β-complementary Sequences and Peak-to-Mean Envelope Power Ratio Reduction in OFDM

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***Abstract*— In this paper we introduce a novel sequence "β-complementary sequence" to encode the OFDM signals, by which one can substantially increase the code rate while enjoy a tight PMEPR of at most 2. On the other hand, such encoding by β-complementary sequences has very good trade-off performance between PMEPR and code rate. This observation follows from the properies of β-complementary sequences investigated by us, the numerical results based on these properties and the comparison with the well discussed Golay complementary sequences and the generalized Golay complementary sequences (called "G_N-complementary sequences' in this paper).**

I. Introduction

In muticarrier communications, the orthogonal frequency division multiplexing (OFDM) has been made use for the wireless large-area networks (LANs) by the international standards IEEE 802.11 and ETSI BRAN committees, since it provides great immunity to impulse noise and fast fades and eliminates the need for equalizers, while efficient hardware implementations can be realized using fast Fourier transform (FFT).

However, a major drawback of OFDM signals is the high peak to mean envelope power ratio (PMEPR) of the uncoded OFDM signal. Numbers of PMEPR reduction schemes have been proposed based on the oversampled sequence [11], [15], [17], [18], [19], [21]. On the other hand, several coding schemes to reduce the PMEPR of the OFDM waveform have been studied in [7], [9], [10], [13], [20], [22]. Some of them meanwhile enjoys the large Euclidean distance and efficient soft-decision decoding algorithm.

An idea introduced in [1] and developed in [14] is to use the Golay complementary sequences [8] to encode the OFDM signals with PMEPR of at most 2. Recently Davis and Jedwab [6] made further advances on this work and observed that the 2^h-ary Golay complementary sequences of length 2^m [8] can be obtained from certain second order cosets of the classical first order Reed-Muller code. As consequence of this intrinsic observation, Davis and Jedwab [7] were able to obtain, for a small number of carriers ($n \leq 32$), a range of binary, quaternary and actuary OFDM codes with good error-correcting capabilities, efficient encoding and decoding, reasonable code rates, and controlled PMEPR of at most 2. A follow-up work done in [12] investigated the trade-offs between code rate and PMEPR by using the complementary set [16], which figured out the codes with PMEPR of at most the exponential of 2 in the second order Reed-Muller code. Since this trade-off only slightly increases the code rate by relaxing the PMEPR, it remains an open problem to discover low PMEPR error-correcting code constructions for a moderately large subcarrers n.

In this paper, we introduce a novel sequence "β-complementary sequences" to encode the OFDM signal, which substantially increases the code rate. Meanwhile such code also enjoys good performance of trade-off between code rate and PMEPR. At first, we will investigate the properties of β-complementary sequences, and demonstrate numerical results based on these properties. Then we make comparison with the well discussed Golay complementary sequences and the generalized Golay complementary sequences (called "G_N-complementary sequences" in this paper).

II. β-complementary sequences

Before proceeding further, let us introduce the OFDM signals, the PMEPR and the related concepts at first.

A. *Preliminaries*

Let j be the imaginary unit, i.e., $j^2 = -1$. The n subcarrier complex baseband OFDM signal can be represented as

$$s(t) = \sum_{k=-\infty}^{\infty} \sum_{\ell=0}^{n-1} c_{k,\ell} e^{j2\pi(f_c+\ell\Delta f)t} g[t-k(T+T_g)], \quad (1)$$

where $0 \leq t < T$, $c_{k,\ell}$ is the data symbol for the ℓ-th subcarrier and the k-th OFDM symbol, the frequency separation between any two adjacent subcarriers is $\Delta f = 1/T$, and f_c is the carrying frequency that is much larger than Δf ($f_c \gg \Delta f$). The unit rectangular pulse $g(t)$ is of duration $T + T_g$, where T_g is known as the guard interval. For the PMEPR problem, it is enough to consider only a single OFDM symbol since there is almost no overlap between different OFDM symbols. In practice, filtering can cause some intersymbol interference, which will be neglected in this discussion. The guard interval T_g is used to repeat parts of each OFDM signal, and has no effect on PMEPR. For the convenience of discussion, we may set $T_g = 0$ and $c_{0,\ell} = c_\ell$.

