# A New Class of Space-Time Codes via Orthogonal Designs, Circulant Basis, and Kronecker Product

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***Abstract*— We present a unified algebraic structure of space-time block codes (STBC) with orthogonality called orthogonality-embedded space-time (OEST) codes. Previously known codes, including orthogonal, quasi-orthogonal, semi-orthogonal, and non-orthogonal rate-one circulant space-time codes, are special cases of OEST codes. To construct OEST codes, the generalized complex or real orthogonal designs are employed with two main differences: (1) each data symbol is replaced by a circulant matrix; (2) the scalar product is replaced by the Kronecker product. We show that each group of transmitted symbols embedded in the circulant matrices can be separately detected without any interference from other groups. Signal rotations are used to obtain full diversity and optimal coding gain.**

## I. Introduction

Space-time block codes (STBC) are designed to exploit the diversity and/or the capacity of multiple-input multiple-output (MIMO) wireless fading channels. Among the existing codes, orthogonal STBC (OSTBC) [1]–[3] are ones of the most well-known STBC. The columns of the OSTBC code matrix are all orthogonal to each other allowing single-symbol detection. However, orthogonality entails low code rates [4]; a code rate of one symbol per channel use (pcu) with complex constellations exists for 2 transmit (Tx) antennas only.

Quasi-orthogonal STBC (QSTBC) (e.g. [5]) has therefore been proposed to increase the code rate of OSTBC. The columns of QSTBC code matrices are non-orthogonal in pairs and QSTBC admit low decoding complexity. Nevertheless, the rate-one QSTBC exist for 4 Tx antennas only. Recently, semi-orthogonal algebraic STBC (SAST) has been introduced [6] providing rate-one STBC for any number of Tx antennas. In the SAST code matrices, the left-half columns are orthogonal to right-half columns. This fact leads to the separation decoding of the transmitted symbols into two groups. In general, OSTBC, QSTBC and SAST code matrices exhibits different degrees of column orthogonality and decoding complexity. The question is that whether a general algebraic structure of those codes exists and how to decode this generalized code?

We address those questions in this paper. We present a new class of STBC called orthogonality-embedded space-time (OEST) code. The generalized complex or real orthogonal designs are exploited as the structural basis. Each transmitted symbol is replaced by a circulant matrix and the scalar product is replaced by the Kronecker product [7]. OEST codes subsume OSTBC, QSTBC [5], SAST and rate-one circulant ST codes [8], [9]. The general decoder of OEST codes is derived. OEST codes bring new insights on the class of ST codes with flexible rate-performance-decoding complexity tradeoffs. Other related problems such as unitary space-time modulation and channel information feedback can be further developed systematically from the OEST framework.

## II. Preliminaries

### A. System Model

We consider data transmission over a quasi-static Rayleigh flat fading channel. The transmitter and receiver are equipped with $M$ Tx and $N$ receive (Rx) antennas. The receiver, but not the transmitter, completely knows the channel gains.

A general representation of STBC in the linear dispersion form is given below [10]:

$$\mathcal{X}_M = \sum_{k=1}^{K} (s_k A_k + s_k^* B_k) \tag{1}$$

where $A_k$ and $B_k$, $(k = 1, 2, \cdots, K)$ are $T \times M$ constant basis matrices, superscript $^*$ denotes conjugate [1]. The average energy of code matrices $X \in \mathcal{X}$ is constrained such that $\mathcal{E}_{\mathcal{X}} = \mathbb{E}[\mathrm{trace}(X^\dagger X)] = \mathbb{E}[\|X\|_{\mathsf{F}}^2] = T$, where $\|X\|_{\mathsf{F}}$ denotes Frobenius norm of matrix $X$ [7]. The code rate $\mathsf{R}_{\mathcal{X}_M}$ of a STBC $\mathcal{X}_M$ designed for $M$ Tx antennas, in symbols per channel use (pcu) is defined by $\mathsf{R}_{\mathcal{X}_M} = K/T$.

We next review the constructions and properties of OSTBC [3], [4] and rate-one circulant codes [8], [9], which are necessary to design OETS codes.

### B. Orthogonal Space-Time Block Codes

In the following, we just consider the generalized complex orthogonal designs [3], [4] only. The results can be easily extended to generalized real orthogonal designs.

