Infinite Series Representations of the Trivariate and Quadrivariate Nakagami-*m* Distributions

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***Abstract*— In this paper, using Miller's approach and Dougall's identity, we derive new infinite series representations for the quadrivariate Nakagami-*m* joint density function, cumulative distribution function (cdf) and characteristic functions (chf). The classical joint density function of exponentially correlated Nakagami-*m* variables can be identified as a special case of the joint density function obtained here. Our results are based on the most general arbitrary correlation matrix possible. Moreover, the trivariate density function, cdf and chf for an arbitrary correlation matrix are also derived from our main result. Bounds on the series truncation error are also presented. Finally, we develop several representative applications: the outage probability of triple branch selection combining (SC), the moments of the equal gain combining (EGC) output signal to noise ratio (SNR) and the moment generation function of the generalized SC(2,3) output SNR in an arbitrarily correlated Nakagami-*m* environment. Simulation results are also presented to verify the accuracy of our theoretical results.**

***Index Terms*— Characteristic function (chf), constant correlation, exponential correlation, fading, quadrivariate Nakagami-*m* distribution, trivariate Nakagami-*m* distribution.**

I. INTRODUCTION

SINCE multipath fading (or simply fading) is an important radio propagation mechanism affecting wireless communications, several statistical models have been used to describe the received envelope. For example, the Rayleigh, Rician and Weibull distributions are used to characterize the received envelope over a small geographical area (distances on the order of half the signal wavelength) and the log normal distribution, over large areas. The Nakagami-*m* distribution [1] is a more general model which includes a variety of fading environments. Furthermore, [2], [3] demonstrate that it is more flexible and fits more accurately to the experimental data for many physical propagation channels than other distributions.

The joint probability density function (pdf) of N correlated Nakagami variables occurs frequently in many performance analysis problems [4]-[8]. The bivariate Nakagami density function is given in [1]. Tan and Beaulieu [9] derive an infinite series representation for bivariate Nakagami cdf. Blumenson and Miller [10] derive the joint pdf of p generalized Rayleigh variables when the underlying Gaussian variables have tridiagonal form of inverse covariance matrix. Miller in [11] derives the joint pdf of three generalized Rayleigh variables when the underlying Gaussian variables have an arbitrary covariance matrix. Karagiannidis *et al.* extend Blumenson and Miller's result [10] to come up with the multivariate Nakagami-*m* distribution function for exponentially correlated underlying Gaussian variables [12]. Green's matrix approach is used in [13] to approximate an arbitrary correlation matrix to a matrix with a tridiagonal form of inverse such that the results given in [12] can be harnessed for applications. A comprehensive analysis of the multivariate Rayleigh distribution can be found in [14]. Recently, Chen and Tellambura [15] derive new infinite series for quadrivariate Rayleigh distribution for possibly the most general correlation matrix using Miller's approach [16].

In this paper, we use Miller's approach [11] and Dougall's identity [17] to obtain new infinite series representations for the joint pdf, cdf and chf of four correlated Nakagami-*m* variables when m is either an integer or half integer such that $m \geq 3/2$. Careful study of the related literature reveals that in all of the above studies, a restriction is placed on the inverse correlation matrix. The restriction is that a certain number of off-diagonal terms of the inverse correlation matrix must be zero. The inverse correlation matrix used in this paper is the most general arbitrary matrix that appears to be analytically tractable - just one off-diagonal term of the inverse correlation matrix must be zero - and it is the same restriction considered in [15]. The trivariate joint pdf, cdf and chf for an arbitrary correlation matrix are presented as special cases of our more general result. The trivariate scenario is discussed in detail with respect to the exponential [4] and constant correlation models since no result for trivariate Nakagami-*m* distribution for constant correlation model appears in the open literature. However, obtaining the quadrivariate Nakagami-*m* distribution for an arbitrary correlation matrix seems intractable. The bound on the series truncation error and numerical results are presented to show the convergence behavior of the series. Furthermore, the outage probability of three branch SC, moments of the three branch EGC output SNR and the output moment generation function of the generalized SC (2,3) receiver are derived as applications.

This paper is organized as follows. The main derivation of quadrivariate Nakagami-*m* distribution is given in Section II. The cdf, chf and simplifications to existing cases are also

Manuscript received March 28, 2006; revised September 2, 2006; accepted October 31, 2006. The associate editor coordinating the review of this paper and approving it for publication was A. Molisch.

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Digital Object Identifier 10.1109/TWC.2007.060120.

presented. Section III discusses the trivariate Nakagami-*m* distribution with emphasize on the exponential and constant correlation models. In addition, brief numerical results are presented to show the infinite series' convergence behavior as a function of the fading and correlation parameters. Section IV deals with the applications of the new results and the numerical and simulation results. Finally, concluding remarks are made in Section V.

II. QUADRIVARIATE DISTRIBUTION

Let $\{\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3, \mathbf{X}_4\}$ be four zero mean Gaussian vectors with $\mathbf{X}_i = (x_{1i}\, x_{2i}\, \ldots \ldots\, x_{2mi})^T$ for all $1 \leq i \leq 4$. Here $(\cdot)$ denotes the transpose of a matrix. Let $\mathbf{V}_j = (x_{j1}\, x_{j2}\, x_{j3}\, x_{j4})$, $1 \leq j \leq 2m$ be independent four dimensional zero mean Gaussian vectors composed of the jth components of $\mathbf{X}_i$. In this display, the columns are the $2m$-dimensional Gaussian vectors

$$\begin{array}{c|cccc} & \mathbf{X}_1 & \mathbf{X}_2 & \mathbf{X}_3 & \mathbf{X}_4 \\ \hline \mathbf{V}_1 & x_{11} & x_{12} & \ldots & x_{14} \\ \mathbf{V}_2 & x_{21} & x_{22} & \ldots & x_{24} \\ & \ldots & \ldots & \ldots & \ldots \\ \mathbf{V}_{2m} & x_{2m,1} & x_{2m,2} & \ldots & x_{2m,4} \end{array} \tag{1}$$

and the rows $\mathbf{V}_j$ are independent from each other and with identical covariance matrix $\mathbf{M}_4$. The inverse correlation matrix of $\mathbf{V}_j$ is

$$\mathbf{W}_4 = \mathbf{M}_4^{-1} = \begin{pmatrix} w_{11} & w_{12} & w_{13} & w_{14} \\ w_{12} & w_{22} & w_{23} & w_{24} \\ w_{13} & w_{23} & w_{33} & w_{34} \\ w_{14} & w_{24} & w_{34} & w_{44} \end{pmatrix}. \tag{2}$$

The derivation of the joint pdf becomes analytically tractable when one or more off-diagonal terms of $\mathbf{W}_4$ are zero. For example, in the following previous work [7], [8], [10], [12], [13], the inverse correlation matrix is restricted to be tri-diagonal, i.e., $w_{kl} = 0$ for $|k - l| > 1$.