For an M-ary phase modulation OFDM, $c_\ell \in \xi^{\mathbb{Z}_M} = \{\xi^k : k \in \mathbb{Z}_M\}$, where $\xi = \exp(2\pi j/M)$ and $\mathbb{Z}_d = \{0, \cdots, M -$

1}. Let $\theta = 2\pi\Delta f t$. Suppose that $f_c = K\Delta f$ with some large $K \in \mathbb{Z}$, the integer set. For a codeword $b = (b_0, \ldots, b_{n-1})$ with $b_\ell \in \xi^{\mathbb{Z}_M}$, the complex envelope (1) can be therefore reduced to

$$s_b(\theta) = \sum_{\ell=K}^{K+n-1} b_\ell e^{j\ell\theta}, \quad \theta \in [0, 2\pi). \tag{2}$$

The instantaneous power of the complex envelope $s_b(\theta)$ is defined by

$$P_b(\theta) = |s_b(\theta)|^2 \tag{3}$$

So the peak-to-mean power ratio (PMEPR) of the codeword b is defined by

$$\mathrm{PMEPR}(b) = \frac{1}{n} \sup_{0 \le \theta < 2\pi} |s_b(\theta)|^2. \tag{4}$$

Obviously, in such PMEPR problem, one can assume $K = 0$.

A $\xi^{\mathbb{Z}_M}$-sequence a of length n is called a Golay complementary sequence [8] if there is a $\xi^{\mathbb{Z}_M}$-sequence b of length n such that

$$P_a(\theta) + P_b(\theta) = 2n. \tag{5}$$

It is easy to see $\mathrm{PMEPR}(a) \le 2$ if a is a Golay complementary sequence. A generalization of Golay complementary sequence is the complementary set [16]. A set of $\xi^{\mathbb{Z}_M}$-sequences $a^0, \cdots, a^{N-1}$ of length n is said to be a complementary set if

$$P_{a^0}(\theta) + \cdots + P_{a^{N-1}}(\theta) = Nn. \tag{6}$$

Any sequence in the complementary set is called a G_N- or $G_N(n)$-complementary sequence. Obviously, $\mathrm{PMEPR}(a) \le N$ if a is a G_N-complementary sequence, and any G_2-complementary sequence is a Golay complementary sequence. In this paper, we will further generalize the G_N-complementary sequences, by which the code rate will be substantially increased.

B. β-complementary sequences

In Golay complementary set, one actually use the sequences $a^0 \in \xi^{\mathbb{Z}_M}$ only to encode the OFDM signals. Therefore the other sequences $a^1, \cdots, a^{N-1}$ can be any sequences in $\mathbb{C}^n$. This is one of the primary motivations to introduce the β-complementary sequence. Note that $s_b(\theta)$ defined in (2) and $P_b(t)$ defined in (3) are also well defined for any sequence $b \in \mathbb{C}^n$. We therefore give the following definition.

Definition 1: A $\xi^{\mathbb{Z}_M}$-sequence a of length n is said to be a β- or $\beta(n)$-complementary sequence for some $\beta \ge 1$ if there is a sequence $b \in \mathbb{C}^n$ such that

$$P_a(\theta) + P_b(\theta) = \beta n,$$

It is easy to see $\mathrm{PMEPR}(a) \le \beta$ if a is a β-complementary sequence. In the following, we will also show an intrinsic property of β-complementary sequences which claims that any $\xi^{\mathbb{Z}_M}$-sequence a such that $\mathrm{PMEPR}(a) \le \beta$ is a β-complementary sequence. At first we show some fundamental properties of β-complementary sequences.

C. Some fundamental properties of β-complementary sequences

For a sequence $a = (a_0, \cdots, a_{n-1}) \in \mathbb{C}^n$, the aperiodic auto-correlation of a is defined as

$$R_a(\ell) = \begin{cases} \sum_{k=0}^{n-\ell-1} a_{k+\ell}\bar{a}_k, & 0 \le \ell < n, \\ 0, & \text{otherwise.} \end{cases}$$

where $\bar{a}_k$ is the complex conjugate of a_k. Immediately we can use the aperiodic auto-correlation to describe β-complementary sequences.