*Proposition 1:* Let $\mathcal{O}_Q$ be an STBC with $R$ rows, $Q$ columns, and $K$ symbols per code matrix. $\mathcal{O}_Q$ is an OSTBC if and only if its basis matrices $A_k$ and $B_k$ in (1) satisfy

$$A_i^\dagger A_i + B_i^\dagger B_i = \boldsymbol{I}_Q, \qquad i = 1, 2, \cdots, K \tag{2a}$$

$$A_i^\dagger A_j + B_j^\dagger B_i = \mathbf{0}_Q, \qquad 1 \le i < j \le K \tag{2b}$$

$$A_i^\dagger B_j + A_j^\dagger B_i = \mathbf{0}_Q, \qquad i, j = 1, 2, \cdots, K. \tag{2c}$$

[1]From now on, superscripts $^\mathsf{T}$ and $^\dagger$ denote matrix transpose and transpose conjugate. The $n \times n$ identity and all-zero matrices are denoted by $\boldsymbol{I}_n$ and $\mathbf{0}_n$, respectively. $\mathbb{E}[\cdot]$ denotes average. A mean-$m$ and variance-$\sigma^2$ circularly complex Gaussian random variable is written by $\mathcal{CN}(m, \sigma^2)$. The minimum Euclidean distance of a constellation is given by $d_{\min}$

*Proposition 2:* The maximal code rate of OSTBC for $Q = 2a-1$ or $Q = 2a$, where $a$ is any positive integer, is $\frac{a+1}{2a}$.

Assume that the data symbols are drawn from a constellation with unit average power. To guarantee the average power constraint, the OSTBC code matrices are contracted by a constant $\kappa$. We can show that $\kappa = \frac{1}{Q\mathsf{R}_{\mathcal{O}_Q}}$.

The coding gain [1] of OSTBC can be easily found to be

$$G_{\mathcal{O}_Q} = \frac{1}{Q\mathsf{R}_{\mathcal{O}_Q}} d_{\min}^2. \tag{3}$$

## C. Rate-one circulant STBC

The idea to employ circulant matrices [11] to build rate-one ST codes has appeared in [8], [9]. We call such codes circulant ST codes. The code matrix of circulant ST codes is

$$C = \begin{bmatrix} u_1 & u_2 & \cdots & u_P \\ u_P & u_1 & \cdots & u_{P-1} \\ \vdots & \vdots & & \vdots \\ u_2 & u_3 & \cdots & u_1 \end{bmatrix}. \tag{4}$$

To achieve full diversity, the transmitted signals are designed as follows. Let $P$ be the number of Tx antennas, let $\boldsymbol{s} = [s_1, s_2, \cdots, s_P]^{\mathsf{T}}$ be a data vector of $P$ symbols to be rotated before transmission. The transmitted vector $\boldsymbol{u}$ is given by

$$\boldsymbol{u} = \Theta \boldsymbol{s} \tag{5}$$

where $\Theta = \mathrm{diag}[\,1,\ \phi^{1/P},\ \cdots,\ \phi^{(P-1)/P}\,]$ and $\phi$ is called a Diophantine number [8]. A rate-one circulant ST code (or linear threaded algebraic ST (LTAST)) matrix is given by [8]

$$\mathcal{C}_P = \frac{1}{\sqrt{P}} C\,. \tag{6}$$

The coding gain of rate-one LTAST code is upper-bounded by [8, Eq. (7)]

$$G_{\mathcal{C}_P} \le \frac{1}{P} d_{\min}^2\,. \tag{7}$$

Some optimal values of $\phi$ so that the maximal coding gain can be obtained are specified as follows [8, Theorem 2].

- If $P = 2^r, r \ge 1$, then $\phi = \mathrm{j}$ for $\mathsf{S}$ carved from the ring of Gaussian integers or,
- If $P = 2^{r_0}3^{r_1}, r_0, r_1 \ge 0$, then $\phi = e^{2\mathrm{j}\pi/6}$ and constellations $\mathsf{S}$ carved from the ring of Eisenstein integers.

[8, Theorem 1] also suggests how to select $\phi$ for PSK constellations; however, computer search is required to find the $\phi$ that maximizes the coding gain. For a special case with $P = 2$, we have the following result without proof for brevity.

