A water-filling algorithm for receive antenna selection based on mutual information maximization

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***Abstract*— Despite high spectral efficiencies of multiple-input multiple-output systems, they suffer from high cost and complexity due to the use of multiple radio frequency chains. A possible solution is to select a subset of the available antennas at transmitter and/or receiver based on a criterion such as capacity maximization or error-rate minimization. In this paper, we propose a receive antenna selection algorithm to maximize instantaneous mutual information. The algorithm is based on singular value decomposition and iterative water-filling. Although sub-optimal, our algorithm performs close to optimal selection based on exhaustive search.**

I. Introduction

Multiple-input multiple-output (MIMO) wireless communication systems, employing multiple antennas at both the transmitter and the receiver, achieve remarkably high spectral efficiencies in rich scattering multipath environments. However, they suffer from high cost and complexity due to multiple radio frequency chains at both link ends (such as amplifiers, mixers and analog-to-digital converters). A powerful solution is to select a subset of the available antennas at transmitter and/or receiver [1]–[3], that is a limited number of transmit/receive chains are dynamically multiplexed among several available transmit/receive antennas.

Exhaustive search based on maximum output signal-to-noise (SNR) is proposed in [4] when signal detection is performed by the linear receivers. Since exhaustive search is computationally expensive for large MIMO systems, several sub-optimal algorithms with lower complexity have been derived at the expense of performance. A selection algorithm based on accurate approximation of the conditional probability on quasi-static MIMO systems is derived in [5]. The authors in [6] develop upper and lower bounds for channel capacity with antenna selection. Near-optimal approaches for capacity maximization are given in [7], [8]. In [9], the authors formulate receive antenna selection as a convex relaxation problem and employ interior point algorithms (specifically, the barrier method), where only an approximate solution is obtained. Although their approach has the total complexity $O(N_r^{3.5})$ for the total number of N_r receive antennas, their iterative method might be difficult to be implemented in real-time systems.

In this paper, we propose an algorithm for receive antenna selection using a simple modified water-filling rule. The performance of our algorithm is close to optimal selection based on exhaustive search. Moreover, it is easy to implement because of its utilization of well-known tools such as singular value decomposition (SVD) and water-filling. Our algorithm can also be extended for capacity maximization when implementing practical receivers such as minimum mean square error (MMSE) or ordered successive interference cancelation (OSIC) [9].

Notation: Bold symbols denote matrices or vectors. $(\cdot)^T$, $(\cdot)^H$ and $(\cdot)^*$ denote transpose, conjugate transpose and conjugate, respectively. The sets of real numbers and complex numbers are $\mathbb{R}$ and $\mathbb{C}$. Complex $K \times 1$ vectors and $M \times N$ matrices are denoted by $\mathbb{C}^K$, $\mathbb{C}^{M\times N}$ respectively. A circularly symmetric complex Gaussian variable with mean μ and variance σ^2 is denoted by $z \sim \mathcal{CN}(\mu, \sigma^2)$. An $N \times N$ identity matrix is I_N.

II. System Model

We consider a MIMO system with total of N_t transmit and N_r receive antennas, where $N_r \geq N_t$. For each transmission epoch, a set of $M < N_r$ receive antennas is chosen for signal reception. This letter considers the case $M \geq N_t$ only. For spatial multiplexing, the number of receive antennas should be at least the number of transmit antennas. If $M < N_t$, the system will be rank-deficient. The channel gain h_{ij} is the complex path gain from the jth transmit antenna to the ith receive antenna. All the channel gains form the channel matrix $\mathbf{H} = [h_{i,j}] \in \mathbb{C}^{N_r \times N_t}$, where $h_{i,j} \sim \mathcal{CN}(0, 1)$ are identically independent distributed (i.i.d.). Moreover, $\mathbf{H}$ is known to the receiver, but not to the transmitter. N_t symbols from the input signal constellation are transmitted through the channel and the received signal samples are given by

$$\mathbf{y} = \mathbf{H}\mathbf{x} + \mathbf{z} \quad (1)$$

where $\mathbf{x} \in \mathbb{C}^{\mathbf{N_t}}$ whose elements x_i are the transmitted signals from antenna i with $\mathbb{E}\{|x_i|^2\} = E_s \ \forall i$; $\mathbf{y} \in \mathbb{C}^{\mathbf{N_r}}$ has its entries $y_j, j = 1, \ldots, N_r$ being the received signals of the jth antenna. E_s is the average energy per receive antenna. The Gaussian noise vector $\mathbf{z} \in \mathbb{C}^{N_r}$ consists of i.i.d. $\mathcal{CN}(0, N_0)$ variables so that $\mathbb{E}[\mathbf{z}\mathbf{z}^\dagger] = N_0 I_{N_r}$.

