merely from 0.5366 to 0.6688, while that of β-complementary sequences increases from 0.5639 to 0.97. When $\mathrm{PMEPR} = 8$ or 16, the code rates of G_N-complementary sequences increase only slightly, while those of β-complementary sequences rise to almost 1. This numerical result implies that β-complementary sequences have very good performance in terms of the tradeoff between PMEPR and code rate, compared to the G_N-complementary sequences, and suggests a way to determine an efficient encoding

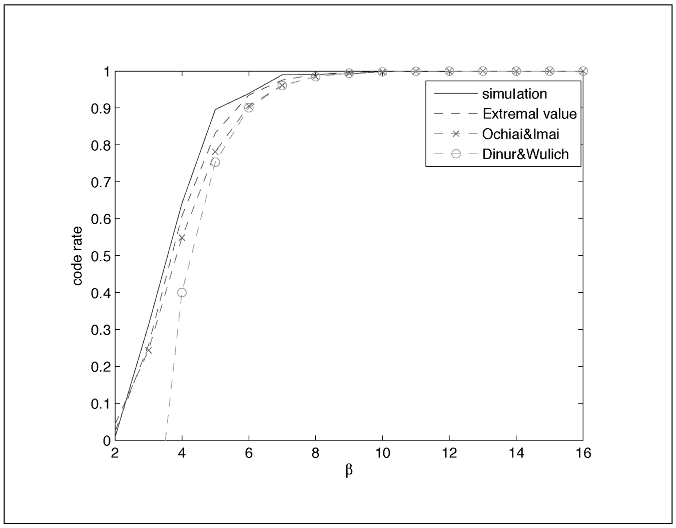

Figure 3: *CDF of PMEPR for binary OFDM signal for $n = 16$. The estimates derived in [19] and [20] provide good approximations of the actual CDF. However, the estimate derived in this paper by random phase and extremal theory approximates the actual distribution much better.*

scheme with low PMEPR based on some subset of β-complementary sequences.

Fig. 3 shows the distribution of code rate versus β for β-complementary sequences of length 16. We also compare the results with those derived in [19] and [20]. From this figure, we observe that the estimate from (14) approximates the actual CDF very well. However, there is a difference between the estimate and the actual CDF, which comes from (12), where we approximately consider $\tilde{s}_n(t)$ as a complex Gaussian process, and (13), where we approximately use the limit distribution "double exponential distribution" as the distribution of $\max_{t\in[0,n]} \chi^2(t)$. Further estimation of the approximation errors in (12) and (13) will reduce the difference between the estimate and the actual distribution. However, this is obviously not an easy problem in the theory of stochastic processes and extremal theory, and it remains open.

V Conclusion and open problems

In this paper, we introduced novel sequences called "β-complementary sequences" to encode OFDM signals in a way that can substantially increase code rate while tightly bounding the PMEPR. As the subcarrier n increases in size, the code rate of β-complementary sequences will decline like that of Golay complementary sequences. However, the code rate of β-complementary sequences declines very slowly compared to that of Golay complementary sequences. Since the code rate of Golay complementary sequences is prohibitively low for a moderately large subcarrier (e.g., $n > 32$), the β-complementary sequences are suitable for use in encoding both small and large subcarrier OFDM signals, while maintaining a PMEPR of at most 2. On the other hand, the β-complementary sequences also enjoy good performance in terms of the tradeoff between PMEPR and code rate, compared to the G_N-complementary sequences. Slightly relaxing the PMEPR in β-complementary sequences will greatly increase the code rate to combat the slow increment when the PMEPR is relaxed in G_N-complementary sequences. This discussion suggests a way to construct a low-PMEPR encoding scheme for OFDM signals based on some subset of β-complementary sequences.

For a moderately large subcarrier OFDM signal, one can create a look-up table to encode the OFDM symbols. However, for large subcarrier OFDM signals, the question of finding an efficient way to generate sufficient numbers of β-complementary sequences remains open. We have developed certain recursive generating formulas. If one also considers decoding, the Hamming distance is a critical factor affecting the decoding efficiency. Therefore, understanding the Hamming distance is another open problem for the use of β-complementary sequences in OFDM. In addition, we conjecture that Theorem 1 and Corollary 1 also hold for any β-sequences in $\xi^{\mathbb{Z}_M}$.

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