*Proposition 3:* Consider the rate-one circulant ST codes for $P = 2$. One of the two transmitted symbols is drawn from an $M$-ary PSK constellation $\mathsf{S}$ and the other one is drawn from $e^{\mathrm{j}\alpha}\mathsf{S}$. The coding gain of circulant codes is maximized if and only if the rotation angle $\alpha$ is $\frac{(2k+1)\pi}{M}$ for $k = 0, 1, ..., M-1$ if $M$ is even and $\frac{(2k+1)\pi}{M}$, $k = 0, 1, ..., 2M-1$ if $M$ is odd.

We next briefly review some important properties of circulant matrices, which are not recognized in [8], [9].

## D. Properties of Circulant Matrices

A $P \times P$ matrix $C$ is called a circulant matrix if

$$C = \begin{bmatrix} x_1 & x_2 & \cdots & x_P \\ x_P & x_1 & \cdots & x_{P-1} \\ \vdots & \vdots & & \vdots \\ x_2 & x_3 & \cdots & x_1 \end{bmatrix}. \tag{8}$$

Let $\boldsymbol{x} = [x_1\ x_2\ \cdots\ x_P]^{\mathsf{T}}$. We can add the argument $\boldsymbol{x}$ to $C$ as $C(\boldsymbol{x})$.

*Proposition 4:* Basic properties of the circulant matrices:

- P1 $C$ is circulant if and only if $C^\dagger$ is circulant.
- P2 if $A$ and $B$ are circulants of the same order, and $\alpha_1$ and $\alpha_2$ are two scalars, then the matrices $A^{\mathsf{T}}$, $\alpha_1 A + \alpha_2 B$, $AB$ are circulants.
- P3 All of the right circulants of the same order commute, i.e. $AB = BA$.

Let $\alpha = [0\ 1\ 0\ \cdots\ 0]^{\mathsf{T}}$ be a $P \times 1$ vector. A matrix $\pi$ called *forward shift permutation* [11] is defined as $\pi = C(\alpha)$. The main property of $\pi$ is given in the following [11, p. 27].

$$\pi^{\mathsf{T}} = \pi^\dagger = \pi^{-1} = \pi^{P-1}. \tag{9}$$

The circulant matrix in (4) can be represented as [11, p. 68]

$$C(\boldsymbol{x}) = \sum_{i=1}^{P} x_i \pi^{i-1}. \tag{10}$$

From Proposition 4 and (10), we obtain

$$[C^\dagger(\boldsymbol{x})C(\boldsymbol{x})]_{ij} = \boldsymbol{x}^{\mathsf{T}}\pi^i\pi^{-j}\boldsymbol{x}^* = \boldsymbol{x}^\dagger\pi^{j-i}\boldsymbol{x}\,. \tag{11}$$

# III. Constructions and Properties of OEST Codes

## A. Constructions of OEST Codes

Let the number of Tx antennas be $M = PQ$, where $P$ and $Q$ are positive integers, and let $A_k$ and $B_k$ $(k = 1, 2, \cdots, K)$ be the $R \times Q$ basis matrices of OSTBC $\mathcal{O}_Q$. A block of $K \times P$ input data symbols are divided into $K$ data vectors $\boldsymbol{s}_k$, each of size $P \times 1$. Each of the $K$ data vectors is rotated as specified in (5); then the resulting $K$ vectors $\boldsymbol{u}_k$ are used to build $K$ circulant code matrices $\mathcal{C}_k = C(\boldsymbol{u}_k)$ as in (6).

Now we present the *main results* of the paper. Two constructions of OEST codes are proposed as follows:

*Construction I*:

$$\mathcal{D} = \sqrt{\frac{\kappa}{P}} \sum_{k=1}^{K} \Big( A_k \otimes \mathcal{C}_k + B_k \otimes \mathcal{C}_k^\dagger \Big), \tag{12}$$

*Construction II*:

$$\mathcal{D} = \sqrt{\frac{\kappa}{P}} \sum_{k=1}^{K} \Big( \mathcal{C}_k \otimes A_k + \mathcal{C}_k^\dagger \otimes B_k \Big)\,. \tag{13}$$

where $\otimes$ denotes the Kronecker product [7].

In can be shown that Construction I and II are permutation equivalent [12, corollary 4.3.10]. We will therefore derive the properties of the OEST codes for Construction I only.

