MIMO-OFDM Channel Estimation in the Presence of Frequency Offsets

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Abstract—Optimal pilot design and placement for channel estimation in Multiple-input Multiple-output (MIMO) Orthogonal Frequency-Division Multiplexing (OFDM) systems with frequency offsets are considered. Both the single-frequency-offset case and the multiple-frequency-offset case are treated. We show that the Constant-Envelope (CE) condition is sufficient but not necessary for pilot design, and that pilots with multiple envelopes can also achieve the optimal performance in terms of the Mean Square Error (MSE) minimization, provided that an additional constraint on the pilot placement is satisfied simultaneously. New pilot designs, which take into account the multiple-frequency-offset case, are proposed to eliminate Inter-Pilot-Interference (IPI) and to optimize the MSE performance. The Least-Squares (LS) and Linear Minimum Mean Square Error (LMMSE) channel estimators for the multiple-frequency-offset case are designed for uncorrelated and correlated MIMO-OFDM channels, respectively. The LMMSE estimator requires the channel covariance matrix. Both optimal adaptive pilot power allocation and suboptimal uniform pilot power allocation are developed for the proposed LMMSE estimator. The adaptive allocation performs 4 dB better than the uniform allocation in the high noise region, but they both perform identically in the low noise region. Performance comparisons are made against several previous pilot designs due to [1], [2]. The proposed LMMSE estimator significantly outperforms the LS estimator.

Index Terms—Frequency offset, channel estimation, MIMO, OFDM.

I. INTRODUCTION

O NSIDERABLE research has focused on Multiple-input Multiple-output (MIMO) technology for increasing the wireless system capacity. Compared to a single-input single-output (SISO) system, a MIMO system can improve the capacity by a factor of the minimum number of transmit and receive antennas. Space-time coding, including space-time block codes (STBC) and space-time trellis codes (STTC) [3], can extract transmit diversity in MIMO systems.

The frequency-selective fading MIMO channel can be transformed into a set of flat-fading MIMO channels by using Orthogonal Frequency-Division Multiplexing (OFDM). This transformation achieves a high capacity at a low cost of equalization and demodulation [4], [5]. However, just as with the SISO-OFDM, MIMO-OFDM systems too are sensitive to frequency offset. Many SISO-OFDM frequency offset estimators have been proposed [6]–[8]. A synchronization algorithm for MIMO-OFDM systems is proposed in [9], where identical timing offset and frequency offset with respect to each transmit-receive antenna pair are assumed. Parameters estimation of MIMO flat-fading channels is discussed in [10], where frequency offsets for different transmit-receive antennas are assumed to be different, and the Cramer-Rao Lower Bound (CRLB) for either the frequency offsets or channel estimation variance error is derived.

Another prevalent impairment is the channel estimation error, which degrades the bit error rate of MIMO-OFDM systems. Robust channel estimation for OFDM systems is discussed in [11]. Optimal pilot design and placement for channel estimation is developed in terms of minimization of the CRLB [12]. Optimal training signal design for frequency-selective block fading channel estimation in MIMO-OFDM systems is discussed in [13]. Some other constraints for training signal designs, such as low peak-to-average energy ratio (PAR) and robustness to frequency offsets, are also considered in [13]. Barhumi, Leus and Moonen (BLM) propose a high-quality channel estimator for a MIMO-OFDM channel based on frequency-domain uniformly placed pilots (the frequency offset and channel correlation are not considered) [1]. Hu improves the BLM pilots with a nonuniform placement design, which mitigates the performance loss due to the nonmodulated pilots in virtual subcarriers [14]. MIMO-OFDM channel estimation for a correlated channel is discussed in [15], [16]. However, neither [15] nor [16] considers the effect of frequency offset. Training sequence design for MIMO channel estimation in the presence of a single frequency offset is discussed in [17], [18]. In [17], optimal training signal design for MIMO-OFDM channel estimation while considering a single frequency offset and phase noise is proposed, and its performance is much more robust than the algorithms that consider only the effect of frequency offset. In [18], the estimation of frequency-selective channel and frequency offset in either SISO or MIMO systems is discussed. By using the exact CRLB as a metric, a power efficient training preamble is designed in [18] to reduce the complexity of estimation. Joint frequency offset and channel estimation with either single or multiple frequency offsets for MIMO frequency selective fading channels is discussed in [2], where pilots for different transmit antennas are orthogonal in the time-domain, and pilot optimization in terms of MSE reduction is also studied.

This paper considers optimal pilot design and placement for channel estimation in Multiple-input Multiple-output (MIMO) Orthogonal Frequency-Division Multiplexing (OFDM) systems in the presence of frequency offsets. Both the single-frequency-offset case and the multiple-frequency-offset case
are treated. Each frequency offset is modeled as an Independent and Identically Distributed (i.i.d.) random variable (RV). In our proposed scheme, several subcarriers are allocated to each transmit antenna to transmit either pilots or data symbols. The pilots of different transmit antennas may be either orthogonal (in the frequency-domain or time-domain) or not. We show that the Constant-Envelope (CE) condition is sufficient but not necessary for pilot design, and that pilots with multiple envelopes can also achieve the optimal performance in terms of the Mean Square Error (MSE) minimization, provided that an additional constraint on the pilot placement is satisfied simultaneously. New pilot designs, which take into account the multiple-frequency-offset case, are proposed to eliminate Inter-Pilot-Interference (IPI) and to optimize the MSE performance. The Least-Squares (LS) and Linear Minimum Mean Square Error (LMMSE) channel estimators for the multiple-frequency-offset case are designed for uncorrelated and correlated MIMO-OFDM channels, respectively. The LMMSE estimator requires the channel covariance matrix. Both optimal adaptive pilot power allocation and suboptimal uniform pilot power allocation are developed for the proposed LMMSE estimator. Performance comparisons are made against several previous pilot designs due to [1], [2]. The estimator of [1] is extended to consider multiple frequency offsets. The estimator of [2] is only for the single frequency offset case. The remainder of this paper is organized as follows. A brief review of the OFDM channel model is given in Section II, and the LS frequency offset and LMMSE channel estimation are discussed in Section III. The optimal pilot design and placement for LS channel estimation in the presence of multiple frequency offsets are discussed in Section IV, and an LMMSE estimator for a correlated MIMO-OFDM channel is proposed in Section V. Numerical results are given in Section VI, followed by conclusions in Section VII.