Proposition 1: A $\xi^{\mathbb{Z}_M}$-sequence is a $\beta(n)$-complementary sequence if and only if there is a sequence $b \in \mathbb{C}^n$ such that $R_b(0) = (\beta-1)n$ and $R_a(\ell) + R_b(\ell) = 0$ for $\ell = 1, \cdots, n-1$.

Proof: For any $b \in \mathbb{C}^n$,

$$P_b(\theta) = R_b(0) + \sum_{\ell=1}^{n-1} \left[R_b(\ell)e^{j\ell\theta} + \bar{R}_b(\ell)e^{-j\ell\theta} \right].$$

Denote the real part and imaginary part of $R_b(\theta)$ by $R_b^R(\theta)$ and $R_b^I(\theta)$ respectively. Then

$$P_b(\theta) = R_b(0) + 2\sum_{\ell=1}^{n-1} \left[R_b^R(\ell)\cos \ell\theta + R_b^I(\ell)\sin \ell\theta \right].$$

Since $\{1, \cos\theta, \cdots, \cos(n-1)\theta, \sin\theta, \cdots, \sin(n-1)\theta\}$ is an orthogonal system in the square integrable function space $(L^2[0, 2\pi])$, we conclude that the condition $P_a(\theta) + P_b(\theta) = \beta n$ is equivalent to the conditions $R_a(0) + R_b(0) = \beta n$ and $R_a^R(\ell) + R_b^R(\ell) = 0 = R_a^I(\ell) + R_b^I(\ell)$ for $\ell = 1 \cdots, n-1$. By noting that $R_a(0) = n$ when a is a $\xi^{\mathbb{Z}_M}$-sequences, one has the equivalent conditions $R_b(0) = (\beta-1)n$ and $R_a(\ell) + R_b(\ell) = 0$ for $\ell = 1, \cdots, n-1$. This completes the proof.

The following property tells how to produce new β-complementary sequences from a known β-complementary sequence.

Proposition 2: If $a = (a_0, \cdots, a_{n-1})$ is a β-complementary sequence, then for any $\zeta \in \xi^{\mathbb{Z}_M}$, $\zeta a = (\zeta a_0, \cdots, \zeta a_{n-1})$, $\bar{a}^r = (\bar{a}_{n-1}, \cdots, \bar{a}_0)$, $\bar{a} = (\bar{a}_0, \cdots, \bar{a}_{n-1})$, $a^r = (a_{n-1}, \cdots, a_0)$ and $a^\zeta = (a_0\zeta^0, \cdots, a_{n-1}\zeta^{n-1})$ are also β-complementary sequences.

D. An intrinsic property of binary β-complementary sequences

Restrict to the binary phase shift keying (BPSK), i.e., the $\xi^{\mathbb{Z}_2}$-sequences, one will find that the β-complementary sequences comprises all sequences with PMEPR of at most β. But we can not yet extend this result to the M-ary phase shift keying (MPSK).

Theorem 1: A $\xi^{\mathbb{Z}_2}$-sequence a of length n is a β-complementary sequence if and only if $\mathrm{PMEPR}(a) \le \beta$.

Proof: We only need to show that a is a β-complementary sequence if $\mathrm{PMEPR}(a) \le \beta$, since the necessity has been shown in subsection II-B. We separate the proof into several steps. Due to the limited pages, we will omit the details

This immdediately implies that G_N-complementary sequence is a special β-complementary sequence.

Corollary 1: A $\xi^{\mathbb{Z}_2}$-sequence is a β-complementary sequence with $\beta = N$ if a is a G_N-complementary sequence.
Proof: It is easy to see that PMEPR$(a) \leq N$ if a is a G_N-complementary sequence. Then Theorem 1 immediately shows that a is a β complementary sequence with $\beta = N$.