In our results, the restriction is only $w_{14} = 0$ [15]. This appears to be the most general inverse covariance matrix that can be handled by the Miller's approach. Clearly, our results are more general than [7], [8], [10], [12], [13]. However, the solution for an arbitrary covariance matrix (i.e., not subject to $w_{14} = 0$) seems intractable.

The amplitudes $r_i = |\mathbf{X}_i|$ $(1 \leq i \leq 4)$, being the square root of sum of squares of $2m$ zero mean i.i.d. Gaussian random variables, are Nakagami-*m* random variables [1]. Here $|\cdot|$ denotes the norm of a column vector . The joint pdf of $\{\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3, \mathbf{X}_4\}$ is clearly given by

$$\begin{aligned} f(\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3, \mathbf{X}_4) &= \prod_{j=1}^{2m} f(\mathbf{V}_j) \\ &= \frac{W_4^m}{(2\pi)^{4m}} \exp\left\{-\frac{1}{2}\sum_{j=1}^{2m} \mathbf{V}_j \mathbf{W}_4 \mathbf{V}_j^T\right\}. \end{aligned} \tag{3}$$

Expanding the quadratic form in (3) and interchanging $\mathbf{V}_j$'s by $\mathbf{X}_i$ (see the display in (1)), we find that

$$\begin{aligned} f(\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3, \mathbf{X}_4) &= \frac{W_4^m}{(2\pi)^{4m}} \exp\left\{-\frac{1}{2}\sum_{i=1}^{4} w_{ii} r_i^2\right\} \\ &\times \exp\left(-w_{23}\mathbf{X}_2^T \mathbf{X}_3\right) \exp\left\{-\mathbf{X}_1^T\left(w_{12}\mathbf{X}_2 + w_{13}\mathbf{X}_3\right)\right\} \\ &\times \exp\left\{-\mathbf{X}_4^T\left(w_{24}\mathbf{X}_2 + w_{34}\mathbf{X}_3\right)\right\}. \end{aligned} \tag{4}$$

>From this pdf (4), we need to integrate out $\mathbf{X}_i$, $1 \leq i \leq 4$, subject to the constraints $r_i = |\mathbf{X}_i|$, which will yield the joint pdf of correlated Nakagami variables $\{r_1, r_2, r_3, r_4\}$ [11]. Consequently, we write the joint Nakagami pdf as given in (5), where $d\sigma_{x_i}$[1], $1 \leq i \leq 4$ are the elements of surface area and W_4 is the determinant of the square matrix $\mathbf{W}_4$. The second integral in (5) can be evaluated as [11, eq.2.2.9]

$$\begin{aligned} \int_{|\mathbf{X}_1|=r_1} \exp\left\{-\mathbf{X}_1^T\left(w_{12}\mathbf{X}_2 + w_{13}\mathbf{X}_3\right)\right\} d\sigma_{x_1} &= (2\pi r_1)^m \\ \times \left|w_{12}\mathbf{X}_2 + w_{13}\mathbf{X}_3\right|^{1-m} I_{m-1}\left(r_1\left|w_{12}\mathbf{X}_2 + w_{13}\mathbf{X}_3\right|\right) \end{aligned} \tag{6}$$

where $I_n(x)$ is the nth order modified Bessel function of the first kind [18], and the third integral follows the same form. Furthermore, the right hand side of (6) can be rewritten using the generalized Neumann addition formula [18, p. 365] when $m \geq 3/2$ being an integer or half integer, as given in (7), where $\Gamma(x)$ is the gamma function [19], $C_n^\lambda(x)$ denotes the Gegenbauer or ultraspherical polynomials [19] and θ is the angle between the vectors $\mathbf{X}_2$ and $\mathbf{X}_3$. The ultraspherical polynomials of degree n are the coefficients of r^n in the

[1] If $\mathbf{X} = \{x_1, x_2, \cdots, x_n\}$ is an n-dimensional vector, then $\int_{|\mathbf{X}|=r} d\sigma_x$ represents the surface area of the sphere of radius r. The integral is taken over the surface of the sphere where $d\sigma_x$ is the element of surface area.

$$\begin{aligned} f(r_1, r_2, r_3, r_4) = \frac{W_4^m}{(2\pi)^{4m}} \exp\left\{-\frac{1}{2}\left(w_{11}r_1^2 + w_{22}r_2^2 + w_{33}r_3^2 + w_{44}r_4^2\right)\right\} \int_{|\mathbf{X}_2|=r_2} \exp\left(-w_{23}\mathbf{X}_2^T\mathbf{X}_3\right) d\sigma_{x_2} \\ \times \int_{|\mathbf{X}_1|=r_1} \exp\left\{-\mathbf{X}_1^T\left(w_{12}\mathbf{X}_2 + w_{13}\mathbf{X}_3\right)\right\} d\sigma_{x_1} \int_{|\mathbf{X}_4|=r_4} \exp\left\{-\mathbf{X}_4^T\left(w_{24}\mathbf{X}_2 + w_{34}\mathbf{X}_3\right)\right\} d\sigma_{x_4} \int_{|\mathbf{X}_3|=r_3} d\sigma_{x_3}. \end{aligned} \tag{5}$$

$$\begin{aligned} (2\pi r_1)^m \left|w_{12}\mathbf{X}_2 + w_{13}\mathbf{X}_3\right|^{1-m} I_{m-1}\left(r_1\left|w_{12}\mathbf{X}_2 + w_{13}\mathbf{X}_3\right|\right) = \frac{2^{m-1}(2\pi)^m\Gamma(m-1)r_1}{\left(w_{12}w_{13}r_2r_3\right)^{m-1}} \sum_{k=0}^{\infty}(m+k-1)I_{m+k-1}\left(w_{12}r_1r_2\right) \\ \times I_{m+k-1}\left(w_{13}r_1r_3\right) C_k^{m-1}\left(\cos\theta\right). \end{aligned} \tag{7}$$

power-series expansion given by [19]

$$\left(1-2xr+r^2\right)^{-\lambda}=\sum_{n=0}^{\infty}C_n^{\lambda}(x)r^n$$

where $\lambda > -\frac{1}{2}$. The following closed-form expression is available:

$$C_n^{\lambda}(\cos\theta)=\sum_{\substack{k,l=0\\k+l=n}}^{n}\frac{\Gamma(\lambda+k)\Gamma(\lambda+l)}{k!l![\Gamma(\lambda)]^2}\cos(k-l)\theta.$$