II. MIMO-OFDM SIGNAL MODEL

Input data bits of MIMO-OFDM are mapped to complex symbols drawn from a typical signal constellation, e.g., phase-shift keying (PSK) or quadrature amplitude modulation (QAM). An OFDM symbol is generated by taking the Inverse Discrete Fourier Transform (IDFT) of $N$ input sub-symbols, where $N$ is the IDFT size, and the IDFT matrix $F$ is defined as 

$$F_{nk} = \frac{1}{\sqrt{N}} e^{\frac{j2\pi nk}{N}} \text{ for } 0 \leq n, k \leq N - 1.$$ 

1 This case can occur when, for example, MIMO OFDM is used in a multiple user scenario. This case can also occur in cooperative relaying, which can be seen as virtual MIMO systems. The single frequency offset case is a special case of this case.

2 Notation: $(\cdot)^{-1}$, $(\cdot)^T$ and $(\cdot)^H$ are the inverse, transpose and complex conjugate transpose of a matrix. The imaginary unit is $j = \sqrt{-1} \in \mathbb{R}$, and $\Re\{x\}$ and $\Im\{x\}$ are the real and imaginary part of $x$, respectively. A circularly symmetric complex Gaussian RV $w$ with mean $m$ and variance $\sigma^2$ is denoted by $w \sim \mathcal{CN}(m, \sigma^2)$. A real Gaussian RV $x$ with mean $m$ and variance $\sigma^2$ is denoted by $x \sim \mathcal{N}(m, \sigma^2)$. $I_N$ is the $N \times N$ identity matrix, and $O_N$ is the $N \times N$ all-zero matrix. $N_{\infty}$ is the $N \times 1$ all-zero vector. $a[i]$ is the $i$-th entry of vector $a$, and $\|a\|_2^2 = \sum_{i=1}^{\infty} |a[i]|^2$. $[B]_{mn}$ is the $mn$-th entry of matrix $B$. $(\cdot)_n^\bot$ represents the remainder after division of $x$ by $n$. $\Re\{x\}$ and $\Im\{x\}$ are the mean and variance of $x$. 

$N_t$ transmit antennas and $N_r$ receive antennas, a $N \times 1$ vector $x_i(z)$ is used to represent the $z$-th block of frequency-domain symbols sent by the $i$-th transmit antenna, where $i \in \{1, 2, \ldots, N_t\}$. In the following sections, the temporal index $z$ will be omitted for the sake of simplicity. Without loss of generality, each entry of $x_i$ is assumed to be an i.i.d. RV with mean zero and variance $\sigma^2 = \frac{E_z}{N N_t}$, where 

$$E_z = \sum_{i=1}^{N_t} E\left\{ ||x_i||^2 \right\}$$ 

is the total transmit power.

By using $h_{k,i}(z)$ to represent the discrete-time impulse response of the $z$-th tap channel between the $i$-th transmit and the $k$-th receive antenna, the related channel response vector can be represented as 

$$h_{k,i} = \left[ h_{k,i}(0), h_{k,i}(1), \ldots, h_{k,i}(L_{\text{max}} - 1) \right] \in \mathbb{C}^{L_{\text{max}} \times 1},$$

representing the maximum length of all channels. The corresponding frequency-domain channel attenuation matrix is given by 

$$H_{k,i} = \left[ \sum_{d=0}^{L_{\text{max}}-1} h_{k,i}(d) e^{-\frac{j2\pi d}{N}} \right] \text{ representing the channel attenuation at the } n\text{-th subcarrier.}$$

In the following sections, $\psi_{k,i}$ and $\varepsilon_{k,i}$ are used to represent the initial phase and the normalized frequency offset of the frequency offset normalized to a subcarrier spacing of OFDM symbols (where $q_k$ is longer than the $k$-th receive antenna response of the delayed channel at the $k$-th receive antenna, by taking the DFT operation on the $N$-th symbol sent by the $k$-th transmit antenna). 

$$H_{k,i} = \left[ \sum_{d=0}^{L_{\text{max}}-1} h_{k,i}(d) e^{-\frac{j2\pi d}{N}} \right] \text{ representing the channel attenuation at the } n\text{-th subcarrier.}$$

By considering the channel attenuations and frequency offsets, the $q$-th received vector can be represented as 

$$y(q) = [y_T^T(q), y_T^T(q), \ldots, y_N^T(q)]^T,$$ 

where 

$$y_T(q) = \sum_{i=1}^{N_t} E_{k,i} F h_{k,i} x_i(q) + w_k(q),$$

and $w_k(q)$ is a vector of additive white Gaussian noise (AWGN) with $w_k(q)[i] \sim \mathcal{CN}(0, \sigma^2)$. 

At the $k$-th receive antenna, by taking the DFT operation to the received vector, we obtain 

$$y_k(q) = F_{\text{max}}H_{\text{max}} \tilde{h}_{k,i} \left( \sum_{i=1}^{N_t} \sqrt{N} E_{k,i} F h_{k,i} x_i(q) \right) + \sum_{i=1}^{N_t} \sqrt{N} E_{k,i} F h_{k,i} \tilde{h}_{k,i} x_i(q)$$

$$+ \sum_{i=1}^{N_t} \sqrt{N} E_{k,i} F h_{k,i} \tilde{h}_{k,i} x_i(q) + \sum_{i=1}^{N_t} \sqrt{N} E_{k,i} F h_{k,i} \tilde{h}_{k,i} x_i(q),$$

where $E_{k,i}^T = F^H E_{k,i} F$ is a circulant matrix, $F_{\text{Lmax}}$ is the first $L_{\text{max}}$ rows of $F$, and 

$$X_i(q) = X_i^0(q) + X_i^1(q) = \text{diag}\{x_i^0(q)\} + \text{diag}\{x_i^1(q)\} \text{ with } x_i^0(q) \text{ and } x_i^1(q) \text{ being some } N \times 1 \text{ data and pilot vectors, respectively. Note that the averaged power of the data and pilot may be different. With the pilots modulated over consecutive symbols } (M \geq 1), \text{ the received vector becomes}$$

$$r_k = \left[ r_k^T(0), \ldots, r_k^T(M-1) \right]^T = P_k h_k + D_k h_k + \eta_k,$$
where \( P_k = \begin{bmatrix} P_{k,1}(0) & \cdots & P_{k,N_t}(0) \\ \vdots & \ddots & \vdots \\ P_{k,1}(M-1) & \cdots & P_{k,N_t}(M-1) \end{bmatrix} \), \\
\( D_k = \begin{bmatrix} D_{k,1}(0) & \cdots & D_{k,N_t}(0) \\ \vdots & \ddots & \vdots \\ D_{k,1}(M-1) & \cdots & D_{k,N_t}(M-1) \end{bmatrix} \), \\
\( h_k = \left[ h_{k,1}^T, \ldots, h_{k,N_t}^T \right]^T \) and \( \eta_k = [\eta_k^T(0), \ldots, \eta_k^T(M-1)]^T \).

