# Unitary Signal Constellations for Differential Space-Time Modulation

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***Abstract*— In this letter, we introduce two matrix-signal constellations for differential unitary space time modulation. We also derive an approximation of the upper bound on the symbol error probability. The new constellations generalize several previously reported constellations and yield better performance when the number of transmitter antennas and the constellation size increase.**

***Index Terms*— Differential unitary space time codes, pairwise error probability, diversity product, union bound.**

## I. Introduction

DIFFERENTIAL unitary space time modulation (DUSTM) has been proposed for use with an unknown, slow, flat multiple-input multiple output (MIMO) fading channel [1], [2], [3]. The signal constellation consists of a set of unitary matrices and the design objective is to maximize the diversity product among all the members of the unitary constellation. This design goal leads to the minimization of the block error probability in the high signal-to-noise ratio (SNR) region.

Based on maximizing the diversity product, several unitary constellations have been proposed [3], [4], [5] (due to space limitation, other references are omitted). The design in [3] results in cyclic diagonal matrices with $M$ parameters, where $M$ is the number of transmit antennas. The parameters are numerically optimized to maximize the diversity product. In [5], [4], the cyclic design is augmented with additional multiplying matrices; the design of [4] is limited to three to six transmit antennas. Instead of maximizing the diversity product, Wang et al. [6] minimize the union bound on the block error probability by taking into consideration the number of receive and transmit antennas and the operating SNR.

In this letter, we give two new unitary signal constellations; the first one is a simple generalization of [5] and the second one is based on [7]. When $M$ is even, the first is a special case of the second. We also give an approximate union bound.

## II. System Model and DUSTM

We consider a wireless system in a Rayleigh flat-fading channel with $M$ transmit and $N$ receive antennas. The $T \times N$ complex received signal matrix $\mathbf{Y}_\tau$ is [6]

$$\mathbf{Y}_\tau = \sqrt{\rho}\mathbf{S}_\tau\mathbf{H}_\tau + \mathbf{W}_\tau, \qquad \tau = 0, 1, \dots \tag{1}$$

where $\mathbf{S}_\tau$ is the $T \times M$ complex transmitted signal matrix at time index $\tau$, $H_\tau$ is the $M \times N$ channel matrix, and $W_\tau$ is the $T \times N$ additive noise matrix. The entries of both the channel and noise matrices are independent identically distributed complex Gaussian $\mathcal{CN}(0, 1)$ variables[1]. The transmitted signal energy is normalized so that $\rho$ is the average SNR per receiver (i.e $\mathrm{E}[\sum_{i=1}^{M} |s_{t,i}|^2] = 1$ for any $t$).

Hereafter, we only consider square signal matrices ($M = T$). To transmit a data sequence of integers $d_1, d_2, \dots$ with $d_t \in \{0, \dots, L-1\}$, each $d_t$ is mapped to a distinct unitary matrix signal $\mathbf{\Phi}_{d_t}$ drawn from a unitary space-time matrix constellation $\mathcal{U}$, i.e. $\mathcal{U} = \{\mathbf{\Phi}_1, \mathbf{\Phi}_2, \dots, \mathbf{\Phi}_L\}$. The data rate is given by $R = \log_2 L/M$. In differential unitary space-time modulation, the transmitted signal matrix is

$$\mathbf{S}_\tau = \begin{cases} \mathbf{\Phi}_{d_\tau}\mathbf{S}_{\tau-1}, & \tau = 1, 2\dots \\ \mathbf{I}_M, & \tau = 0. \end{cases} \tag{2}$$

Assuming that the channel remains constant for at least two block intervals (i.e., $\mathbf{H}_\tau = \mathbf{H}_{\tau-1}$), it has been shown in [6] that the pairwise error probability (PEP) is given by

$$P_{ll'} = \Pr(\mathbf{\Phi}_l \longrightarrow \mathbf{\Phi}_{l'}) = \frac{1}{\pi}\int_0^{\frac{\pi}{2}} \prod_{i=1}^{M}\left(1 + \frac{\gamma\lambda_i}{4\sin^2\theta}\right)^{-N} d\theta \tag{3}$$

where $\gamma = \frac{\rho^2}{1+2\rho}$ and $\{\lambda_i\}$ is the $i$-th eigenvalue of the matrix $\Delta_{ll'} = (\mathbf{\Phi}_l - \mathbf{\Phi}_{l'})(\mathbf{\Phi}_l - \mathbf{\Phi}_{l'})^H$.