The argument used in the proof of Theorem 1 also implies the following results about combination of β-complementary sequences

Corollary 2: If a $\xi^{\mathbb{Z}_2}$-sequences a is a $\beta(n)$-complementary sequence, then the concation sequence $a \vee a = (a_0, \cdots, a_{n-1}, a_0, \cdots, a_{n-1})$ and the interleaving sequence $a \wedge a = (a_0, a_0, \cdots, a_{n-1}, a_{n-1})$ are $4\beta(2n)$-complementary sequences.
Proof: There is a sequence $b \in \mathbb{C}^n$ such that

$$P_a(\theta) + P_b(\theta) = \beta n.$$

Then

$$P_{a\vee a}(\theta)+P_{b\vee b}(\theta) = |1+z^n|^2(P_a(\theta)+P_b(\theta)) = \beta n|1+e^{jn\theta}|^2.$$

Let $p(\theta) = 4\beta n - \beta n|1 + e^{jn\theta}|^2 + P_{b\vee b}(\theta)$. Then $P(\theta)$ is a non-negative trigonometric ppolynomial of degree $4n - 2$. By the argument in the proof of theorem 1, we conclude that there is a $b' \in \mathbb{C}^{2n}$ such that $P_{b'}(\theta) = p(\theta)$, that is

$$P_{a\vee a}(\theta) + P_{b'}(\theta) = 4\beta n.$$

Similarly, we can prove that the interleaving sequence $a \wedge a$ is a $4\beta(2n)$-complementary sequence. This completes the proof.

E. Numerical results

Consider the simplest $\xi^{\mathbb{Z}_2}$-sequences, i.e., the binary sequences. Computer exhaustive searching shows that there are respectively 4, 8, 64, 608 and 149184 available codewords for $n = 2, 4, 8, 16, 32$. Therefore the corresponding code rates are respectively 1, 0.75, 0.75, 0.578and 0.58 for $n = 2, 4, 8, 16, 32$. The distribution of code rate vs length of binary β-complementary sequences is plotted in Fig. 1 by the diamond solid line. In Fig. 1, we also plot the distribution of code rate vs the binary Golay complementary sequences by the triangle dashed line. By [7], the code rates of binary Golay complementary sequences are $[\lfloor \log_2(m!) \rfloor + m + 1]/2^m$ where $m = \log_2 n$. Therefore the code rates of binary Golay complementary sequences are respectively $1, 0.75, 0.5, 0.3125$ and 0.1875 for $n = 2, 4, 8, 16, 32$. From Fig. 1, both lines are decreasing as the number of subcarriers increases. But code rate of binary Golay sequences decreases very fast, while that of β-complementary sequences relatively persists. It implies that the β-complementary sequences provide a high code rate for the moderately large subcarrier in OFDM. Meanwhile, we find that the diamond solid line lies above the triangle dashed line, and the difference is more than 0.38 for $n = 32$, which means that there are much more codes with PMEPR of at most 2 is available beyond the Golay complementary sequences. Therefore it is expectable to figure out an efficient encoding method using these codewords. This is another primary motivation to introduce the β-complementary sequences.

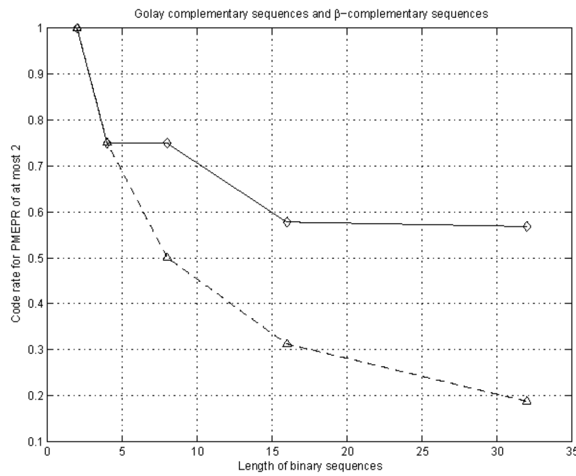

Fig. 1. The diamond solid line is the code rate versus the length of the β-complementary sequences for PMEPR of at most 2. The triangle dashed line is the code rate versus length of the Golay complementary sequences for PMEPR of at most 2.