### III. LS FREQUENCY OFFSET AND CHANNEL ESTIMATION IN MIMO-OFDM SYSTEMS

This section and the next assume that the channel coefficients for the different transmit and receive antennas are independent. Section V considers correlated MIMO-OFDM channels.

Using \( r_k \), the frequency offsets and channel coefficients are jointly estimated as

\[
\{\hat{\varepsilon}_{k,1}, \ldots, \hat{\varepsilon}_{k,N_t}, \hat{h}_k\} = \arg \min_{\varepsilon_{k,1}, \ldots, \varepsilon_{k,N_t}, h_k} \| \hat{P}_k \hat{h}_k \|_2^2,
\]

where \( \hat{P}_k = \sqrt{N} \begin{bmatrix} \Gamma_{k,1}(0) & \cdots & \Gamma_{k,N_t}(0) \\ \vdots & \ddots & \vdots \\ \Gamma_{k,1}(M-1) & \cdots & \Gamma_{k,N_t}(M-1) \end{bmatrix} \), \\
\( \Gamma_{k,i}(m) = E_{\text{cir}}^{\text{H}} \left\{ \left( i \pi \frac{m}{N} + \hat{\varepsilon}_{k,i} \right) \right\} \), and \\
\( \hat{E}_{k,i} = \text{diag} \left\{ 1, e^{j2\pi \frac{1}{N}}, \ldots, e^{j2\pi \frac{i-1}{N}} \right\} \). By transmitting the pilot vector \( m_k(n) = \sum_{i=1}^{N_t} E_{k,i} h_{k,i} x_i^p(n) \), from [19, page 926], for an unbiased estimator, the CRLB for a variance error of \( \hat{\varepsilon}_{k,i} \) is given by

\[
\text{Var} \{ \hat{\varepsilon}_{k,i} \} \geq \frac{\sigma_w^2}{\sum_{n=0}^{M-1} \left\| \frac{\partial m_k(n)}{\partial \varepsilon_{k,i}} \right\|_2^2}
\]

\[
= \frac{\sigma_w^2}{\sum_{n=0}^{M-1} \left\| A F h_{k,i} x_i^p(n) \right\|_2^2}
\]

\[
= \frac{3N_t \sigma_w^2}{2\pi^2(N-1)(2N-1)E_p}.
\]

The second condition in Proposition 1 can be derived as follows:

\[
\text{trace} \left\{ \left( P_k^H P_k \right)^{-1} V_k \Phi_k \right\} = 0
\]

\[
\Leftrightarrow \text{trace} \left\{ \left( P_k^H P_k \right)^{-1} V_k \Phi_k \right\} = 0
\]

\[
\Leftrightarrow \left( P_k^H P_k \right)^{-1} V_k \Phi_k = 0
\]

where \( \Phi_k = \mathbb{E} \{ h_k h_k^H \} \), and \( V_k = P_k^H D_k (V_k \) represents the power spread of \( D_k \) to the signal space of \( P_k \). When in the presence of frequency offsets, \( \text{trace} \left\{ \left( P_k^H P_k \right)^{-1} V_k \Phi_k \right\} = 0 \) is achieved if and only if the following two conditions are satisfied simultaneously:

**Proposition 1:**

1. \( x_i^p(n)x_i^p(m)^H(q) = 0 \) for each \( 1 \leq i, k \leq N_t, 0 \leq m, q \leq M-1 \); i.e., the subcarrier symbols allocated to data symbols are orthogonal to each pilot subcarrier space of each transmit antenna.
2. \( P_k^H D_k = (P_k^H P_k)^{-1} P_k^H D_k = O_{L_{\max} N_t} \).

The condition in Proposition 1 can be derived as follows:

\[
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\]

\[
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\]

\[
\Leftrightarrow \left( P_k^H P_k \right)^{-1} V_k \Phi_k = O_{L_{\max} N_t}.
\]
Since both $P_k^H P_k$ and $\Phi_k$ are non-zero Hermitian matrices, the only solution of (9) is $V_k = O_{L_{max} N_t}$, which proves the second condition in Proposition 1.

IV. OPTIMAL PILOT DESIGN AND PLACEMENT FOR LS CHANNEL ESTIMATION IN THE PRESENCE OF MULTIPLE FREQUENCY OFFSETS

In the multiple frequency offsets case, define $\lambda^2_{p,k,j}$, $\lambda^2_{h,k,j}$ and $\lambda^2_{h,h,k,j}$ as the $j$-th eigenvalue of $P_k^H P_k$, $V_k V_k^H$ and $\Phi_k$, respectively, where both $\lambda^2_{p,k,j}$ and $\lambda^2_{h,k,j}$ are functions of $\varepsilon_{k,1}, \ldots, \varepsilon_{k,N_t}$, and, therefore, (8) can be rewritten as

$$MSE \left( \hat{h}_{k|LS} \right) = \frac{1}{L_{max} N_t} \left( \sum_{j=0}^{L_{max} N_t - 1} \lambda^2_{p,k,j} \lambda^2_{h,k,j} + \sum_{j=0}^{L_{max} N_t - 1} \sigma^2_w \lambda^2_{p,k,j} \right),$$

s.t. $\sum_{j=0}^{L_{max} N_t - 1} \lambda^2_{p,k,j} = L_{max} E_p$.