The fourth integral in (5) represents the surface are of a $2m$ dimensional sphere, which can be evaluated using [11, eq.1.5.19]. Upon the evaluation of these integrals, (5) can be written as shown in (8) at the bottom. The product of two ultraspherical polynomials can be written using the Dougall's identity given in [17, eq.6.8.4] as

$$C_p^{\lambda}(x)C_q^{\lambda}(x)=\sum_{n=0}^{min(p,q)}a(n,p,q)C_{p+q-2n}^{\lambda}(x)$$

where

$$a(n,p,q)=\frac{(p+q+\lambda-2n)(\lambda)_n(\lambda)_{p-n}(\lambda)_{q-n}}{(p+q+\lambda-n)n!(p-n)!(q-n)!(\lambda)_{p+q-n}}\times\frac{(2\lambda)_{p+q-n}(p+q-2n)!}{(2\lambda)_{p+q-2n}}$$

with $(\lambda)_n=\frac{\Gamma(\lambda+n)}{\Gamma(\lambda)}$ denotes the Pochhammer symbol [19] and $min(p,q)$ selects the minimum of p,q. We make use of the Dougall's identity in (8) to yield (9). The integral in (9) may be solved using [11, eq.2.2.26] to yield the quadrivariate Nakagami-*m* density as given in (10), where $\binom{n}{r}=\frac{n!}{r!(n-r)!}$. To the best of our knowledge (10) is a novel result. The above result would have been impossible unless $w_{14}=0$ and in general it is possible to obtain results analogous to (10) if at least one of the elements $w_{12},w_{13},w_{14},w_{23},w_{24},w_{34}$ is zero. In [12] and [13] the authors make the three elements w_{13},w_{14} and w_{24} equal to zero simultaneously to come up with tridiagonal form of inverse covariance matrix. So it is evident that our formulation includes all those scenarios as special cases. This makes our result best possible with the available tools.

The quadrivariate Rayleigh density derived by Chen and Tellambura [15] for the same scenario can be considered as the case when $m=1$. But, their result cannot be obtained directly from (10) due to the fact that (8) does not hold for $m=1$ [19]. However, the degenerated cases of (10) are valid for all $m\geq 1$ as shown below. Moreover, if a given covariance matrix does not match with the criteria mentioned above, we can use a constrained least square approach as given by [15] to find the best approximate matrix. Similar kind of arguments based on Green's matrix approach are used in [13] to approximate a given covariance matrix with a matrix having tridiagonal form of inverse.

As a sanity check, we next consider several special cases of (10).

A. Independent Nakagami-m Distributions

It is obvious that $\mathbf{W}_4$ is a diagonal matrix having $\{w_{11},w_{22},w_{33},w_{44}\}$ along the main diagonal if all Nakagami variables are independent. Since all off-diagonal elements are zero, letting k,l,n equal to zero and using the following identity involving the modified Bessel function of the first kind

$$\lim_{\alpha\to 0}\frac{I_n(\alpha x)}{\alpha^n}=\frac{x^n}{2^n\Gamma(n+1)} \tag{11}$$

$$f\left(r_1,r_2,r_3,r_4\right)=\frac{W_4^m 2^{m-1}\Gamma^2(m-1)r_1r_3r_4\exp\left\{-\frac{1}{2}\left(w_{11}r_1^2+w_{22}r_2^2+w_{33}r_3^2+w_{44}r_4^2\right)\right\}}{(2\pi)^m\Gamma(m)\left(w_{12}w_{13}w_{24}w_{34}\right)^{m-1}r_2^{2m-2}}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}(m+k-1)$$
$$\times(m+l-1)I_{m+k-1}\left(w_{12}r_1r_2\right)I_{m+k-1}\left(w_{13}r_1r_3\right)I_{m+l-1}\left(w_{24}r_2r_4\right)I_{m+l-1}\left(w_{34}r_3r_4\right)$$
$$\times\int_{|\mathbf{X}_2|=r_2}\exp\left(-w_{23}\mathbf{X}_2^T\mathbf{X}_3\right)C_k^{m-1}\left(\cos\theta\right)C_l^{m-1}\left(\cos\theta\right)d\sigma_{x_2}. \tag{8}$$

$$f\left(r_1,r_2,r_3,r_4\right)=\frac{W_4^m 2^{m-1}\Gamma^2(m-1)r_1r_3r_4\exp\left\{-\frac{1}{2}\left(w_{11}r_1^2+w_{22}r_2^2+w_{33}r_3^2+w_{44}r_4^2\right)\right\}}{(2\pi)^m\Gamma(m)\left(w_{12}w_{13}w_{24}w_{34}\right)^{m-1}r_2^{2m-2}}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{n=0}^{min(k,l)}a(n,k,l)$$
$$\times(m+k-1)(m+l-1)I_{m+k-1}\left(w_{12}r_1r_2\right)I_{m+k-1}\left(w_{13}r_1r_3\right)I_{m+l-1}\left(w_{24}r_2r_4\right)$$
$$\times I_{m+l-1}\left(w_{34}r_3r_4\right)\int_{|\mathbf{X}_2|=r_2}\exp\left(-w_{23}\mathbf{X}_2^T\mathbf{X}_3\right)C_{k+l-2n}^{m-1}\left(\cos\theta\right)d\sigma_{x_2}. \tag{9}$$

$$f\left(r_1,r_2,r_3,r_4\right)=\frac{W_4^m 2^{m-1}\Gamma^2(m-1)r_1r_2r_3r_4\exp\left\{-\frac{1}{2}\left(w_{11}r_1^2+w_{22}r_2^2+w_{33}r_3^2+w_{44}r_4^2\right)\right\}}{\Gamma(m)\left(w_{12}w_{13}w_{23}w_{24}w_{34}\right)^{m-1}\left(r_2r_3\right)^{m-1}}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{n=0}^{min(k,l)}(-1)^{k+l}a(n,k,l)$$
$$\times(m+k-1)(m+l-1)\binom{k+l+2m-2n-3}{2m-3}I_{m+k-1}\left(w_{12}r_1r_2\right)I_{m+k-1}\left(w_{13}r_1r_3\right)I_{m+l-1}\left(w_{24}r_2r_4\right)$$
$$\times I_{m+l-1}\left(w_{34}r_3r_4\right)I_{m+k+l-2n-1}\left(w_{23}r_2r_3\right). \tag{10}$$

we can simplify (10) as

$$f(r_1,r_2,r_3,r_4)=\prod_{i=1}^{4}\frac{2}{\Gamma(m)}\left(\frac{w_{ii}}{2}\right)^m r_i^{2m-1}\exp\left(-\frac{w_{ii}}{2}r_i^2\right). \tag{12}$$

Therefore, we get a product of four Nakagami-*m* probability density functions.

B. Exponentially Correlated Nakagami-m Variables

The covariance matrix of this model is given as [4], [12] $m_{i,j}=\rho^{|i-j|}$, where $0\le\rho<1$. Since $\mathbf{W}_4$ is tridiagonal, $w_{24}=w_{13}=0$, $w_{11}=w_{44}=\frac{1}{1-\rho^2}$, $w_{22}=w_{33}=\frac{1+\rho^2}{1-\rho^2}$ and $w_{12}=w_{23}=w_{34}=\frac{-\rho}{1-\rho^2}$. Substituting those values in (10) and using (11), we can write the joint pdf of exponentially correlated quadrivariate Nakagami variables as

$$\begin{aligned}f(r_1,r_2,r_3,r_4)&=\frac{r_1^m r_4^m r_2 r_3}{2^{m-1}\Gamma(m)\rho^{3(m-1)}\left(1-\rho^2\right)^3}\\&\quad\times\exp\left\{-\frac{r_1^2+r_4^2}{2\left(1-\rho^2\right)}-\frac{\left(1+\rho^2\right)\left(r_2^2+r_3^2\right)}{2\left(1-\rho^2\right)}\right\}\\&\quad\times I_{m-1}\left(\frac{\rho r_1 r_2}{1-\rho^2}\right)I_{m-1}\left(\frac{\rho r_2 r_3}{1-\rho^2}\right)I_{m-1}\left(\frac{\rho r_3 r_4}{1-\rho^2}\right).\end{aligned} \tag{13}$$

This expression exactly matches with the result given in [12, eq.3].