On the other hand we can consider the trade-off between the PMEPR and code rate as in [12]. We focus on the binary sequences of length $n = 16$. The distributions of code rate vs PMEPR for β-complementary sequences and vs G_N-complementary sequences $(N = \beta)$ are plotted in Fig. 2. When one relaxes the PMEPR, the code rate of G_N-complementary sequence slowly increases, while that of β-complementary sequences increases very fast. For example, when the PMEPR is relaxed from 2 to 4, the code rate of G_N-complementary sequences just increases from 0.5366 to 0.6688, while that of β-complementary sequences from 0.5639 to 0.97. When PMEPR$= 8, 16$, the code rates of G_N-complementary sequences just slightly increase, while those of β-complementary sequences are almost 1. This numerical result implies that, β-complementary sequences have very good performance in trade-off between PMEPR and code rate compared to the G_N-complementary sequences. Therefore it suggest a way to figure out an efficient encoding scheme with low PMEPR based on some subset of β-complementary sequences.

III. Conclusion and open problems

In this paper, we introduce the novel sequences "β-complementary sequences" to encode the OFDM signals, by which code rate can be substantially increased while the PMEPR is tightly bounded. As the subcarrier n increases, the code rate of β-complementary sequences will decline as that of Golay complementary sequences. But the code rate of β-complementary sequences declines very slowly compared to that of Golay complementary sequences. Since the code rate of Golay complementary sequences is prohibitively low for a moderately large subcarrier (e.g., $n > 32$), the β-complementary sequences are suitable to be used to encode

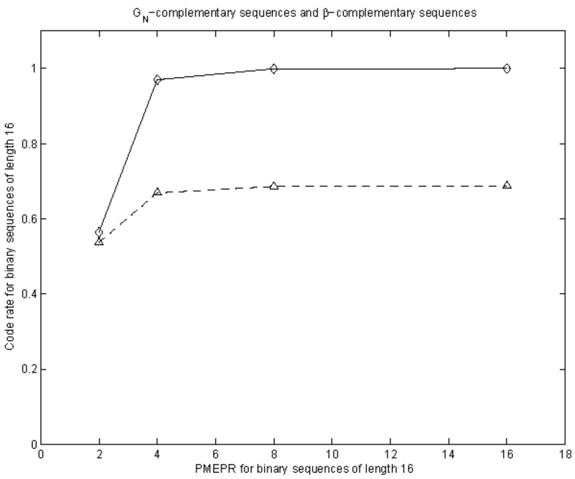

Fig. 2. The diamond solid line is the code rate versus the PMEPR of the $\beta(16)$-complementary sequences. The triangle dashed line is the code rate versus the PMEPR of the $G_N(16)$-complementary sequences with $N = \beta$.

both small and large subcarrier OFDM signals while enjoys the PMEPR of at most 2. On the other hand, the β-complementary sequences also enjoy the good performance in trade-off between PMEPR and code rate compared to the G_N-complementary sequences. Slightly relaxing the PMEPR in β-complementary sequences will greatly increase the code rate, which combats the slow increment when relaxing the PMEPR in G_N-complementary sequences. These discussion suggests a way to construct a low PMEPR encoding scheme for OFDM signals based on some subset of β-complementary sequences.

For a moderately large subcarrier OFDM signal, one can make a look-up table to encode the OFDM symbols. However, for large subcarrier OFDM signal, it remains open to find an efficient way to generate sufficient number of β-complementary sequences. Certain recursive generating formula has been expected by us. If one also consider decoding, the Hamming distance is a critical factor to affect the decoding efficiency. Therefore to understand the Hamming distance is another open problem for using β-complementary sequences in OFDM. In addtion, we conjecture that Theorem 1 and Corollary 1, 1 also hold for any β-sequences in $\xi^{\mathbb{Z}_M}$.

IV. Acknowledgement

This work is supported by NSF China #60672067, by NSF Shanghai #062R14041, by Shanghai-Canada NRC #06SN07112, by Cultivation Fund of the Key Scientific and Technical Innovation Project, Ministry of Education of China #706022, and by Program for New Century Excellent Talents in University (NCET-06-0386).

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