(10)

The optimal $\lambda^2_{p,k,j}$ that minimizes $MSE \left( \hat{h}_{k|LS} \right)$ can be derived based on (10). Unfortunately, for a frequency selective fading MIMO-OFDM channel where $\lambda^2_{h,k,0} \neq \lambda^2_{h,k,1} \neq \ldots \neq \lambda^2_{h,k,L_{max} N_t - 1}$, obtaining a closed-form resolution of $\lambda^2_{p,k,j}$ appears intractable, if not impossible. A closed-form resolution of $\lambda^2_{p,k,j}$ is achievable when $\lambda^2_{h,k,0} = \lambda^2_{h,k,1} = \ldots = \lambda^2_{h,k,L_{max} N_t - 1} = \lambda^2_{h,k}$, i.e., $\Phi_k = \lambda^2_{h,k} I_{L_{max} N_t}$. From the definition of $V_k$, we can readily represent its $j$-th eigenvalue as $\lambda^2_{h,k,j} = \alpha_{d,k,j}^2 \lambda^2_{p,k,j}$, which means that $\lambda^2_{p,k,j}$ is proportional to $\lambda^2_{h,k,j}$ (Note that $\alpha_{d,k,j}^2$ here is a function of the frequency offset with $\alpha_{d,k,j} \leq \lambda^2_{d,k,j}$, and that $\lambda^2_{d,k,j}$ is the $n$-th eigenvalue of $D_k D_k^H$). Equality holds only if the subcarriers allocated to the data and pilots for each transmit antenna are totally overlapped.) For MIMO-OFDM systems with frequency offsets, the interference of the pilots, contributed by the data subcarriers, is noise-like; therefore, it is reasonable to assume that the average interference power of each pilot subcarrier is identical, i.e., $\alpha_{d,k,0} = \alpha_{d,k,1} = \ldots = \alpha_{d,k,L_{max} N_t - 1} = \alpha_{d,k}^2$, and the optimal $\lambda^2_{p,k,j}$ can be obtained as

$$\lambda^2_{p,k,j} = \frac{E_p}{N_t} 0 \leq j \leq L_{max} N_t - 1.$$  

(11)

A. The Optimal Pilot Design and Placement

Before designing pilots that satisfy (11), let us first analyze $P_k^H P_k$:

$$P_k^H P_k = \begin{bmatrix} G_{k,1,1} & \cdots & G_{k,1,N_t} \\ \vdots & \ddots & \vdots \\ G_{k,N_t,1} & \cdots & G_{k,N_t,N_t} \end{bmatrix},$$

(12)

where $G_{k,m,n} = \sum_{i=0}^{M-1} F_{(L_{max})} X_m^{H(i)} E^\text{cir}_{k,m,n} X_n^{H(i)} F_{(L_{max})}^H$.

(13)
be optimal in a single frequency offset case. Therefore, we can simplify the condition 2 of Preposition 2 as
\[
\sum_{i=0}^{M-1} F_{(L_{\text{max})}} X_m^H (i) X_n^H (i) F_{(L_{\text{max})}} = O_{L_{\text{max}}}, \quad \text{(SFO)};
\]
\[
F_{(L_{\text{max})}} P_{mn} F_{(L_{\text{max})}} = O_{L_{\text{max}}}, \quad \text{(MFO)},
\]
where \( P_{mn} = \sum_{i=0}^{M-1} X_m^H (i) E_{k,m,n}^p X_n^H (i) \), and (17b) implies (17a). Also define a \( N_p \times N_p \) matrix \( P_{mn} \), which is generated by deleting all the zero rows and columns of \( P_{mn} \). In Appendix A, it is shown that when pilot subcarriers allocated to the different transmit antennas are orthogonal in the frequency-domain, \( G_{k,m,n} \equiv O_{L_{\text{max}}} \) for each \( m \neq n \), and that, therefore, all pilots and their placements that satisfy (18) are optimal pilots in terms of the MSEE:

\[
\arg \left\{ X_m^p (i) \right\}_{\theta_m,z, \theta_m,z} = \frac{2 \pi \theta_m, K_p m}{N}, \quad \text{s.t.} \quad L_{\text{max}} \leq N_p \leq N, \quad \frac{N}{N_p} = \text{integer}; \]
\[
K_p \geq L_{\text{max}}, \quad \theta_m,z \theta_m,z, n - m \neq \text{integer}; \]
\[
\left| \left[ X_m^p (i) \right]_{\theta_m,z, \theta_m,z} \right|^2 = \frac{E_p}{N_p N_t} \sum_{i=1}^{M} \left| X_m^p (i) \right|_{\theta_m,z, \theta_m,z}^2 = \frac{E_p}{N_t}, \quad \text{for each } 1 \leq z \leq N_p, 1 \leq m \leq N_t, 0 \leq i \leq M - 1.
\]

When \( \theta_m,z = \theta_n,z \neq m = \theta_z \) for each \( 1 \leq m, n \neq m \leq N_t \) and \( 1 \leq z \leq N_p \), the \( N_t \) transmit antennas share \( N_p \) pilot subcarriers. From (34) in Appendix A, we know that in order to make \( G_{k,m,n} = O_{L_{\text{max}}} \), \( \mu_{k,m,n} \) should be satisfied for each \( 1 \leq u, t \leq L_{\text{max}} \). One way to achieve this result is to make \( P_{mn} = O_{N_p} \), which requires the pilots transmitted by different transmit antennas be orthogonal in the time-domain. These time-domain orthogonal pilots achieve optimal MSEE performance, but at the cost of low spectral efficiency.

B. The Suboptimal Pilot Design and Placement to Minimize the Expectation of MSEE

Note that the MSEE given by (8) is a function of \( \varepsilon_{k,1}, \ldots, \varepsilon_{k,N_t} \). In the multiple frequency offsets case, designing the optimal pilots that satisfy
\[
F_{(L_{\text{max})}} \sum_{i=0}^{M-1} X_m^H (i) E_{k,m,n}^p X_n^H (i) F_{(L_{\text{max})}} = O_{L_{\text{max}}},
\]
\[
F_{(L_{\text{max})}} P_{mn} F_{(L_{\text{max})}} = O_{L_{\text{max}}},
\]
is difficult, if the pilots transmitted by the different transmit antennas are neither orthogonal in the time-domain nor orthogonal in the frequency-domain.