From [3] and [8], in order to minimize the PEP at high SNR, one can maximize the diversity product $\zeta$, which is defined as

$$\zeta(\mathcal{U}) = \min_{0\le l \le l' \le L-1} \zeta_{ll'} = \frac{1}{2}\min_{l\ne l'} |\det(\mathbf{\Phi}_l - \mathbf{\Phi}_{l'})|^{\frac{1}{M}}. \tag{4}$$

## III. Approximate Union Bound

In [6], instead of the diversity product, the union bound on the block error probability is the design objective. Thus we derive an easy-to-compute approximation of the PEP for the rapid evaluation of the union bound.

Substituting $\sin\theta = t$ in (3) and using the Gaussian quadrature rules [9], the pairwise error probability (3) may be rewritten as

$$P_{ll'} = \frac{1}{2n}\sum_{i=1}^{n} \frac{1}{\det[I + \frac{\gamma}{4x_i^2}\Delta_{ll'}]^N} + R_n \tag{5}$$

where $x_i = \cos(2i-1)\pi/2n$ and $R_n$ is a remainder term. Numerical experiments show that the choice of about 9 terms ($n = 9$) is sufficient for the remainder term to be negligible. Since the above PEP approximation is very accurate, we combine it with the union bound on the overall block error

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[1]In this letter we use the following notations: $(\cdot)^H$ denotes conjugate transpose. The trace, determinant and the Frobenius norm of matrix $\mathbf{A}$ are $\mathrm{trace}(\mathbf{A})$, $\det(\mathbf{A})$ and $\|\mathbf{A}\|_F^2 = \mathrm{tr}(\mathbf{A}\mathbf{A}^H)$. $E[\cdot]$ represents expectation over the random variables within the bracket. A circularly complex Gaussian variable with mean $\mu$ and variance $\sigma^2$ is denoted by $z \sim \mathcal{CN}(\mu, \sigma^2)$. Matrix $I_M$ denotes the $M \times M$ identity matrix.

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probability. With equally-likely transmission of all the space-time signals $\mathbf{\Phi}_l$, the union bound becomes

$$P_{UB} = \frac{1}{18L}\sum_{l=0}^{L-1}\sum_{l\neq l'}^{L-1}\sum_{i=1}^{9}\frac{1}{\det[I+\frac{\gamma}{4x_i^2}\Delta_{ll'}]^N}. \quad (6)$$

Unlike the diversity product which ignores the SNR, (6) takes into account the operational SNR and number of receive antennas as well. Thus minimizing the union bound (6) may be a useful design objective.

## IV. DUSTM Constellation Design

We next develop the two new signal constellations and prove several properties of these.

Consider rotation matrix given by

$$RF_M(\underline{k}\theta) = \begin{pmatrix} RF_2(k_1\theta) & \dots & 0 \\ \vdots & \ddots & \\ 0 & \dots & RF_2(k_{\frac{M}{2}}\theta) \end{pmatrix}_{M\times M} \quad (7)$$

where

$$RF_2(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

and $\underline{k} = \{k_1, k_2, \dots, k_{\frac{M}{2}}\}$ is a set of rotation factors. Our proposed DUSTM constellation $\mathcal{U} = \{\mathbf{\Phi}_l | l = 0, \dots, L-1\}$ consists of the following unitary matrices:

$$\mathbf{\Phi}_l = \begin{pmatrix} e^{j\theta_L\mu_1} & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & e^{j\theta_L\mu_M} \end{pmatrix}^l .[RF_M(\underline{k}\theta_L)]^l \quad (8)$$

where $l = 0, \dots, L-1$ and $\theta_L = \frac{2\pi}{L}$. Clearly, this constellation is characterized by $\frac{3}{2}M$ parameters. When all $k_i$'s are the same, our proposed constellation reduces to the constellation in [5]. When all $k_i$'s are set to zero, (8) reduces to the diagonal cyclic constellation of [3]. Since our constellation has more parameters, we would expect better performance than previous designs; for example, it outperforms those in [5] and [3] in terms of the maximum diversity product. In comparison to [4], our constellation is simple and is available for any number of transmit antennas $M$ (not limited to $M \leq 6$).

The design goal is to find the optimum set of parameters $\underline{\mu} = \{\mu_1, \cdots, \mu_M\}$ and $\underline{k} = \{k_1, \cdots, k_{M/2}\}$ that yield the largest diversity product (4) or the smallest union bound (6) depending on the case. Since analytical determination of the optimums appears intractable, we resort to exhaustive computer search for optimum parameters. Thus, candidates for the best set of $\underline{\mu}$ and $\underline{k}$ are exhaustively generated and examined for performance ( maximum $\zeta$ or minimum $P_{UB}$) and held if they yield better performance than previous best candidate set.

Since the computational complexity grows exponentially with the increase of $M$ and $L$, it can be reduced by applying the following theorems.

*Theorem 4.1:* For an even number of transmit antennas, the diversity product between the $l$-th and $l'$-th unitary matrices in (8) depends only on $(l'-l) \mod L$.