Following [9], we define the diagonal matrix Δ of size $N_r \times N_r$ with entries

$$\Delta_{ii} = \begin{cases} 1, & \text{if } i^{th} \text{ receive antenna selected} \\ 0, & \text{otherwise.} \end{cases} \quad (2)$$

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Clearly, $\mathrm{tr}(\Delta) = \sum_i^{N_r} \Delta_{ii} = M < N_r$ equals the number of receive antennas selected for signal reception. The MIMO signal model can now be re-written incorporating receive antenna selection as

$$\mathbf{y} = \Delta \mathbf{H}\mathbf{x} + \mathbf{z}. \tag{3}$$

The channel mutual information with receive antenna selection can now be expressed as in [9]

$$C = \log_2 \det(I_{N_r} + \gamma \Delta \mathbf{H}\mathbf{H}^H) \tag{4}$$

where $\gamma = E_s/N_0$ is SNR per symbol transmitted. We can see that the mutual information is a function of antennas selected (through the matrix Δ in (4)). For brevity, we let $\gamma = 1$. The optimization problem is to pick the M receive antennas such that they maximize the mutual information in (4). It is equivalent to find the matrix Δ such that

$$\Delta = \arg \max_{\substack{\Delta_{ii} \in \{0,1\} \\ \sum_i \Delta_{ii} = M}} \log_2 \det(I_{N_r} + \Delta \mathbf{H}\mathbf{H}^H).$$

The binary variables $\{\Delta_{ii}, i = 1, \ldots, N_r\}$ make the antenna selection problem an NP-hard combinatorial optimization, i.e. an exhaustive search to evaluate all the $\binom{N_r}{M}$ antenna subsets may be needed to pick the optimal solution. One efficient way to deal with the combinatoric nature of variables is to relax the binary constraints by letting $\Delta_{ii} \in [0, 1]$. The problem of receive antenna selection for mutual information maximization is then approximated by the following optimization problem

$$\begin{aligned} \max \quad & \log\det(I_{N_r} + \Delta \mathbf{H}\mathbf{H}^H) \\ \text{subject to} \quad & \Delta \text{ is diagonal}, \\ & 0 \le \Delta_{ii} \le 1, \quad i = 1, ..., N_r, \\ & \mathrm{tr}\,(\Delta) = \sum_{i=1}^{N_r} \Delta_{ii} = M. \end{aligned} \tag{5}$$

Since the variables are now continuous, it is likely that the optimal solution Δ is fractional. The antenna subset is then obtained by choosing the receive antennas corresponding to M largest values of $\Delta_{ii}, i = 1, \ldots, N_r$.

III. A WATER-FILLING BASED ALGORITHM FOR MUTUAL INFORMATION MAXIMIZATION

In [9], the authors use the barrier algorithm to solve the convex relaxation (5). However, the barrier method involves a variable number of Newton steps, which inhibits its practicability. Here, we will provide a direct approach to find an approximate solution of (5). To reduce complexity and ensure implementation ease, we trade off performance compared with the optimal solution based on exhaustive search.

First, let the SVD of $\mathbf{H}\mathbf{H}^H = \mathbf{U}_H \Sigma_H^2 \mathbf{U}_H^*$ with unitary matrix $\mathbf{U}_H \in \mathcal{C}^{N_r \times N_r}$ and diagonal matrix Σ_{LH} with elements being the singular values of $\mathbf{H}\mathbf{H}^H$:

$$\Sigma_H = \mathrm{diag}[\Sigma_{LH} \quad 0] \in \mathcal{R}^{N_r \times N_r}, \quad 0 < \Sigma_{LH} \in \mathcal{R}^{N_t \times N_t}. \tag{6}$$