However, orthogonal pilot placement considerably reduces the spectral efficiency, and in this subsection, we consider only the case of \( \theta_m,z = \theta_n,z \neq m = \theta_z \) for each \( (1 \leq m, n \neq m \leq N_t, 1 \leq z \leq N_p) \) with \( M < N_t \). Since the IPI cannot be totally eliminated in this case, we can get

\[
\text{suboptimal pilots only in terms of the MSEE. These suboptimal pilots can be designed by minimizing the expectation of MSEE } \left\{ h_k | \varepsilon_{k,1}, \ldots, \varepsilon_{k,N_t} \right\}; \quad \text{i.e.,}
\]
\[
P_k \text{subopt} = \arg \min_{P_k} \mathbb{E} \left\{ \text{MSEE } \left\{ h_k | \varepsilon_{k,1}, \ldots, \varepsilon_{k,N_t} \right\} \right\}, \quad \text{s.t.} \quad \text{trace } \left\{ P_k^H P_k \right\} = L_{\text{max}} E_p,
\]
where the expectation is with respect to \( \varepsilon_{k,1}, \ldots, \varepsilon_{k,N_t} \). When the eigenvalues of \( P_k \) are identical, the minimum MSEE is achieved, which requires that
\[
\sum_{i=0}^{M-1} P_{k,m,n}^H (i) P_{k,m,n} (i) = O_{L_{\text{max}}}, \quad \text{(SFO)};
\]
\[
\arg \min_{P_k} \left\{ \mathbb{E} \left\{ \text{trace } \left\{ \sum_{i=0}^{M-1} P_{k,m,n}^H (i) P_{k,m,n} (i) \right\} \right\} \right.\]
(20a) can be achieved by all the pilots that satisfy (18). To resolve (20b), we have

\[
\mathbb{E} \left\{ \text{trace} \left\{ \sum_{i=0}^{M-1} \mathbf{P}_{k,m}(i) \mathbf{P}_{k,n}(i) \right\} \right\} = \text{trace} \left\{ \sum_{i=0}^{M-1} \mathbf{F}(L_{max}) \mathbf{X}^{pH}_{m}(i) \mathbf{F}^{H}_{p}(i) \mathbf{F}^{H}_{l(L_{max})} \right\},
\]

where \( \mathbf{A}_{p} \equiv \text{diag} \left\{ 0, -\frac{4\pi^{2} \times 4}{N^{2}} \sigma_{2}^{2}, \cdots, -\frac{4\pi^{2} (N-1)^{2}}{N^{2}} \sigma_{2}^{2} \right\} \).

To minimize (21), it requires

\[
\sum_{m=1}^{N_{i}} \sum_{i=0}^{M-1} \mathbf{F}(L_{max}) \mathbf{X}^{pH}_{m}(i) \mathbf{F}^{H}_{p}(i) = \frac{E_{p}}{N_{t}} \mathbf{B}_{L} \cdots \mathbf{B}_{L},
\]

where \( \mathbf{B}_{L} = \left[ \mathbf{I}_{L_{max}}, \cdots, \mathbf{I}_{L_{max}}, \mathbf{0}_{L_{max} \times (N_{p}-L_{max}N_{i})} \right] \), and \( N_{p} \geq L_{max}N_{i} \) should be satisfied. By resolving (20) to (22), the suboptimal pilots and their placements are

\[
\begin{align*}
\arg & \left\{ \mathbb{E} \left( \text{trace} \left\{ \sum_{i=0}^{M-1} \mathbf{P}_{k,m}(i) \mathbf{P}_{k,n}(i) \right\} \right) \right\} = \frac{2\pi\theta_{x}(m-1)L_{max}}{N_{i}}, \\
\text{s.t.} & \quad L_{max}N_{i} \leq N_{p} \leq N, \\
& \quad N_{p} = \text{integer}; \\
& \quad (\theta_{2} - \theta_{1} \cdot 1)_{N} = \cdots = (\theta_{N_{p}} - \theta_{N_{p} - 1} \cdot 1)_{N} = (\theta_{1} \cdot 1 - \theta_{N_{p}} \cdot 1)_{N}, l = 1, 2, \ldots, L_{max} - 1; \\
& \quad \theta_{x}(n-m)L_{max} \neq \text{integer}; \\
& \quad ||\mathbf{X}^{pH}_{m}(i)||_{\theta_{x} \theta_{z}} = \cdots = ||\mathbf{X}^{pH}_{m}(i)||_{\theta_{x} \theta_{z}} \geq 0; \\
& \quad \sum_{i=1}^{M} ||\mathbf{X}^{pH}_{m}(i)||_{\theta_{x} \theta_{z}}^{2} = \frac{E_{p}}{N_{p}N_{t}}; \\
& \quad \sum_{i=1}^{M} \sum_{i=1}^{N_{p}} ||\mathbf{X}^{pH}_{m}(i)||_{\theta_{x} \theta_{z}}^{2} = \frac{E_{p}}{N_{t}}.
\end{align*}
\]

V. LINEAR MINIMUM MEAN SQUARE ERROR (LMMSE) ESTIMATOR BY EXPLOITING THE CHANNEL CORRELATION

Sections III to IV considered pilot design for an uncorrelated MIMO-OFDM channel. However, channel correlation is usually exploited to improve the performance, as discussed in [15], [16].

A. LMMSE Estimator

Define the received vector on the \( N_{r} \) receive antennas as \( \mathbf{r} = [r_{1}^{T}, \cdots, r_{N_{r}}^{T}]^{T} = \mathbf{P} \mathbf{h} + \mathbf{D} + \mathbf{\eta} \), where \( \mathbf{P} = \text{diag} \{ \mathbf{P}_{1}, \cdots, \mathbf{P}_{N_{r}} \}, \mathbf{D} = \text{diag} \{ \mathbf{D}_{1}, \cdots, \mathbf{D}_{N_{r}} \}, \mathbf{h} = [h_{1}^{T}, \cdots, h_{N_{r}}^{T}]^{T}, \mathbf{\eta} = [\eta_{1}^{T}, \cdots, \eta_{N_{r}}^{T}]^{T} \). Based on it, we define

\[
\hat{\mathbf{h}} = \mathbf{P}^{T} \mathbf{r} = \mathbf{h} + \mathbf{P}^{T} \mathbf{D} + \mathbf{P}^{T} \mathbf{\eta},
\]

where \( \mathbf{P}^{T} = (\mathbf{P}^{H} \mathbf{P})^{-1} \mathbf{P}^{H} \). The channel correlation matrix is given by

\[
\Phi = \mathbb{E} \{ \mathbf{h} \mathbf{h}^{H} \} = \left[ \begin{array}{cccc} \Phi_{1,1} & \Phi_{1,2} & \cdots & \Phi_{1,N_{r}} \\
\Phi_{2,1} & \Phi_{2,2} & \cdots & \Phi_{2,N_{r}} \\
\vdots & \vdots & \ddots & \vdots \\
\Phi_{N_{r},1} & \Phi_{N_{r},2} & \cdots & \Phi_{N_{r},N_{r}} \end{array} \right],
\]

where \( \Phi_{k,i} = \mathbb{E} \{ \mathbf{h}_{k} \mathbf{h}_{i}^{H} \} = \mathbb{E} \{ \mathbf{h}_{k,m} \mathbf{h}_{i,m}^{H} \} = \theta_{k,m} \delta_{i,m} \).