There exist several different constructions of OSTBC (e.g. [3], [4]). In combination with the circulant codes, there are a large number of OEST codes' variants, which can be implemented for the same number of Tx antennas. Additionally, OEST codes subsume some existing STBC as we show below.

*OSTBC*: If $P = 1$, the circulant matrix $\mathcal{C}_k$ reduces to a single symbol $u_k$, OEST codes are just the original OSTBC.

*QSTBC*: If $Q = 2$, Construction II is identical with the QSTBC codes given by Tirkkonen *et al.* [5].

*Proof:* The QSTBC in [5] is known as ABBA codes. Its construction for $M = 2Q$ Tx antennas is $\mathcal{Q} = \begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{B} & \mathcal{A} \end{bmatrix}$, where $\mathcal{A}$ and $\mathcal{B}$ are two matrices of an OSTBC $\mathcal{O}_Q$. Thus $\mathcal{A}$ and $\mathcal{B}$ can be represented as

$$\mathcal{A} = \sum_{k=1}^{K} (s_k A_k + s_k^* B_k),\ \mathcal{B} = \sum_{k=1}^{K} \left(s_{k+K} A_k + s_{k+K}^* B_k\right)$$

where $A_k$ and $B_k$, $(k = 1, 2, \cdots, K)$ are the basis matrices of OSTBC for $Q$ Tx antennas. Hence, we have

$$\begin{aligned} \mathcal{Q} &= \sum_{k=1}^{K} \underbrace{\begin{bmatrix} s_k & s_{k+K} \\ s_{k+K} & s_k \end{bmatrix}}_{\mathcal{C}_k} \otimes A_k + \sum_{k=1}^{K} \underbrace{\begin{bmatrix} s_k^* & s_{k+K}^* \\ s_{k+K}^* & s_k^* \end{bmatrix}}_{\mathcal{C}_k^\dagger} \otimes B_k \\ &= \sum_{k=1}^{K} (\mathcal{C}_k \otimes A_k + \mathcal{C}_k^\dagger \otimes B_k). \end{aligned} \tag{14}$$

Eq. (14) is identical to the OEST Construction II (13). □

*SAST codes* [6]: If $Q = 2$, and the basis matrices are the ones of the Alamoutic code with

$$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, B_1 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, B_2 = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix},$$

then from Construction I, we have OEST for $Q = 2$ as

$$\mathcal{D} = \begin{bmatrix} \mathcal{C}_1 & \mathcal{C}_2 \\ -\mathcal{C}_2^\dagger & \mathcal{C}_1^\dagger \end{bmatrix}. \tag{15}$$

It is exactly the structure of SAST codes [6].

*Rate-one circulant codes* [8], [9]: In this case, $Q = 1$, $A_1 = \boldsymbol{I}_1$, $B_1 = \mathbf{0}_1$.

## B. Properties of OEST codes

*Theorem 1:* The rate of OEST codes for $M = PQ$ Tx antennas is equal to the rate of OSTBC for $Q$ Tx antennas, i.e. $\mathsf{R}_{\mathcal{D},M} = \mathsf{R}_{\mathcal{O},Q}$. The upper bound of the code rate for $Q = 2a - 1$ or $Q = 2a$ is $\frac{a+1}{2a}$.

*Proof:* Since OEST codes have size $RP \times QP$, their code rate for $M = QP$ Tx antennas is equal to the rate of OSTBC for $Q$ Tx antennas. The upper bound of the code rate is directly from Proposition 2. □

*Theorem 2:* If the circulant ST code used to construct OEST codes achieves full diversity, then so do the OEST codes.