The quadrivariate cdf is given by definition [20]

$$F(r_1,r_2,r_3,r_4)=\int_0^{r_1}\int_0^{r_2}\int_0^{r_3}\int_0^{r_4}f(y_1,y_2,y_3,y_4)\times dy_1dy_2dy_3dy_4. \tag{14}$$

Substituting (10) in (14), followed by expansion of the Bessel function terms with equivalent infinite series and subsequent term by term integration assuming uniform convergence, we get the cdf as given in (15), where

$$\begin{aligned}\kappa_1&=i_1+i_2+m+k\\\kappa_2&=i_1+i_3+i_5+k+m+l-n\\\kappa_3&=i_2+i_4+i_5+k+m+l-n\\\kappa_4&=i_3+i_4+m+l\end{aligned}$$

and $\gamma(a,x)=\int_0^x t^{a-1}\exp(-t)dt$ is the incomplete gamma function [19]. The joint chf of quadrivariate Nakagami distribution is defined as [20]

$$\psi(v_1,v_2,v_3,v_4)=E\left\{\exp j\left(v_1r_1+v_2r_2+v_3r_3+v_4r_4\right)\right\} \tag{16}$$

where $j=\sqrt{-1}$ and $E\{\cdot\}$ denotes the mathematical expectation. Following the same line of arguments as for the cdf derivation and using [21, eq.3.462.1], the chf can be written as given in (17), where $D_v(x)$ is the parabolic cylinder function [21]. Equations (15) and (17) can easily be simplified to the respective cdf and chf for independent and exponential correlation cases.

C. Truncation Error

Let us assume that the cdf series (15) is truncated with K,L,I_1,I_2,I_3,I_4,I_5 in the variables k,l,i_1,i_2,i_3,i_4 and i_5 respectively. Then the remaining terms comprise the truncation error, E_T, which can be expressed using the approach given in [15]. After some manipulations and when ξ is given by

$$\begin{aligned}\xi&=\frac{W_4^m\Gamma^2(m-1)a(n,k,l)(m+k-1)(m+l-1)}{\Gamma(m)i_1!i_2!i_3!i_4!i_5!\Gamma\left(i_1+m+k\right)\Gamma\left(i_2+m+k\right)}\\&\quad\times\frac{\binom{k+l+2m-2n-3}{2m-3}w_{12}^{2i_1+k}w_{13}^{2i_2+k}w_{24}^{2i_3+l}w_{34}^{2i_4+l}w_{23}^{2i_5+k+l-2n}}{\Gamma\left(i_3+m+l\right)\Gamma\left(i_4+m+l\right)\Gamma\left(i_5+m+k+l-2n\right)}\\&\quad\times\frac{\Gamma(\kappa_1)\Gamma(\kappa_2)\Gamma(\kappa_3)\Gamma(\kappa_4)}{w_{11}^{\kappa_1}w_{22}^{\kappa_2}w_{33}^{\kappa_3}w_{44}^{\kappa_4}}\end{aligned}$$

$$\begin{aligned}F(r_1,r_2,r_3,r_4)&=\frac{W_4^m\Gamma^2(m-1)}{\Gamma(m)}\sum_{k,l=0}^{\infty}\sum_{n=0}^{min(k,l)}\sum_{i_1,i_2,i_3,i_4,i_5=0}^{\infty}\frac{(-1)^{k+l}a(n,k,l)(m+k-1)(m+l-1)}{i_1!i_2!i_3!i_4!i_5!w_{11}^{\kappa_1}w_{22}^{\kappa_2}w_{33}^{\kappa_3}w_{44}^{\kappa_4}}\\&\quad\times\frac{\binom{k+l+2m-2n-3}{2m-3}w_{12}^{2i_1+k}w_{13}^{2i_2+k}w_{24}^{2i_3+l}w_{34}^{2i_4+l}w_{23}^{2i_5+k+l-2n}}{\Gamma\left(i_1+m+k\right)\Gamma\left(i_2+m+k\right)\Gamma\left(i_3+m+l\right)\Gamma\left(i_4+m+l\right)\Gamma\left(i_5+m+k+l-2n\right)}\gamma\left(\kappa_1,\frac{w_{11}r_1^2}{2}\right)\gamma\left(\kappa_2,\frac{w_{22}r_2^2}{2}\right)\\&\quad\times\gamma\left(\kappa_3,\frac{w_{33}r_3^2}{2}\right)\gamma\left(\kappa_4,\frac{w_{44}r_4^2}{2}\right)\end{aligned} \tag{15}$$

$$\begin{aligned}\psi(v_1,v_2,v_3,v_4)&=\frac{16W_4^m\Gamma^2(m-1)}{\Gamma(m)}\exp\left\{-\frac{1}{4}\left(\frac{v_1^2}{w_{11}}+\frac{v_2^2}{w_{22}}+\frac{v_3^2}{w_{33}}+\frac{v_4^2}{w_{44}}\right)\right\}\sum_{k,l=0}^{\infty}\sum_{n=0}^{min(k,l)}\sum_{i_1,i_2,i_3,i_4,i_5=0}^{\infty}(-1)^{k+l}\\&\quad\times\frac{a(n,k,l)(m+k-1)(m+l-1)\binom{k+l+2m-2n-3}{2m-3}\Gamma\left(2\kappa_1\right)\Gamma\left(2\kappa_2\right)\Gamma\left(2\kappa_3\right)\Gamma\left(2\kappa_4\right)}{i_1!i_2!i_3!i_4!i_5!\Gamma\left(i_1+m+k\right)\Gamma\left(i_2+m+k\right)\Gamma\left(i_3+m+l\right)\Gamma\left(i_4+m+l\right)\Gamma\left(i_5+m+k+l-2n\right)}\\&\quad\times\frac{w_{12}^{2i_1+k}w_{13}^{2i_2+k}w_{24}^{2i_3+l}w_{34}^{2i_4+l}w_{23}^{2i_5+k+l-2n}}{2^{\kappa_1+\kappa_2+\kappa_3+\kappa_4}w_{11}^{\kappa_1}w_{22}^{\kappa_2}w_{33}^{\kappa_3}w_{44}^{\kappa_4}}D_{-2\kappa_1}\left(-\frac{jv_1}{\sqrt{w_{11}}}\right)D_{-2\kappa_2}\left(-\frac{jv_2}{\sqrt{w_{22}}}\right)D_{-2\kappa_3}\left(-\frac{jv_3}{\sqrt{w_{33}}}\right)D_{-2\kappa_4}\left(-\frac{jv_4}{\sqrt{w_{44}}}\right)\end{aligned} \tag{17}$$