Since (5) is a convex optimization problem on Δ, even when we relax the trace constraint $\mathrm{tr}(\Delta) = M$ to $\mathrm{tr}(\Delta) \le M$, the optimal solution always achieves $\mathrm{tr}(\Delta) = M$. Eq. (5) becomes

$$\begin{aligned} &\max_{\substack{0 \le \Delta_{ii} \le 1 \\ \mathrm{tr}(\Delta) \le M}} \log\det(I_{N_r} + \Delta \mathbf{H}\mathbf{H}^H) = \\ &\max_{\substack{0 \le \Delta_{ii} \le 1 \\ \mathrm{tr}(\Delta) \le M}} \log\det(I_{N_r} + \Delta \mathbf{U}_H \Sigma_H^2 \mathbf{U}_H^*) \end{aligned} \tag{7}$$

Since $\mathrm{tr}(\mathbf{A}\mathbf{B}) = \mathrm{tr}(\mathbf{B}\mathbf{A})$ for matrices $\mathbf{A}$ and $\mathbf{B}$ with appropriate size, we have

$$\begin{aligned} &\max_{\substack{0 \le \Delta_{ii} \le 1 \\ \mathrm{tr}(\Delta) \le M}} \log\det(I_{N_r} + \Delta \mathbf{U}_H \Sigma_H^2 \mathbf{U}_H^*) = \\ &\max_{\substack{0 \le \Delta_{ii} \le 1 \\ \mathrm{tr}(\Delta) \le M}} \log\det(I_{N_r} + \Sigma_H \mathbf{U}_H^* \Delta \mathbf{U}_H \Sigma_H). \end{aligned} \tag{8}$$

By introducing new variable $\mathbf{X} = \mathbf{U}_H^* \Delta \mathbf{U}_H \in \mathcal{C}^{N_r \times N_r}$, we find that it has the following properties:

- $\mathrm{tr}(\mathbf{X}) = \mathrm{tr}(\Delta)$ [10],
- $\mathbf{X}$ is positive definite but may not be diagonal even if Δ is,
- $0 \le \mathbf{X}_{ii} \le 1$.

The last property derives from the fact that $0 \le \mathbf{X}_{ii} = \sum_j \|\mathbf{U}_{Hij}^*\|^2 \Delta_{jj} \le \sum_j \|\mathbf{U}_{Hij}^*\|^2 = 1$. The optimization problem can now be approximately expressed in terms of new variable $\mathbf{X}$:

$$\begin{aligned} &\max_{\substack{\mathbf{X} \ge 0,\ 0 \le \mathbf{X}_{ii} \le 1 \\ \mathrm{tr}(\mathbf{X}) \le M}} \log\det(I_{N_r} + \Sigma_H \mathbf{X} \Sigma_H) \\ \Longleftrightarrow &\max_{\substack{\mathbf{X} \ge 0,\ 0 \le \mathbf{X}_{ii} \le 1 \\ \mathrm{tr}(\mathbf{X}) \le M}} \log\det(I_{N_t} + \Sigma_{LH} \mathbf{X}_L \Sigma_{LH}) \end{aligned} \tag{9}$$

where $\mathbf{X} = \begin{bmatrix} \mathbf{X}_L & * \\ * & * \end{bmatrix}$, $\mathbf{X}_L \in \mathcal{C}^{N_t \times N_t}$. Note that (9) and (8) are not necessarily equivalent. Since $\mathrm{tr}(\mathbf{X}_L) \le \mathrm{tr}(\mathbf{X})$, at optimality $\mathbf{X} = \begin{bmatrix} \mathbf{X}_L & * \\ * & 0 \end{bmatrix}$. Then

$$(9) \Longleftrightarrow \max_{\substack{\mathbf{X}_L \ge 0,\ 0 \le \mathbf{X}_{Lii} \le 1 \\ \mathrm{tr}(\mathbf{X}_L) \le M}} \log\det(I_{N_t} + \Sigma_{LH} \mathbf{X}_L \Sigma_{LH}). \tag{10}$$

We remove the constraint $\mathbf{X}_{Lii} \le 1\ \forall i$. By modifying the conventional water-filling rule, we will ensure that $\mathbf{X}_{Lii} \le 1 \forall i$

$$\max_{\substack{\mathbf{X}_L \ge 0,\ 0 \le \mathbf{X}_{Lii} \\ \mathrm{tr}(\mathbf{X}_L) \le M}} \log\det(I_{N_t} + \Sigma_{LH} \mathbf{X}_L \Sigma_{LH}) \tag{11}$$