*Proof:* From (12) we have

$$\begin{aligned} \frac{\mathcal{D}^\dagger \mathcal{D}}{\kappa/P} &= \sum_{i=1}^{K} (A_i \otimes \mathcal{C}_i + B_i \otimes \mathcal{C}_i^\dagger)^\dagger \cdot \sum_{j=1}^{K} (A_i \otimes \mathcal{C}_i + B_i \otimes \mathcal{C}_i^\dagger) \\ &= \sum_{i=1}^{K}\sum_{j=1}^{K} (A_i^\dagger A_j) \otimes (\mathcal{C}_i^\dagger \mathcal{C}_j) + \sum_{i=1}^{K}\sum_{j=1}^{K} (B_i^\dagger B_j) \otimes (\mathcal{C}_i \mathcal{C}_j^\dagger) \\ &\quad + \underbrace{\sum_{i=1}^{K}\sum_{j=1}^{K} (A_i^\dagger B_j) \otimes (\mathcal{C}_i^\dagger \mathcal{C}_j^\dagger)}_{\mathbf{0}_M} + \underbrace{\sum_{i=1}^{K}\sum_{j=1}^{K} (B_i^\dagger A_j) \otimes (\mathcal{C}_i \mathcal{C}_j)}_{\mathbf{0}_M} \\ &= \sum_{k=1}^{K} \boldsymbol{I}_Q \otimes (\mathcal{C}_k^\dagger \mathcal{C}_k) = \boldsymbol{I}_Q \otimes \left( \sum_{k=1}^{K} \mathcal{C}_k^\dagger \mathcal{C}_k \right). \end{aligned} \tag{16}$$

For two distinct code matrices $D$ and $\hat{D}$, then

$$P_D = (D - \hat{D})^\dagger (D - \hat{D}) = \frac{\kappa}{P} \boldsymbol{I}_Q \otimes \left( \sum_{k=1}^{K} \Delta_{\mathcal{C}_k}^\dagger \Delta_{\mathcal{C}_k} \right) \tag{17}$$

where $\Delta_{\mathcal{C}_k} = \mathcal{C}_k - \hat{\mathcal{C}}_k$. Since $D \neq \hat{D}$, there exists at least one pair of $\mathcal{C}_i$ and $\hat{\mathcal{C}}_i$ such that $\mathcal{C}_i \neq \hat{\mathcal{C}}_i$ or $\Delta_{\mathcal{C}_i}^\dagger \Delta_{\mathcal{C}_i}$ is positive definite. Then the matrix $\left( \sum_{k=1}^{K} \Delta_{\mathcal{C}_k}^\dagger \Delta_{\mathcal{C}_k} \right)$ is always positive definite for any pairs of distinct code matrices. Therefore, the matrix $P_D$ is always of full rank. □

The coding gain of OEST codes immediately follows

$$G_{\mathcal{D}_M} = \min_{D \neq \hat{D}} \det P_D = \frac{\kappa}{P} \min_{D \neq \hat{D}} \left[ \det \left( \sum_{k=1}^{K} \Delta_{\mathcal{C}_k}^\dagger \Delta_{\mathcal{C}_k} \right) \right]^{\frac{Q}{M}}. \tag{18}$$

In the worst-case, where there only exists one pair of $\mathcal{C}_i$ and $\hat{\mathcal{C}}_i$ such that $\mathcal{C}_i \neq \hat{\mathcal{C}}_i$, the coding gain is

$$G_{\mathcal{D}_M} = \frac{\kappa}{P} \min_{\mathcal{C}_i \neq \hat{\mathcal{C}}_i} \left[ \det \left( \Delta_{\mathcal{C}_i}^\dagger \Delta_{\mathcal{C}_i} \right) \right]^{1/P} = \kappa G_{\mathcal{C}_P}. \tag{19}$$

Thus from (7), one can use the optimal rotation of circulant ST codes in Section II-C to maximize the coding gain of OEST codes. The coding gain of OEST codes is upper-bounded by

$$G_{\mathcal{D}_M} \leq \frac{1}{Q \mathsf{R}_{\mathcal{O}_Q}} \frac{d_{\min}^2}{P} = \frac{d_{\min}^2}{M \mathsf{R}_{\mathcal{O}_Q}}. \tag{20}$$

From (16), if $M = PQ$ columns of OEST code matrices are divided into $Q$ groups, each of $P$ consecutive columns, then the columns of a group are orthogonal to the columns of the other groups. We next derive a decoder exploiting this property such that the transmitted symbols can be divided into $K$ subgroups with lower decoding complexity. For the sake of clarity and simplicity, we just consider the case with $N = 1$ Rx antennas. The generalization for $N \geq 1$ is straightforward.