the truncation error can be expressed as

$$\begin{aligned}|E_T| < &\sum_{k=0}^{K-1}\sum_{l=0}^{L-1}\sum_{n=0}^{min(k,l)}\sum_{i_1=0}^{I_1-1}\sum_{i_2=0}^{I_2-1}\sum_{i_3=0}^{I_3-1}\sum_{i_4=0}^{I_4-1}\sum_{i_5=I_5}^{\infty}\xi\\
&+\sum_{k=0}^{K-1}\sum_{l=0}^{L-1}\sum_{n=0}^{min(k,l)}\sum_{i_1=0}^{I_1-1}\sum_{i_2=0}^{I_2-1}\sum_{i_3=0}^{I_3-1}\sum_{i_4=I_4}^{\infty}\sum_{i_5=0}^{\infty}\xi\\
&+\sum_{k=0}^{K-1}\sum_{l=0}^{L-1}\sum_{n=0}^{min(k,l)}\sum_{i_1=0}^{I_1-1}\sum_{i_2=0}^{I_2-1}\sum_{i_3=I_3}^{\infty}\sum_{i_4=0}^{\infty}\sum_{i_5=0}^{\infty}\xi\\
&+\sum_{k=0}^{K-1}\sum_{l=0}^{L-1}\sum_{n=0}^{min(k,l)}\sum_{i_1=0}^{I_1-1}\sum_{i_2=I_2}^{\infty}\sum_{i_3=0}^{\infty}\sum_{i_4=0}^{\infty}\sum_{i_5=0}^{\infty}\xi\\
&+\sum_{k=0}^{K-1}\sum_{l=0}^{L-1}\sum_{n=0}^{min(k,l)}\sum_{i_1=I_1}^{\infty}\sum_{i_2=0}^{\infty}\sum_{i_3=0}^{\infty}\sum_{i_4=0}^{\infty}\sum_{i_5=0}^{\infty}\xi\\
&+\sum_{k=0}^{K-1}\sum_{l=L}^{\infty}\sum_{n=0}^{min(k,l)}\sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty}\sum_{i_3=0}^{\infty}\sum_{i_4=0}^{\infty}\sum_{i_5=0}^{\infty}\xi\\
&+\sum_{k=K}^{\infty}\sum_{l=0}^{\infty}\sum_{n=0}^{min(k,l)}\sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty}\sum_{i_3=0}^{\infty}\sum_{i_4=0}^{\infty}\sum_{i_5=0}^{\infty}\xi. \qquad (18)\end{aligned}$$

Further simplification of (18) is a difficult task. A tighter bound can be obtained with the approach due to Tan and Beaulieu [9] at the expense of more mathematical rigor.

III. Trivariate Distribution

Here we obtain the trivariate Nakagami-*m* density for an arbitrary correlation matrix from (10) since it is not available in the open literature. If $\{r_1, r_2, r_3\}$ are independent from r_4, then we can write the quadrivariate density as a product of trivariate and a univariate density function. Equating w_{24}, w_{34} to zero and using (10) with $l = n = 0$, the trivariate Nakagami density for $m \geq 3/2$ can be written as

$$\begin{aligned}f(r_1, r_2, r_3) = &\frac{W_3^m r_1 r_2 r_3}{(m-1)\left(w_{12}w_{13}w_{23}\right)^{m-1}}\\
&\times \exp\left\{-\frac{1}{2}\left(w_{11}r_1^2 + w_{22}r_2^2 + w_{33}r_3^2\right)\right\}\\
&\times \sum_{k=0}^{\infty}(-1)^k(m+k-1)\binom{2m+k-3}{2m-3}\\
&\times I_{m+k-1}\left(w_{12}r_1r_2\right) I_{m+k-1}\left(w_{13}r_1r_3\right)\\
&\times I_{m+k-1}\left(w_{23}r_2r_3\right) \qquad (19)\end{aligned}$$

where W_3 denotes the determinant of the inverse covariance matrix corresponding to the trivariate case. It should be noted that no restriction is imposed on the covariance matrix in this derivation. Our result is exactly equivalent to the previous result given in [11, eq.2.2.18] for trivariate generalized Rayleigh density function. The trivariate joint density for an arbitrary covariance matrix with $m = 1$ is given in [16].

A. Joint cdf and chf

Following the same line of arguments as for the derivation of trivariate cdf from quadrivariate case, we can obtain the trivariate cdf from (15) as

$$\begin{aligned}F(r_1, r_2, r_3) = &\frac{W_3^m}{(m-1)}\sum_{k,p,q,r=0}^{\infty}\frac{(-1)^k(m+k-1)}{p!q!r!\Gamma(p+m+k)}\\
&\times \frac{\binom{k+2m-3}{2m-3}w_{12}^{2p+k}w_{13}^{2q+k}w_{23}^{2r+k}}{\Gamma(q+m+k)\Gamma(r+m+k)w_{11}^{\varepsilon_1}w_{22}^{\varepsilon_2}w_{33}^{\varepsilon_3}}\gamma\left(\varepsilon_1, \frac{w_{11}r_1^2}{2}\right)\\
&\times \gamma\left(\varepsilon_2, \frac{w_{22}r_2^2}{2}\right)\gamma\left(\varepsilon_3, \frac{w_{33}r_3^2}{2}\right) \qquad (20)\end{aligned}$$

where

$$\begin{aligned}\varepsilon_1 &= p + q + m + k\\
\varepsilon_2 &= p + r + m + k\\
\varepsilon_3 &= q + r + m + k.\end{aligned}$$

The joint characteristic function can also be derived in similar manner as

$$\begin{aligned}\psi(v_1, v_2, v_3) = &\frac{8W_3^m\Gamma(m-1)}{\Gamma(m)}\\
&\times \exp\left\{-\frac{1}{4}\sum_{i=1}^{3}\frac{v_i^2}{w_{ii}}\right\}\sum_{k,p,q,r=0}^{\infty}\frac{(-1)^k(m+k-1)}{p!q!r!2^{\varepsilon_1+\varepsilon_2+\varepsilon_3}}\\
&\times \frac{\binom{2m+k-3}{2m-3}\Gamma(2\varepsilon_1)\Gamma(2\varepsilon_2)\Gamma(2\varepsilon_3)w_{12}^{2p+k}w_{13}^{2q+k}w_{23}^{2r+k}}{\Gamma(m+k+p)\Gamma(m+k+q)\Gamma(m+k+r)w_{11}^{\varepsilon_1}w_{22}^{\varepsilon_2}w_{33}^{\varepsilon_3}}\\
&\times D_{-2\varepsilon_1}\left(\frac{-jv_1}{\sqrt{w_{11}}}\right)D_{-2\varepsilon_2}\left(\frac{-jv_2}{\sqrt{w_{22}}}\right)D_{-2\varepsilon_3}\left(\frac{-jv_3}{\sqrt{w_{33}}}\right). \qquad (21)\end{aligned}$$

Next we consider (20) with respect to the exponential and constant correlation models.