By substituting constellation (8) in formula (4), the diversity product can be written as

$$\begin{aligned}\zeta_{ll'} &= \frac{1}{2}|\det(\mathbf{\Phi}_l - \mathbf{\Phi}_{l'})|^{\frac{1}{M}} \\ &= \frac{1}{2}\prod_i |1 - (e^{j\Delta_l\Theta_L\mu_i} + e^{j\Delta_l\Theta_L\mu_{i+1}})\cos k_i\Delta_l\Theta_L \\ &\qquad + e^{j\Delta_l\Theta_L(\mu_i+\mu_{i+1})}|^{\frac{1}{M}}\end{aligned} \quad (9)$$

where $1 \leq i \leq M-1$, $i$ is odd and $\Delta_l = l' - l$. It is clear, therefore, that $\zeta_{ll'}$ depends only on the difference between $l$ and $l'$. As a result, it is sufficient to consider $\zeta_{0l'}$ for $l' = 1, 2, \cdots, L-1$ to find the diversity product for a particular sets of parameter $\underline{\mu}$ and $\underline{k}$.

*Theorem 4.2:* Assume all the conditions of theorem 4.1, $\underline{\mu}$ and $\underline{k}$ should be in either of the below forms,

1) all $\mu_i$'s are even numbers while all $k_i$'s are odd numbers
2) all $\mu_i$'s are odd integers number and all $k_i$'s are even integer numbers.

*Proof:* See [5]. The same argument is applied here just by taking into account the different rotation angles instead of one rotation angle. ■

Unitary signals in (8) are limited to an even number of transmit antennas. We now give a more general constellation based on [7] that can successfully handle both even and odd number of transmit antennas and also includes (8) as a special case (unfortunately, we cannot extend the above two theorems to this case). This constellation has $M$ phase angles $\mu_1, \cdots, \mu_M$ and $M-1$ rotation angles $k_1, \cdots, k_{M-1}$ and is given by

$$\begin{aligned}\mathbf{\Phi}_l = &\begin{pmatrix} e^{j\theta_L\mu_1} & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & e^{j\theta_L\mu_M} \end{pmatrix}^l .[\mathbf{J}_{1,2}(k_1\theta_L)]^l \\ &.[\mathbf{J}_{2,3}(k_2\theta_L)]^l \cdots [\mathbf{J}_{M-1,M}(k_{M-1}\theta_L)]^l\end{aligned} \quad (10)$$

where

$$\mathbf{J}_{i,i+1}(\theta) = \begin{pmatrix} \mathbf{I}_{i-1} & 0 & \cdots & 0 \\ 0 & \cos(\theta) & -\sin(\theta) & 0 \\ \vdots & \sin(\theta) & \cos(\theta) & \vdots \\ 0 & \cdots & 0 & \mathbf{I}_{M-i-1} \end{pmatrix} \quad (11)$$

$\theta_L = \frac{2\pi}{L}$ and $l = 0, \dots, L-1$. When all $k_i$ are set to zero, (10) is exactly same as the diagonal cyclic constellation of [3] and in case of even transmit antenna, if all $k_{2j}$, $j = 1, \cdots, \frac{M-2}{2}$, are set zero, this is an extension of the constellation (8).

*Theorem 4.3:* For proposed unitary matrix $\mathbf{\Phi}_l$ in (10), if $L$ is an even number, at least one parameter must be odd in $\underline{\mu} = \{\mu_1, \cdots, \mu_M\}$ and $\underline{k} = \{k_1, \cdots, k_{M-1}\}$.

*Proof:* Suppose that all parameters $\underline{k}$ and $\underline{\mu}$ are even integer numbers. Thus we observe that $\mathbf{\Phi}_0$ and $\mathbf{\Phi}_{\frac{L}{2}}$ are viewed as the same at the receiver and consequently the receiver cannot distinguished between $\mathbf{\Phi}_0$ or $\mathbf{\Phi}_{\frac{L}{2}}$. Consequently, this set of parameters does not result in the minimum upper bound on PEP or maximum diversity product. ■

In order to further reduce the search space, the number of independent parameters (10) can be decreased. Of course, the

TABLE I
DIVERSITY PRODUCT OF THE OPTIMUM CODES WITH DIFFERENT CONSTELLATION SCHEME $M=6$, $N=1$, $L=16,32$

| M | L | $\zeta(proposed)$ | $\zeta$(in [5]) | cyclic |
|---|---|---|---|---|
| 6 | 16 | 0.5946 | 0.5946 | 0.5066 |
| | 32 | .5577 | .5069 | 0.448 |
| 10 | 16 | 0.5946 | 0.5946 | 0.5623 |
| | 32 | .5655 | .5137 | 0.5131 |