The correlation matrix of \( \mathbf{h} \) is given by

\[
\mathbf{C}_{hh} = \mathbb{E} \{ \mathbf{h} \mathbf{h}^{H} \} = \mathbf{\Omega} \mathbf{\Phi} \mathbf{\Phi}^{H} + \sigma_{w}^{2} (\mathbf{P}^{H} \mathbf{P})^{-1},
\]

where \( \mathbf{\Omega} = \left( \mathbf{I}_{L_{max}N_{r}N_{i}} + (\mathbf{P}^{H} \mathbf{P})^{-1} \mathbf{P}^{H} \mathbf{D} \right) \). We also have

\[
\mathbf{C}_{hh} = \mathbb{E} \{ \mathbf{h} \mathbf{h}^{H} \} = \mathbf{\Omega} \mathbf{\Phi}^{H} \mathbf{h}.
\]

with its covariance matrix of estimation error being given by

\[
\mathbf{C}_{e} = \mathbb{E} \{ \hat{\mathbf{h}} \mathbf{L}_{\text{mmse}} - \mathbf{h} \} \left( \hat{\mathbf{h}} \mathbf{L}_{\text{mmse}} - \mathbf{h} \right)^{H}.
\]

The MSE of \( \hat{\mathbf{h}} \mathbf{L}_{\text{mmse}} \) is MSE (\( \hat{\mathbf{h}} \mathbf{L}_{\text{mmse}} \)) = \( \frac{\text{trace} (\mathbf{C}_{e})}{L_{max}N_{i}N_{r}} \).
B. Adaptive Power Allocation to Minimize MSE

Using the eigenvalue decomposition of $\mathbf{C}_e$, we have

$$\text{MSE} \left( \hat{\mathbf{h}}_{\text{LMMSE}} \right)_{L_{\text{max}}N_tN_r-1} = \sum_{j=0}^{L_{\text{max}}N_tN_r-1} \frac{\sigma_w^2 \lambda_{\phi,j}^2 \lambda_{p,j}^2}{L_{\text{max}}N_tN_r \left( \sigma_w^2 \lambda_{p,j}^2 + \lambda_{\phi,j}^2 (\lambda_{v,j}^2 + \lambda_{v,j}^2) \right)^2},$$

(28)

where $\lambda_{\phi,j}^2$ represents the $j$-th eigenvalue of $\Phi$, $\lambda_{p,j}^2$ and $\lambda_{v,j}^2$ represent the $j$-th eigenvalue of $\mathbf{P}^H \mathbf{P}$ and $\mathbf{V} \mathbf{V}^H$, respectively. By resolving (36) in Appendix B, the optimal $\lambda_{p,j}^2$ to minimize the MSE $\left( \hat{\mathbf{h}}_{\text{LMMSE}} \right)$ can be derived as

$$\lambda_{p,j}^2 = \frac{\left( N_tL_{\text{max}}E_p + \sum_{m=0}^{L_{\text{max}}N_tN_r-1} \sigma_w^2 \lambda_{\phi,m}^2 \right)}{L_{\text{max}}N_tN_r - 1} - \frac{\sigma_w^2 \lambda_{\phi,j}^2}{\lambda_{\phi,j}^2},$$

(29)

where $0 \leq j \leq L_{\text{max}}N_tN_r - 1$. By considering $\sum_{j=0}^{L_{\text{max}}N_tN_r-1} \lambda_{p,j}^2 = N_tL_{\text{max}}E_p$, the minimum MSE $\left( \hat{\mathbf{h}}_{\text{LMMSE}} \right)$ is given by

$$\min \left\{ \text{MSE} \left( \hat{\mathbf{h}}_{\text{LMMSE}} \right) \right\} = \sum_{j=0}^{L_{\text{max}}N_tN_r-1} \frac{\sigma_w^2 \lambda_{\phi,j}^2}{N_tL_{\text{max}}E_p + \sum_{m=0}^{L_{\text{max}}N_tN_r-1} \sigma_w^2 \lambda_{\phi,m}^2}. $$

(30)

As compared to (30), when uniform power allocation to the pilots is applied; i.e., when $\mathbf{P}^H \mathbf{P} = E_p N_tL_{\text{max}}N_tN_r$, the suboptimal MSE $\left( \hat{\mathbf{h}}_{\text{LMMSE}} \right)$ is given by

$$\text{MSE} \left( \hat{\mathbf{h}}_{\text{LMMSE}} \left| \mathbf{P}^H \mathbf{P} = \frac{E_p}{N_t} \mathbf{I}_{L_{\text{max}}N_tN_r} \right. \right) = \sum_{j=0}^{L_{\text{max}}N_tN_r-1} \frac{\sigma_w^2 \lambda_{\phi,j}^2}{N_tL_{\text{max}} \left( N_t \sigma_{\phi,j}^2 + \lambda_{\phi,j}^2 \right) E_p}. $$

(31)

VI. NUMERICAL RESULTS

A multipath-fading channel with path gains $h_{k,i}(l) = e^{-l}$ and the channel correlation coefficient $\rho_{k,i,m,n} \neq n = 0.5$ is considered, where $l = 0, 1, 2, \ldots, L_{\text{max}} - 1$, $k, i = 1, 2, \ldots, N_t$ and $m, n = 1, 2, \ldots, N_t$. We also assume that $N_t = 128$, $M = 1, 2, 4$, $L_{\text{max}} = 4$, $N_t = 2, 4$, $N_r = 2, 4$ and $N_p = L_{\text{max}}N_t$. Here, we use the Normalized-Pilot-to-Noise Ratio (NPNR), i.e., $\text{NPNR} = \frac{E_p}{N_tN_p \sigma_w^2}$, instead of the PNR to represent the normalized PNR of each pilot subcarrier for each transmit antenna, and without loss of generality, the average power of each pilot subcarrier is also assumed to be identical to that of each data subcarrier, unless otherwise stated. Multiple frequency offsets are considered for different transmit and receive antennas with $\epsilon_{k,i} \sim \mathcal{N}(0, \sigma^2)$. In a MIMO-OFDM system, pilots transmitted by $N_t$ transmit antennas are modulated into consecutive $M \geq 1$ symbols (Sections III to IV). When $M = 1$, the pilot subcarriers for the different transmit antennas can be either orthogonal in the frequency-domain or not. Fig. 1 to Fig. 5 compares the performance of the proposed LS estimator and the BLM estimator with $M = 1$. Fig. 1 shows that with a single frequency offset, the BLM estimator exhibits a performance floor at high NPNR, and a larger $\sigma^2_p$ implies a worse MSE performance. However, the proposed estimator eliminates this floor, and for different $\sigma^2_p$, it achieves an identical MSE. Fig. 2 to Fig. 5 are for the multiple frequency offsets case. Fig. 2 and Fig. 3 present the simulation results with overlapped pilot subcarriers allocated for the different transmit antennas when $N_t = 2$ and $N_t = 4$, respectively. In the multiple frequency offsets case, both the BLM estimator and the proposed...
Fig. 3. LS channel estimation in multiple frequency offsets MIMO-OFDM systems with $M = 1$, $N_t = 4$ and pilot subcarriers of different transmit antennas are overlapped.