## C. Decoder

Let $\boldsymbol{h} = [h_1\ h_2\ \cdots\ h_M]^\mathsf{T}$ denote the channel gain between $m$th Tx antenna $(m = 1, 2, \cdots, M)$ and the Rx antenna. Let $D \in \mathcal{D}$ be a code matrix, the Rx signal vector $\boldsymbol{y}$ is as [10]

$$\boldsymbol{y} = \sqrt{\frac{\rho\kappa}{P}} D\boldsymbol{h} + \boldsymbol{n}. \tag{21}$$

Let $\boldsymbol{u}_k = [u_{k,1}\ u_{k,2}\ \cdots\ u_{k,P}]^{\mathsf{T}}$ $(k = 1, 2, \cdots, K)$ denote the $k$th input data vector to the circulant ST encoder (6). From (10), (12) and (21), we get

$$\begin{aligned}\boldsymbol{y} &= \sqrt{\frac{\rho\kappa}{P}} \sum_{k=1}^{K} \left[ A_k \otimes \left( \sum_{i=1}^{P} u_{k,p} \pi^{p-1} \right) \right. \\ &\qquad \left. + B_k \otimes \left( \sum_{p=1}^{P} u_{kp}^{*} \pi^{1-p} \right) \right] \boldsymbol{h} + \boldsymbol{n} \\ &= \sqrt{\frac{\rho\kappa}{P}} \sum_{k=1}^{K} \sum_{p=1}^{P} \left[ \left( A_k \otimes \pi^{p-1} \right) \boldsymbol{h} u_{kp} \right. \\ &\qquad \left. + \left( B_k \otimes \pi^{1-p} \right) \boldsymbol{h} u_{kp}^{*} \right] + \boldsymbol{n}\,. \end{aligned} \tag{22}$$

Let $\boldsymbol{e}_{kp} = \left( A_k \otimes \pi^{p-1} \right) \boldsymbol{h}, E_k = \begin{bmatrix} \boldsymbol{e}_{k1} & \boldsymbol{e}_{k2} & \cdots & \boldsymbol{e}_{kP} \end{bmatrix}$, $\boldsymbol{f}_{kp} = \left( B_k \otimes \pi^{1-p} \right) \boldsymbol{h}\,, F_k = \begin{bmatrix} \boldsymbol{f}_{k1} & \boldsymbol{f}_{k2} & \cdots & \boldsymbol{f}_{kP} \end{bmatrix}$. We can rewrite (22) as

$$\begin{aligned}\boldsymbol{y} &= \sqrt{\frac{\rho\kappa}{P}} \begin{bmatrix} E_1 & F_1 & E_2 & F_2 & \cdots & E_K & F_K \end{bmatrix} \\ &\quad \times \begin{bmatrix} \boldsymbol{u}_1^{\mathsf{T}} & \boldsymbol{u}_1^{\dagger} & \boldsymbol{u}_2^{\mathsf{T}} & \boldsymbol{u}_2^{\dagger} & \cdots & \boldsymbol{u}_K^{\mathsf{T}} & \boldsymbol{u}_K^{\dagger} \end{bmatrix}^{\mathsf{T}} + \boldsymbol{n}\,. \end{aligned} \tag{23}$$

Furthermore, the following equation is equivalent to (23)

$$\begin{aligned}\begin{bmatrix} \boldsymbol{y} \\ \boldsymbol{y}^{*} \end{bmatrix} &= \sqrt{\frac{\rho\kappa}{P}} \underbrace{\begin{bmatrix} E_1 & F_1 & \cdots & E_K & F_K \\ F_1^{*} & E_1^{*} & \cdots & F_K^{*} & E_K^{*} \end{bmatrix}}_{W} \\ &\quad \times \begin{bmatrix} \boldsymbol{u}_1^{\mathsf{T}} & \boldsymbol{u}_1^{\dagger} & \cdots & \boldsymbol{u}_K^{\mathsf{T}} & \boldsymbol{u}_K^{\dagger} \end{bmatrix}^{\mathsf{T}} + \begin{bmatrix} \boldsymbol{n} \\ \boldsymbol{n}^{*} \end{bmatrix}. \end{aligned} \tag{24}$$

We can show that the columns of matrix $W$ are orthogonal.

*Proof:* We can show that the following equations hold:

$$\begin{bmatrix} E_k \\ F_k^{*} \end{bmatrix}^{\dagger} \begin{bmatrix} E_l \\ F_l^{*} \end{bmatrix} = E_k^{\dagger} E_l + F_k^{\mathsf{T}} F_l^{*} = \boldsymbol{0}_P \quad \text{for } k \neq l, \tag{25a}$$

$$\begin{bmatrix} E_k \\ F_k^{*} \end{bmatrix}^{\dagger} \begin{bmatrix} F_l \\ E_l^{*} \end{bmatrix} = E_k^{\dagger} F_l + F_k^{\mathsf{T}} E_l^{*} = \boldsymbol{0}_P, \tag{25b}$$

We just prove for (25a); (25b) can be shown similarly.