TABLE II
COMPARISON OF CONSTELLATION PARAMETERS AND UNION BOUND FOR ROTATED AND DIAGONAL SIGNAL , $M=3, N=2, L=16$

| Scheme/criterion | $\underline{\mu}$ | $\underline{k}$ | $P_{UB}$ |
|---|---|---|---|
| Diag./ min $P_{UB}$ | $[1,3,7]$ | $[-,-]$ | $5.746e{-}4$ |
| Rot./ max $\zeta$ | $[10,10,9]$ | $[3,12]$ | $2.310e{-}4$ |
| Rot./ min $P_{UB}$ | $[7,7,10]$ | $[12,4]$ | $1.799e{-}4$ |

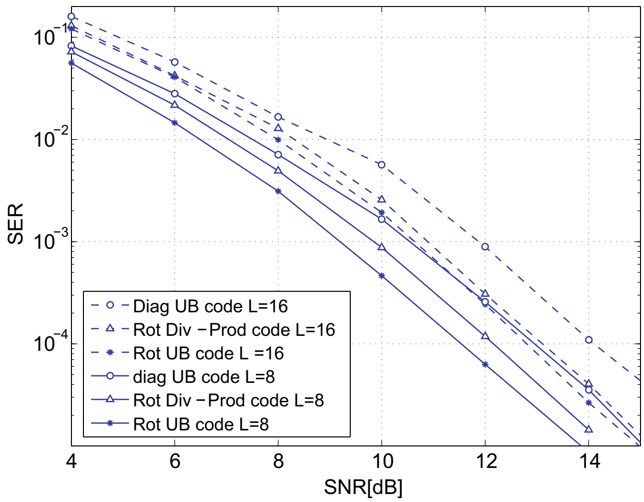


Fig. 1. Symbol error rate of two different constellation with $M=3$, $N=2$ for differential receiver.

achievable diversity product may decrease as well. Following by an idea from [5], if $M$ is even

$$\tilde{\mu}_k = \begin{cases} \mu_1 + 2(k-1) & 1 \le k \le \frac{M}{2}, \\ \mu_2 + 2k - M - 2 & \frac{M}{2} < k \le M \end{cases} \tag{12}$$

and when $M$ is odd

$$\tilde{\mu}_k = \begin{cases} \mu_1 + 2(k-1) & 1 \le k \le \frac{M-1}{2}, \\ \mu_2 & k = \frac{M+1}{2} \\ \mu_3 + 2k - M - 1 & \frac{M+1}{2} < k \le M. \end{cases} \tag{13}$$

The maximum diversity products of our proposed constellation in (10), those in [5] and the diagonal constellation [3] are presented in Table I for a system with 6 or 10 transmit antennas. Due the space limitation, we do not give additional results, but Table I is sufficient to draw the following conclusions. Our constellation improves that of [3] and [5] when $M \geq 16$ and $L \geq 32$. However, there is no improvement when the number of transmit antennas and/or the constellation size are small. This behavior is to be expected given that our constellation incorporates more parameters than [3], [5].

Table II presents the optimum codes that we found from our searches based on optimizing diversity product and minimizing upper bound for rotated signal scheme proposed in (8) and diagonal scheme proposed in [3]. We assumed $M=3$ transmit antennas and $N=2$ receive antennas and an operating SNR of $=12$ dB. Due to continuity, an optimum code in a particular SNR is either optimum or near optimum code within a range of SNR. We list the $P_{UB}$ of the all optimum codes and note that $P_{UB}$ our proposed constellation is smaller than the others.

## V. SIMULATION RESULTS AND DISCUSSION

We simulated codes in Table II and optimum obtained codes for constellation size $L=8$ and found that the proposed constellation in (8) with different rotation angles (2 rotation angles for $M=3$) performs better than the previously proposed constellations. We notice that by applying new constellation and union-bound criteria we achieve coding gain of about 1.5 dB over the code designed in [3] at the SER $10^{-4}$. We have assumed a slow fading channel with Jakes' fading model in which normalized fading parameters $f_dT_s = 1.5 \times 10^{-3}$, where $f_d$ is the Doppler frequency and $T_s$ is the sampling period. we observe that the union-bound based design generally has better performance than the design based on the diversity product in both constellations.

In this letter, we introduced two matrix-signal constellations for differential unitary space time modulation. When the number of transmit antennas $M$ is even, the first is a special case of the second. Since they have $3M/2$ and $2M-1$ parameters, respectively, the search complexity grows rapidly with $M$. We also derived an approximation of the upper bound on the symbol error probability. The new constellations generalize several previously reported constellations and yield better performance when the number of transmitter antennas and the constellation size increase. Since unitary constellations (code books) are required in other applications such as precoder design and limited-feedback systems, the new constellations may prove useful in those cases as well.

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