Fig. 4. LS channel estimation in multiple frequency offsets MIMO-OFDM systems with $M = 1$, $N_t = 2$ and pilot subcarriers of different transmit antennas are orthogonal.

Fig. 5. LS channel estimation in multiple frequency offsets MIMO-OFDM systems with $M = 1$, $N_t = 4$ and pilot subcarriers of different transmit antennas are overlapped.

Fig. 6. LS channel estimation in multiple frequency offsets MIMO-OFDM systems with $M = 4$, $N_t = 4$ and pilot subcarriers of different transmit antennas are orthogonal.

one exhibit a performance floor. However, the latter maintains a performance advantage. For example, when $N_t = 2$ and NPNR=20 dB, the MSE of the proposed estimator is about $1.6 \times 10^{-3}$ (or $5 \times 10^{-3}$) for $\sigma^2 = 10^{-3}$ (or $\sigma^2 = 10^{-2}$), and that for the BLM estimator is about $5.1 \times 10^{-3}$ (or $3.1 \times 10^{-2}$) for $\sigma^2 = 10^{-3}$ (or $\sigma^2 = 10^{-2}$).

Fig. 4 and Fig. 5 compare the proposed estimator and the BLM estimator with orthogonal pilot placement in the frequency-domain. Note that even with orthogonal pilots placement, a performance floor will always appear at the proposed estimator in the presence of multiple frequency offsets, although the proposed estimator still outperforms the BLM estimator. Since in frequency-domain orthogonal pilot placement, the IPI is reduced as compared to that in the overlapped pilot placement, the MSE is smaller than that of the latter, but at the cost of lower spectral efficiency. For example, when $N_t = 4$, $\sigma^2 = 10^{-2}$ and NPNR=20 dB, the MSE for the proposed estimator with overlapped pilots placement is about $1.7 \times 10^{-2}$, whereas that with orthogonal pilots placement is about $1.2 \times 10^{-2}$, as shown in Fig. 3 and Fig. 5.

When $N_t > 1$, pilots for each transmit antenna can be modulated into consecutive $M > 1$ symbols, the envelope can either be CE or not. The simulation results with $N_t = M = 4$ are illustrated in Fig. 6, where frequency-domain orthogonal
pilot placement is assumed for the different transmit antennas in each symbol (a similar result can be achieved in the case of overlapped pilots placement for different transmit antennas). The proposed pilots, either a CE or Multiple-Envelope, can achieve the same performance advantage over that of the BLM estimator, provided that the total pilots power $E_p$ remains fixed. For a given $E_p$, we can also conclude that an identical performance can be achieved in the proposed estimator with either $M = 1$ or $M > 1$. For example, when $N_t = 4$, $\sigma_e^2 = 10^{-2}$ and NPNR=20 dB, an MSE of about $1.5 \times 10^{-2}$ can be achieved with either $M = 1$ or $M = 4$, as shown in Fig.5 and Fig.6. No matter what pilots are used, multiple symbol pilots have no performance advantage over that of single-symbol pilots in terms of MSE, and more seriously, multiple-symbol pilot modulation has lower spectrum efficiency. Multiple-symbol pilot modulation is nevertheless applied for two main reasons: (1) it reduces the peak power of each pilot subcarrier in each symbol; and (2) it has an advantage over single-symbol pilot modulation in tracking the time-variant channel. In the presence of multiple frequency offsets, non-zero Inter-Antenna-Interference will degrade the channel estimation performance if neither time-domain nor frequency-domain orthogonal pilot placement is utilized. To eliminate this Inter-Antenna-Interference, we can make $M = N_t$ and ensure that the pilots and data transmitted by the different transmit antennas are orthogonal in the
time-domain; i.e., when one transmit antenna is transmitting, the other transmit antennas remain silent. Fig. 7 presents the performance analysis of the proposed estimator in two pilot placement cases: the time-domain orthogonal and non-orthogonal pilot placements. The simulation results show that the performance floor at high NPNR can be mitigated in the case of orthogonal pilot placement, but at the cost of lower spectral efficiency than that of non-orthogonal placement.

Note that (11) is optimal in flat fading, but the MSE performance may degrade in frequency selective fading. Fig. 8 shows that a performance degradation of about 0.4 dB (or about 0.9 dB) is achieved in the multipath-fading channel at a high NPNR when $\sigma^2 = 10^{-3}$ (or when $\sigma^2 = 10^{-2}$), as compared to the MSE performance in the flat-fading channels. For a small $\sigma^2$, this performance degradation is negligible.

We also compare the performance of the proposed LS estimator with Cui’s estimator [2] with multiple frequency offsets, and the numerical results are shown in Fig. 9. Note that in Cui’s estimator, the total preamble is used for training only. For a fair comparison, we use 16 pilot subcarriers in the proposed estimator and use a length-16 training sequence in Cui’s estimator, with an identical total power $E_p$ being allocated to the pilots/training in each estimator. If the data subcarriers in the proposed symbol are not modulated, the proposed estimator always outperforms Cui’s estimator at a high NPNR, and a larger performance gain can be achieved as $\sigma^2$ increases. For example, at a high NPNR, the performance improvement of the proposed estimator with $E_d = 0$ over Cui’s estimator is about 0.7 dB (or 2.6 dB) with $\sigma^2 = 10^{-2}$ (or $\sigma^2 = 10^{-1}$). A performance degradation will be achieved in the proposed estimator if $E_d \neq 0$. In this simulation, we assume that $E_d/E_p = 10^{-1}$, and the MSE performance of the proposed estimator is about 0.5 dB worse than that of Cui’s estimator. In other words, Cui’s estimator outperforms the proposed estimator by sacrificing the spectral efficiency.