The size of matrix $Z_{kl} = (E_k^{\dagger} E_l + F_k^{\mathsf{T}} F_l^{*})$ is $P \times P$. The element $[Z_{kl}]_{ij}$ of $Z_{kl}$ can be calculated as

$$\begin{aligned}[Z_{kl}]_{ij} &= \boldsymbol{e}_{ki}^{\dagger} \boldsymbol{e}_{lj} + \boldsymbol{f}_{ki}^{\mathsf{T}} \boldsymbol{f}_{lj}^{*} \\ &= \boldsymbol{h}^{\dagger} (A_k^{\dagger} \otimes \pi^{-i+1})(A_l \otimes \pi^{j-1}) \boldsymbol{h} \\ &\quad + \boldsymbol{h}^{\mathsf{T}} (B_k^{\mathsf{T}} \otimes \pi^{i-1})(B_l^{*} \otimes \pi^{-j+1}) \boldsymbol{h}^{*} \\ &= \boldsymbol{h}^{\dagger} [(A_k^{\dagger} A_l) \otimes (\pi^{j-i})] \boldsymbol{h} + \boldsymbol{h}^{\mathsf{T}} [(B_k^{\mathsf{T}} B_l^{*}) \otimes (\pi^{i-j})] \boldsymbol{h}^{*} \\ &= \boldsymbol{h}^{\dagger} [(A_k^{\dagger} A_l + B_k^{\dagger} B_l) \otimes (\pi^{j-i})] \boldsymbol{h} \\ &= \begin{cases} 0, & k \neq l; \\ \boldsymbol{h}^{\dagger} (\boldsymbol{I}_Q \otimes \pi^{j-i}) \boldsymbol{h}, & k = l. \end{cases} \end{aligned} \tag{26}$$

Thus $Z_{kl} = \boldsymbol{0}_P$ if $k \neq l$ , this completes the proof. □

Since for $k = l$, the matrices $Z_{kk}$ do not depend on the value of $k$; we drop the subscript $k$ for brevity. The entries of $Z$ are

$$z_{ij} = \boldsymbol{h}^{\dagger} (\boldsymbol{I}_Q \otimes \pi^{j-i}) \boldsymbol{h}. \tag{27}$$

Let $\hat{\boldsymbol{h}}_q = \begin{bmatrix} h_{(q-1)P+1} & h_{(q-1)P+2} & \cdots & h_{(q-1)P+P} \end{bmatrix}^{\mathsf{T}}$ for $q = 1, 2, \cdots, Q$. Then $\boldsymbol{h} = \begin{bmatrix} \hat{\boldsymbol{h}}_1^{\mathsf{T}} & \hat{\boldsymbol{h}}_2^{\mathsf{T}} & \cdots & \hat{\boldsymbol{h}}_Q^{\mathsf{T}} \end{bmatrix}^{\mathsf{T}}$, and $z_{ij}$ in (27) can be rewritten as

$$z_{ij} = \sum_{q=1}^{Q} \hat{\boldsymbol{h}}_q^{\dagger} \pi^{j-i} \hat{\boldsymbol{h}}_q. \tag{28}$$

Moreover, from (11), we have an elegant form of $Z$ as

$$Z = \sum_{q=1}^{Q} C^{\dagger}(\hat{\boldsymbol{h}}_q) C(\hat{\boldsymbol{h}}_q). \tag{29}$$

From Proposition 4, $Z$ is also a circulant matrix.