1) Exponential Correlation Model: The inverse covariance matrix of this model given in [4], [12] can be described as $w_{13} = 0$ and $w_{11} = w_{33} = \frac{1}{1-\rho^2}$, $w_{22} = \frac{1+\rho^2}{1-\rho^2}$, $w_{12} = w_{23} = \frac{-\rho}{1-\rho^2}$. Hence (20) can be simplified to

$$\begin{aligned}F(r_1, r_2, r_3) = &\frac{\left(1-\rho^2\right)^m}{\Gamma(m)}\\
&\times \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{\rho^{2(k+l)}}{k!l!\Gamma(k+m)\Gamma(l+m)\left(1+\rho^2\right)^{k+l+m}}\\
&\times \gamma\left(k+m, \frac{r_1^2}{2\left(1-\rho^2\right)}\right)\gamma\left(k+l+m, \frac{\left(1+\rho^2\right)r_2^2}{2\left(1-\rho^2\right)}\right)\\
&\times \gamma\left(l+m, \frac{r_3^2}{2\left(1-\rho^2\right)}\right) \qquad (22)\end{aligned}$$

which agrees with the result given in [12, eq.6].

2) Constant Correlation Model: The correlation among closely placed diversity antennas may be approximated with this correlation model [4]. The correlation matrix of this model is described by $m_{i,j} = \rho(i \neq j)$ and $m_{i,i} = 1$, where

$-\frac{1}{2} < \rho < 1$. Hence, (20) can be simplified to

$$F(r_1, r_2, r_3) = \frac{(1-\rho)^m (1+2\rho)^{2m}}{(m-1)(1+\rho)^{3m}} \times \sum_{k,p,q,r=0}^{\infty} \frac{(m+k-1)\binom{k+2m-3}{2m-3}}{p!q!r!\Gamma(p+m+k)\Gamma(q+m+k)\Gamma(r+m+k)} \times \left(\frac{\rho}{1+\rho}\right)^{\varepsilon_1+\varepsilon_2+\varepsilon_3-3m} \gamma\left(\varepsilon_1, \frac{(1+\rho)r_1^2}{2(1-\rho)(1+2\rho)}\right) \times \gamma\left(\varepsilon_2, \frac{(1+\rho)r_2^2}{2(1-\rho)(1+2\rho)}\right) \gamma\left(\varepsilon_3, \frac{(1+\rho)r_3^2}{2(1-\rho)(1+2\rho)}\right). \quad (23)$$

It should be noted that the expression for joint cdf given in [12] is not valid for the constant correlation case. The result corresponding to $m = 1$ is given in [15, eq.8].

B. Truncation Error

For brevity, we consider the consequences of the truncation of (20) only. The truncation error of the cdf series (20) can be found by limiting the variables k, p, q, r to K, P, Q and R terms respectively. Following the approach given in [15], the truncation error E_T can be expressed as

$$|E_T| < \sum_{k=0}^{K-1}\sum_{p=0}^{P-1}\sum_{q=0}^{Q-1}\sum_{r=R}^{\infty} \epsilon(k,p,q,r) + \sum_{k=0}^{K-1}\sum_{p=0}^{P-1}\sum_{q=Q}^{\infty}\sum_{r=0}^{\infty} \epsilon(k,p,q,r) + \sum_{k=0}^{K-1}\sum_{p=P}^{\infty}\sum_{q=0}^{\infty}\sum_{r=0}^{\infty} \epsilon(k,p,q,r) + \sum_{k=K}^{\infty}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{r=R}^{\infty} \epsilon(k,p,q,r) \quad (24)$$

where

$$\epsilon(k,p,q,r) = \frac{W_3^m (m+k-1)\binom{k+2m-3}{2m-3}\Gamma(\varepsilon_1)\Gamma(\varepsilon_2)\Gamma}{(m-1)p!q!r!\Gamma(p+m+k)\Gamma(q+m+k)} \times \frac{(\varepsilon_3)\, w_{12}^{2p+k} w_{13}^{2q+k} w_{23}^{2r+k}}{\Gamma(r+m+k) w_{11}^{\varepsilon_1} w_{22}^{\varepsilon_2} w_{33}^{\varepsilon_3}}.$$

The error bound (24) may be loose for some r_1, r_2, r_3 since we upper bound the incomplete Gamma function with $\gamma(a, x) \leq \Gamma(a)$. More tighter bound, which reflects the effect of r_1, r_2, r_3 on the convergence rate of (20) can be derived following the approach of Tan and Beaulieu [9]. We do not mention that bound here due to the high complexity involved.

We now give numerical results to illustrate the convergence behavior of the cdf series (20). For simplicity we assume the constant correlation model here, and Table I lists the number of the terms needed in (23) to achieve an accuracy of five significant figures. The dependency of the convergence on r_1, r_2, r_3 is obvious from Table I. The number of terms needed for a given accuracy increases as r_1, r_2, r_3 increase. The increase of correlation also results in the number of terms to be increased. However, the dependency on m is not trivial. As can be seen from the Table I, for some values of r_1, r_2, r_3, the number of terms decreases as m increases. On the contrary, the number of terms increases as m increases after some threshold values of r_1, r_2, r_3.

TABLE I

NUMBER OF TERMS NEEDED IN (23) TO ACHIEVE FIVE SIGNIFICANT FIGURE ACCURACY WHEN $r_1 = r_2 = r_3 = r$.

m	ρ	$r=2$ K,P,Q,R	$r=3$ K,P,Q,R	$r=5$ K,P,Q,R
2	0.1	2, 2, 2, 2	2, 3, 3, 3	2, 3, 3, 3
	0.3	2, 2, 2, 2	4, 4, 4, 4	4, 6, 5, 5
	0.7	3, 3, 3, 3	7, 10, 10, 10	10, 14, 14, 14
	0.9	9, 12, 12, 12	13, 22, 22, 22	18, 42, 42, 42
3	0.1	1, 2, 2, 2	2, 3, 3, 3	3, 4, 4, 4
	0.3	1, 2, 2, 2	3, 4, 4, 4	4, 6, 6, 6
	0.7	3, 4, 4, 4	6, 9, 9, 9	11, 15, 15, 15
	0.9	9, 12, 12, 12	13, 22, 22, 22	20, 44, 44, 44
5	0.1	1, 2, 2, 2	1, 3, 3, 3	3, 4, 4, 4
	0.3	2, 2, 2, 2	3, 4, 4, 4	4, 6, 6, 6
	0.7	2, 3, 3, 3	7, 9, 9, 9	12, 17, 17, 17
	0.9	9, 12, 12, 12	14, 21, 21, 21	24, 47, 47, 47
8	0.1	1, 2, 2, 2	2, 3, 3, 3	3, 4, 4, 4
	0.3	2, 2, 2, 2	3, 4, 4, 4	5, 7, 7, 7
	0.7	2, 3, 3, 3	7, 9, 9, 9	12, 18, 18, 18
	0.9	9, 12, 12, 12	14, 21, 21, 21	27, 48, 48, 48

IV. APPLICATIONS

The new results developed in Section III enable the performance analysis of 3-branch diversity systems in arbitrarily correlated Nakagami fading channels. This section presents three possible applications with numerical results. We also present the outage probability of four branch SC for the special covariance matrix given in [15].