When the covariance matrix is available at both the transmitter and the receiver, an LMMSE estimator can be designed to improve the estimation accuracy. Fig. 10 compares the channel estimation performance of the proposed LS and the proposed LMMSE estimators. Multiple frequency offsets are considered with $\sigma^2 = 10^{-2}$. The LMMSE estimator with CE pilot modulation can considerably reduce the MSE by applying adaptive pilot power allocation at the transmitter. For example, when $N_t = N_r = 2$, the performance advantage of the proposed LMMSE estimator with adaptive pilot power allocation over that with CE pilot modulated at a low NPNR is about 3.1 dB, and its performance advantage over that of the LS estimator is about 5.8 dB.

VII. CONCLUSIONS

The optimal pilot design and placement for channel estimation in MIMO-OFDM with multiple frequency offsets were discussed. The IPI was eliminated in the proposed estimator, and, therefore, a performance advantage over that of conventional estimators was achieved. Given a total pilot power $E_p$, the pilots for the different transmit antennas were modulated into one or consecutive multiple OFDM symbols with each pilot subcarrier in each symbol being modulated as a CE or not (a CE is required in pilot design only when $M = 1$). With the channel covariance matrix known at both

APPENDIX A

ANALYSIS OF $G_{k,m,n \neq m}$

Define $G_{k,m,n} = NF\left(L_{\max}\right)\Pi_{mn}F^H\left(L_{\max}\right)$ where $\Pi_{mn} = \sum_{i=0}^{M-1} F\left(L_{\max}\right)X_{p,m}^H(i)E_{c}\bar{X}_{k,m,n}^H(i)$, where the $\theta_{m,l}\theta_{n,s}$-th element of $\Pi_{mn}$ is non-zero, as given by

$$\Pi_{mn} = \sum_{i=0}^{M-1} \left[X_{p,m}^H(i)\right]_{\theta_{m,l}\theta_{n,s}} \left[E_{c}\bar{X}_{k,m,n}^H(i)\right]_{\theta_{n,s}\theta_{m,l}} \left[X_{p,n}^H(i)\right]_{\theta_{n,s}\theta_{m,l}},$$

(32)

where $1 \leq l, s \leq N_\ell$. Let us consider the following two cases:

1) Pilot Subcarriers for the $m$-th and the $n$-th Transmit Antennas are Orthogonal in the frequency-domain: In this case, $\omega_{m,l} \neq \omega_{n,s}$ for each $l$ and $s$, and it is easy to show that $F\left(L_{\max}\right)\Pi_{mn}F^H\left(L_{\max}\right) = O_{L_{\max}}$, so that $G_{k,m,n \neq m} = O_{L_{\max}}$ for each $m \neq n$.

2) Pilot Subcarriers for the $m$-th and the $n$-th Transmit Antennas are Overlapped in the frequency-domain ($\omega_{m,z} = \omega_{n,z} = \omega_z$ for $z = 1, 2, \cdots, N_p$): Define $\Pi_{mn}$ as

$$\Pi_{mn} = \sum_{i=0}^{M-1} \left[X_{p,m}^H(i)\right]_{\theta_{m,l}\theta_{n,s}} \left[E_{c}\bar{X}_{k,m,n}^H(i)\right]_{\theta_{n,s}\theta_{m,l}} \left[X_{p,n}^H(i)\right]_{\theta_{n,s}\theta_{m,l}},$$

(33)

where $1 \leq l, s \leq N_\ell$. Also define a $N_p \times N_p$ matrix $\Pi_{mn}$, which is generated by deleting all the zero rows and columns of $\Pi_{mn}$. Evidently, $\Pi_{mn} = \Pi^H_{mn}$. $G_{k,m,n}$ can be rewritten as

$$G_{k,m,n} = N \left[\bar{f}_0, \cdots, \bar{f}_{N_p}\right] \Pi_{mn} \left[\bar{f}_0, \cdots, \bar{f}_{N_p}\right]^H = N \left[\mu_1, \cdots, \mu_{L_{\max}}\right] \Pi_{mn} \left[\mu_1, \cdots, \mu_{L_{\max}}\right]^T,$$

(34)

where $\mu^T$ is the $z$-th row of $F\left(L_{\max}\right)$. Therefore, the $u$-th element of $G_{k,m,n}$ is $G_{k,m,n}^\mu_{u,t} = N \mu_u^\mu \Pi_{mn}^u \mu_t$, and $G_{k,m,n} = O_{L_{\max}}$ is achieved only when $\mu_u^\mu \Pi_{mn}^u \mu_t = 0$, $1 \leq u, t \leq L_{\max}$.

APPENDIX B

OPTIMAL EIGENVALUES FOR ADAPTIVE ALLOCATION

Define a cost function,

$$C\left(\lambda^2_{p,0,1, \cdots, \lambda^2_{p,L_{\max}} N_t N_r - 1}\right) = \sum_{j=0}^{L_{\max}N_t N_r - 1} \sigma_{w,j}^2 \lambda^2_{p,j} \lambda^2_{j}$$

$$+ \beta \left(\sum_{j=0}^{L_{\max}N_t N_r - 1} \lambda^2_{p,j} - N_t L_{\max} E_p\right),$$

(35)
where $\beta$ is a positive real coefficient.

To minimize (35), let us take the partial derivative to $C(\lambda_{p,j}^2 \cdots \lambda_{p,j}^{L_{\max}}, N_t, N_{r-1})$ with respect to each $\lambda_{p,j}^2 (0 \leq j \leq L_{\max}N_tN_{r-1})$ and set the result to zero. The result is

$$L_{\max}N_tN_r \left( \frac{2}{\lambda_{p,j}^2} + \frac{2}{\lambda_{\phi,j}^2} \right) \beta - \frac{1}{2} \lambda_{p,j}^2 (4 \lambda_{p,j}^2 - 2 \beta \lambda_{\phi,j}^2) = 0,$$

(36)

st.

$$\sum_{j=0}^{L_{\max}N_tN_{r-1}} \lambda_{p,j}^2 = N_tL_{\max}E_p.$$