The Lagrangian of (11) is

$$\begin{aligned} \mathcal{L}(\mathbf{X}_L, \alpha, \mu) = &- \log\det(I_{N_t} + \Sigma_{LH} \mathbf{X}_L \Sigma_{LH}) - \mathrm{tr}(\mathbf{X}_L D_\alpha) \\ &+ \mu(\mathrm{tr}(\mathbf{X}_L) - M), \\ &\alpha_i \ge 0, \quad \mu \ge 0, \quad D_\alpha = \mathrm{diag}(\alpha). \end{aligned} \tag{12}$$

Note that $-\log\det(I_{N_t} + \Sigma_{LH} \mathbf{X}_L \Sigma_{LH})$ is convex on $\mathbf{X}_L$, so (11) is a convex optimization program. Therefore, the Karush-Kuhn-Tucker (KKT) conditions are necessary and sufficient for the globally optimal solution. We thus find

$$\begin{aligned} 0 &= \frac{\partial \mathcal{L}(\mathbf{X}_L, \alpha, \mu)}{\partial \mathbf{X}_L} \\ &= -\Sigma_{LH}(I_{N_t} + \Sigma_{LH}\mathbf{X}_L\Sigma_{LH})^{-1}\Sigma_{LH} - D_\alpha + \mu\mathbf{I}_{N_t}, \quad (13) \\ 0 &= \alpha_i \mathbf{X}_{Lii}, \quad i = 1, \ldots, N_t, \\ 0 &= \mu(\mathrm{tr}(\mathbf{X}_L) - M). \end{aligned}$$

From (13), it is clear that at optimality, $\mathbf{X}_L$ is diagonal. This is due to all the involved matrices are diagonal.
Then the following water filling solution can be obtained

$$\mathbf{X}_{Lii} = \left(\mu^{-1} - \Sigma_{LHii}^{-2}\right)^+, \quad i = 1, \ldots, N_t, \qquad (14)$$

where $x^+ = \max\{0, x\}$ and μ is chosen to satisfy $\mathrm{tr}(\mathbf{X}_L) = M$. However, due to the nature of the water-filling solution, there is no certainty that (14) will give the optimal $\mathbf{X}_L$ with $\mathbf{X}_{Lii} \leq 1$. Therefore, we propose a modified water-filling rule to achieve this constraint. We have the following observation:

Observation 1: The optimal solution $\mathbf{X}_{Lii}, i = 1, \ldots, N_t$ of the two problems (10) and (11) has the same ordering (increasing or decreasing).

The water-filling rule in (16) to solve (11) optimally now can be modified to accommodate the constraint $\mathbf{X}_{Lii} \leq 1\ \forall i$ which will solve (10) optimally, as follows

- Step 1: Do water-filling as (14) to find $\mathcal{S} = [i|\mathbf{X}_{Lii} > 1]$. If $\mathcal{S}$ is empty, then stop algorithm. Otherwise, set $\mathbf{X}_{Lii} = 1\ \forall i \in \mathcal{S}$.
- Step 2: Compute new total water $M = M - \|\mathcal{S}\|$ where $\|\mathbb{S}\|$ is the number of indices i in $\mathcal{S}$.
- Step 3: Go back Step 1 for the remaining indices $i \notin \mathcal{S}$.

An approximation to the optimal solution of (8) is given by $\mathbf{U}_H \mathrm{diag}[\mathbf{X}_L \quad 0]\mathbf{U}_H^*$ with all the diagonal elements being real.

Simulations show that the iterations almost always stop after the first step, so the second and third steps in the modified water-filling are rarely executed, which reduces complexity.

This solution is not necessarily the optimal solution of our original problem (5) since it is not a diagonal matrix. A direct solution is to take the real diagonal elements of Δ only. Although this will cause a performance loss, simulations results show that this approach nearly achieves optimal performance.