A. Outage Probability of 3-Branch SC

Outage probability is a standard and widely-used performance measure of diversity systems. It is defined as the probability that the output instantaneous SNR γ falls below a certain given threshold γ_{th}. For independent fading, outage expressions have been fully developed (for example see [22] and references therein). This is not however true for correlated fading. Here, we use the joint tri-variate Nakagami cdf (20) to evaluate the outage probability of 3-branch SC in correlated fading channels.

We assume that the noise components at different diversity branches are additive white Gaussian noise (AWGN) with identical power spectral density. Let γ_k and $\bar{\gamma}_k$ denote the instantaneous and the average SNR at the k-th branch ($k = 1, 2, 3$). In SC, the branch with the largest instantaneous SNR is selected as the output, $\gamma_{sc} = \max(\gamma_1, \gamma_2, \gamma_3)$. Using the relation $\gamma_k = \frac{\bar{\gamma}_k}{E(r_k^2)} r_k^2 = \frac{\bar{\gamma}_k}{2m\psi_{kk}} r_k^2$, where r_k is the amplitude of the received signal at the k-th branch, we may obtain the outage probability as

$$P_{out} = \Pr(0 \leq \gamma_{sc} \leq \gamma_{th}) = F\left(\sqrt{\frac{2m\gamma_{th}\psi_{11}}{\bar{\gamma}_1}}, \sqrt{\frac{2m\gamma_{th}\psi_{22}}{\bar{\gamma}_2}}, \sqrt{\frac{2m\gamma_{th}\psi_{33}}{\bar{\gamma}_3}}\right) \quad (25)$$

where $F(\lambda_1, \lambda_2, \lambda_3)$ is the joint cdf of the branch amplitudes (20). Note that the covariance matrix $\mathbf{\Psi}$ specifies the correlation (fading correlation) between three Gaussian samples. The relationship between the envelope correlation and the fading correlation can be found [23, eq.1.5.26]. Thus, the outage can be evaluated in terms of envelope correlation and the average branch SNR.

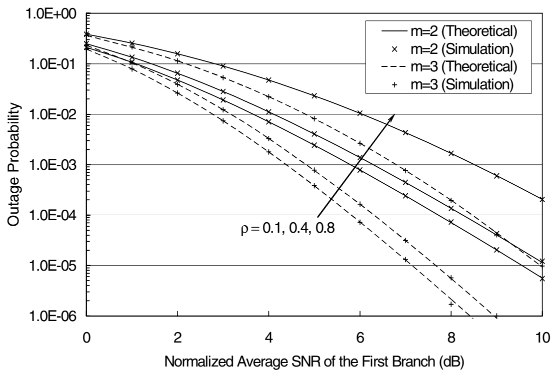

Fig. 1. Outage probability of SC versus the normalized average branch SNR $\bar{\gamma}/\gamma_{th}$ in correlated Nakagami-*m* fading channel.

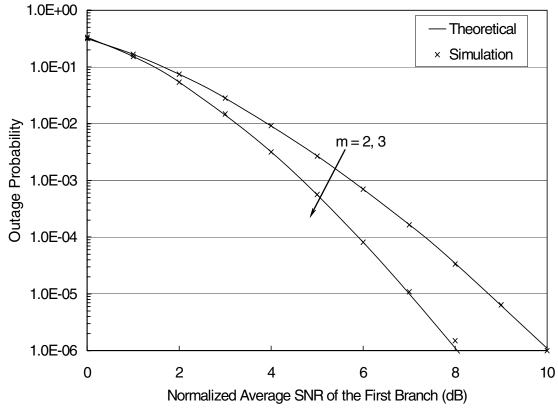

Fig. 2. Outage probability of SC versus the normalized average branch SNR $\bar{\gamma}/\gamma_{th}$ in correlated Nakagami-*m* fading channel.

Let us consider the constant correlation model described by $m_{ij} = \rho(i \neq j)$ and $m_{ii} = 1$, where $-\frac{1}{2} < \rho < 1$. Now, using (24), the outage probability of three branch SC can be found. Fig 1. depicts the impact of correlation on the outage probability of three SC over correlated Nakagami-*m* environment for various values of m. As can be seen from the graphs, the correlation among the branches degrades the performance of SC. The inverse of the covariance matrix

$$\mathbf{M} = \begin{pmatrix} 1 & 0.2920 & 0.2988 & 0.1121 \\ 0.2920 & 0.6602 & 0.2031 & 0.1585 \\ 0.2998 & 0.2031 & 0.7625 & 0.1888 \\ 0.1121 & 0.1585 & 0.1888 & 0.6431 \end{pmatrix} \quad (26)$$

given in [15] satisfies the condition $w_{14} = 0$. Hence the outage probability of four branch SC can be calculated using (16), as shown in Fig. 2. Simulated outage probabilities are also shown. There is an excellent agreement between the theoretical results and the simulation results.

B. Moments of the 3-Branch EGC Output SNR

The moments of the output of a diversity combiner can be used as alternative performance measures to the conventional error-rate analysis. However, a single moment such as the mean SNR is not sufficiently informative and higher order moments can furnish additional information for system design. For example, the variability of the output of a diversity combiner is indicated by the variance. A common moment based measure is known as the amount of fading or coefficient of variation. The new expression (21) enables us to evaluate the moments of the output SNR of a 3-branch EGC system, the output of which can be written as

$$\gamma_{egc} = \frac{(r_1 + r_2 + r_3)^2 E_s}{3N_0} \quad (27)$$

where E_s is the transmitted signal energy and N_0 is the noise power spectral density per branch. The moments of output SNR can be obtained as

$$\begin{aligned} E(\gamma_{egc}^n) &= \left(\frac{E_s}{3N_0}\right)^n E[(r_1 + r_2 + r_3)^{2n}] \\ &= \left(\frac{\bar{\gamma_1}}{6m\psi_{11}}\right)^n \sum_{\substack{k_1,k_2,k_3=0 \\ k_1+k_2+k_3=2n}}^{2n} \frac{(2n)!}{k_1!\,k_2!\,k_3!} E(r_1^{k_1} r_2^{k_2} r_3^{k_3}) \end{aligned} \quad (28)$$

where $E(r_1^{k_1} r_2^{k_2} r_3^{k_3})$ can be computed using (21). To the best of our knowledge, (28) is a new result and provide high order moments of the 3-branch EGC output SNR for the most general case.

C. Output Mgf of 3-Branch GSC

GSC(M, L) achieves a good tradeoff between performance and implementation complexity [24], [25]. However, The distribution theory for order statistics of arbitrarily correlated RVs is not fully developed. Consequently, not many theoretical results are known on how GSC performs over correlated fading channels. For example, Mallik and Win [26] analyze the performance of GSC(M, L) in **equally** correlated Nakagami fading. Our new result (20) enables the performance analysis of 3-branch GSC in arbitrarily correlated Nakagami fading channels.

Since GSC(1,3) and GSC(3,3) are simply SC and MRC, these cases are not treated here. Instead, we consider the GSC(2,3) system, which combines the largest two branch SNRs to form the output:

$$\gamma_{\text{gsc}} = \gamma_{(2)} + \gamma_{(3)} \quad (29)$$

where $\gamma_{(1)} \leq \gamma_{(2)} \leq \gamma_{(3)}$. We derive the joint cdf of $\gamma_{(2)}$ and $\gamma_{(3)}$ via the first principles as [27]

$$\begin{aligned} F_{\gamma_{(2)},\gamma_{(3)}}(\alpha, \beta) &= \Pr(\gamma_1 \leq \beta, \gamma_2 \leq \alpha, \gamma_3 \leq \alpha) \\ &\quad + \Pr(\gamma_1 \leq \alpha, \gamma_2 \leq \beta, \gamma_3 \leq \alpha) \\ &\quad + \Pr(\gamma_1 \leq \alpha, \gamma_2 \leq \alpha, \gamma_3 \leq \beta) \\ &\quad - 2\Pr(\gamma_1 \leq \alpha, \gamma_2 \leq \alpha, \gamma_3 \leq \alpha) \end{aligned} \quad (30)$$

where $\beta \geq \alpha > 0$.

Applying (20) and differentiating (30) with respect to α and β yield the joint pdf of $\gamma_{(2)}$ and $\gamma_{(3)}$ as

$$p_{\gamma_{(2)},\gamma_{(3)}}(x,y) = \frac{W_3^m}{(m-1)} \sum_{k,p,q,r=0}^{\infty} \frac{(-1)^k (m+k-1)}{p!q!r!\Gamma(p+m+k)} \times \frac{\binom{k+2m-3}{2m-3} w_{12}^{2p+k} w_{13}^{2q+k} w_{23}^{2r+k}}{\Gamma(q+m+k)\Gamma(r+m+k) w_{11}^{\varepsilon_1} w_{22}^{\varepsilon_2} w_{33}^{\varepsilon_3}} \times \sum_{\substack{u,v,w=1\\ u\neq v\neq w}}^{3} \gamma(\varepsilon_w, d_w x) d_u^{\varepsilon_u} d_v^{\varepsilon_v} \times \left[x^{\varepsilon_u-1} y^{\varepsilon_v-1} e^{-(xd_u+yd_v)}\right] \tag{31}$$

where $y \geq x > 0$ and $d_k = m\psi_{kk} w_{kk}/\bar{\gamma}_k$ for $k = 1,2,3$.

Using (31), we can obtained the output mgf of GSC(2,3) as

$$\begin{aligned} M_{\text{gsc}}(s) =& E(e^{-s\gamma_{\text{gsc}}}) \\ =& \int_0^\infty \int_x^\infty p_{\gamma_{(2)},\gamma_{(3)}}(x,y) e^{-(x+y)s}\, dy\, dx \\ =& \frac{W_3^m}{(m-1)} \sum_{k,p,q,r=0}^{\infty} \frac{(-1)^k (m+k-1)}{p!q!r!\Gamma(p+m+k)} \\ & \times \frac{\binom{k+2m-3}{2m-3} w_{12}^{2p+k} w_{13}^{2q+k} w_{23}^{2r+k}}{\Gamma(q+m+k)\Gamma(r+m+k) w_{11}^{\varepsilon_1} w_{22}^{\varepsilon_2} w_{33}^{\varepsilon_3}} \\ & \times \sum_{\substack{u,v,w=1\\ u\neq v\neq w}}^{3} \left(\frac{d_v}{d_v+s}\right)^{\varepsilon_v} d_u^{\varepsilon_u} g(u,v,w) \end{aligned} \tag{32}$$

where

$$\begin{aligned} g(u,v,w) &= \int_0^\infty x^{\varepsilon_u-1} e^{-x(d_u+s)} \gamma(\varepsilon_w, xd_w) \\ &\quad \times \Gamma[\varepsilon_v, (d_v+s)x]\, dx \\ &= \frac{d_w^{\varepsilon_w}}{\varepsilon_w} \left[\frac{(\varepsilon_u+\varepsilon_w-1)!(\varepsilon_v-1)!}{(d_u+d_w+s)^{\varepsilon_u+\varepsilon_w}} \right. \\ &\quad \times {}_2F_1\left(\varepsilon_u+\varepsilon_w, 1; \varepsilon_w; \frac{d_w}{d_w+d_u+s}\right) \\ &\quad + \frac{(\varepsilon-1)!(d_v+s)^{\varepsilon_v}}{\varepsilon_v (d+2s)^{\varepsilon}} \\ &\quad \left. F_A\left(\varepsilon; 1,1; \varepsilon_w+1, \varepsilon_v+1; \frac{d_w}{d+2s}, \frac{d_v+s}{d+2s}\right)\right] \end{aligned} \tag{33}$$

where $\varepsilon = \varepsilon_1 + \varepsilon_2 + \varepsilon_3$, $d = d_u + d_v + d_w$, $\Gamma(a,x)$ is the complementary incomplete gamma function, ${}_2F_1(a,b;c;z)$ is the Gauss hypergeometric function which is defined as [21, eq.9.100] and $F_A(\alpha;\beta_1,\ldots,\beta_n;\gamma_1,\ldots,\gamma_n;z_1,\ldots,z_n)$ is the n-th order Appell hypergeometric function [21, (9.180.2)]. Eq. (33) follows from [21, eq.9.236.4] and [28, eq.C.1]. Using the output mgf (32), the performance of various digital modulations with GSC(2,3) may be evaluated.

V. Conclusion

In this paper, we have derived new pdf and cdf for trivariate and quadrivariate Nakagami-*m* distributions. Miller's classical approach and Dougall's identity have been used in deriving the former results. The newly derived trivariate densities are valid for any arbitrary correlation matrix, while the quadrivariate densities are valid for the most general arbitrary class of correlation matrices as mentioned before. Furthermore, the series expressions developed for the joint distributions can be used to analyze the performance of several diversity schemes in correlated fading environments as well as the performance of transmit antenna selection in spatially correlated multiple-input multiple-output wireless systems. For brevity, we consider only a limited number of representative applications including the outage probability of the triple branch SC receiver, the output SNR of triple branch EGC receiver and the moment generating function of the output SNR of generalized SC (2,3) receiver over a correlated Nakagami environment.

Acknowledgment

The authors would like to thank the anonymous reviewers for their critical comments that greatly improved this paper. The government of Finland has provided the doctoral scholarship to the first author.

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