IV. Simulation Results

In this section, we demonstrate the the efficiency of the proposed antenna selection algorithm via Monte-Carlo simulation. Ergodic capacity for MIMO systems is used as the performance evaluation metric, which is obtained by averaging over results from 2000 independent realizations of channel matrix $\mathbf{H}$. The proposed algorithm and the optimal selection are done to maximize the instantaneous mutual information. The optimal set of antennas is obtained by exhaustive search over $\binom{N_r}{M}$ subsets of receive antennas. Figure 1 shows the ergodic capacity vs received SNR per antenna $(= N_t\gamma)$ when $N_t = 2$ and $N_t = 4$, $N_r = 6$, $M = N_t$. The performance of proposed algorithm is almost indistinguishable from optimal selection. In Fig. 2, the number of received antennas N_r are varied while N_t and $M = N_t$ are kept constant. The performance gap between our approach and the optimal one is larger than in Fig. 1 but still quite tolerable for practical implementation. As mentioned before, since our algorithm is not optimal, the results in Fig. 2 are marginally worse. However, our proposed algorithm avoids implementing complex iterative operations.

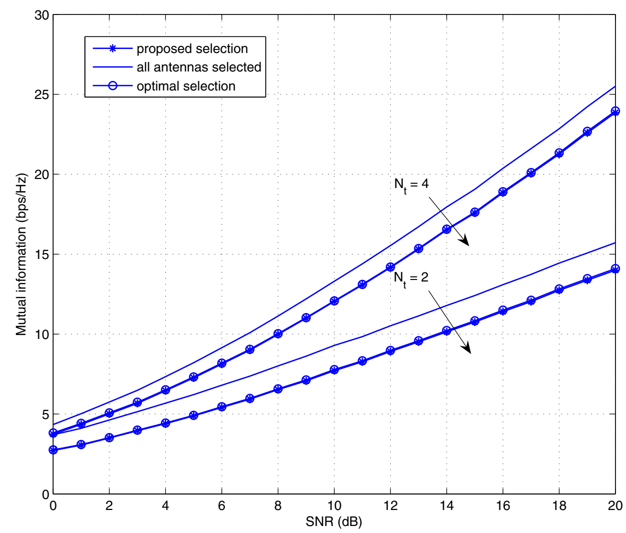

Fig. 1. Ergodic capacity $N_t = 2,\ 4$, $N_r = 6$, $M = N_t$.

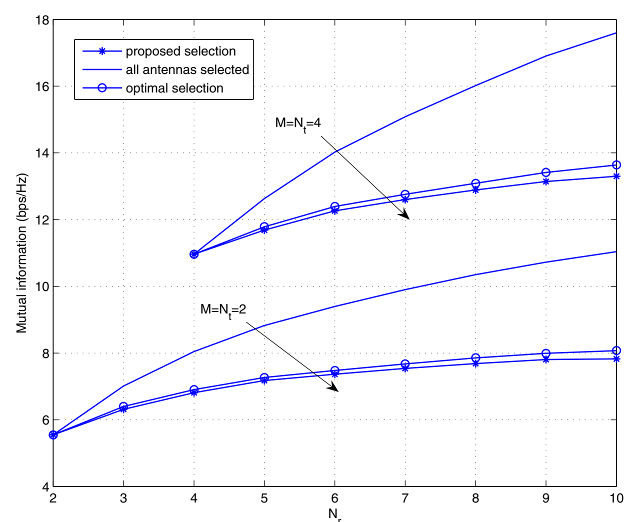

Fig. 2. Ergodic capacity vs N_r, SNR = 10dB N_t = 2,4, $M = N_t$.

V. Conclusion

This paper has presented a direct and attractive approach for receive subset antenna selection to maximize the channel instantaneous mutual information. Our algorithm is based on popular tools i.e. SVD and modified water-filling and does not require iterative Newton steps like [9]. Consequently, our algorithm may be easier to implement in real-time systems. With a total number of N_r receive antennas, the approach [9] has the total complexity $O(N_r^{3.5})$. Since the main part of our algorithm is an SVD, we would expect its complexity be $O(N_r^3)$ for practical purposes. Simulation results show that the performance of our proposed algorithm is close to optimal selection using exhaustive search. The work considers a sub-optimal algorithm that could be more sensitive to imperfect channel estimation. Thus, it would be of further interest to incorporate an imperfect estimation of the channel matrix.

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