To separate the transmitted vector $\boldsymbol{u}_k$ at the receiver, we can multiply the two sides of (24) with $\begin{bmatrix} E_k^{\dagger} & F_k^{\mathsf{T}} \end{bmatrix}$ to get

$$E_k^{\dagger} \boldsymbol{y} + F_k^{\mathsf{T}} \boldsymbol{y}^{*} = Z \boldsymbol{u}_k + \underbrace{(E_k^{\dagger} \boldsymbol{n} + F_k^{\mathsf{T}} \boldsymbol{n}^{*})}_{\hat{\boldsymbol{n}}_k}. \tag{30}$$

The covariance matrix of noise $\mathbb{E}[\hat{\boldsymbol{n}}_k \hat{\boldsymbol{n}}_k^{\dagger}] = Z$ is not an identity matrix, the noise can be whitened by multiplying the two sides of (30) with a whitening matrix $Z^{-\frac{1}{2}}$. The received signal with whitened noise is

$$Z^{-\frac{1}{2}} (E_k^{\dagger} \boldsymbol{y} + F_k^{\mathsf{T}} \boldsymbol{y}^{*}) = Z^{\frac{1}{2}} \boldsymbol{u}_k + \underbrace{Z^{-\frac{1}{2}} \hat{\boldsymbol{n}}_k}_{\hat{\boldsymbol{n}}} \tag{31}$$

where the elements of $\hat{\boldsymbol{n}}$ are $\mathcal{CN}(0, 1)$.

To solve (31), a sphere decoder [13] can be employed.

One can verify the general detection equation (31) for existing codes such as OSTBC and SAST codes.

## IV. OEST Code Examples

Given a value of $M$, one can find the sets of all pairs $\{(P, Q) | P, Q \in \mathbb{N}, PQ = M\}$. Note that one can delete one or several columns of OEST codes for $M$ Tx antennas to construct OEST codes for the smaller numbers of Tx antennas.

Denote OEST codes designed for the set of parameters $P, Q$ as $\mathcal{D}_{P,Q}$. For $M = 6$, there are at least 4 variants as follows.

$$\mathcal{D}_{1,6} = \frac{1}{\sqrt{4}} \mathcal{O}_6 \qquad \text{(see [4, (101)])} \tag{32}$$

$$\begin{aligned}\mathcal{D}_{2,3} &= \frac{\sqrt{2}}{3} \begin{bmatrix} u_1 & u_2 & u_3 & u_4 & u_5 & u_6 \\ u_2 & u_1 & u_4 & u_3 & u_6 & u_5 \\ -u_3^{*} & -u_4^{*} & u_1^{*} & u_2^{*} & 0 & 0 \\ -u_4^{*} & -u_3^{*} & u_2^{*} & u_1^{*} & 0 & 0 \\ -u_5^{*} & -u_6^{*} & 0 & 0 & u_1^{*} & u_2^{*} \\ -u_6^{*} & -u_5^{*} & 0 & 0 & u_2^{*} & u_1^{*} \\ 0 & 0 & u_5^{*} & u_6^{*} & -u_3^{*} & -u_4^{*} \\ 0 & 0 & u_6^{*} & u_5^{*} & -u_4^{*} & -u_3^{*} \end{bmatrix} \\ &\equiv \mathcal{Q}_6 \end{aligned} \tag{33}$$

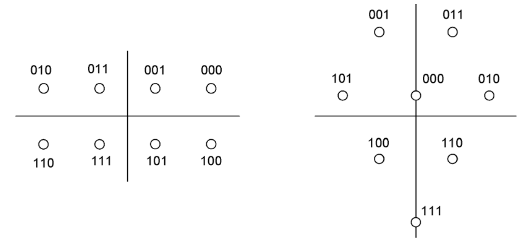
$$\begin{aligned}\mathcal{D}_{3,2} &= \frac{1}{\sqrt{6}} \begin{bmatrix} u_1 & u_2 & u_3 & u_4 & u_5 & u_6 \\ u_3 & u_1 & u_2 & u_6 & u_4 & u_5 \\ u_2 & u_3 & u_1 & u_5 & u_6 & u_4 \\ -u_4^{*} & -u_6^{*} & -u_5^{*} & u_1^{*} & u_3^{*} & u_2^{*} \\ -u_5^{*} & -u_4^{*} & -u_6^{*} & u_2^{*} & u_1^{*} & u_3^{*} \\ -u_6^{*} & -u_5^{*} & -u_4^{*} & u_3^{*} & u_2^{*} & u_1^{*} \end{bmatrix} \\ &= \mathcal{S}_6 \end{aligned} \tag{34}$$

$$\mathcal{D}_{6,1} = \frac{1}{\sqrt{6}} \begin{bmatrix} u_1 & u_2 & u_3 & u_4 & u_5 & u_6 \\ u_6 & u_1 & u_2 & u_3 & u_4 & u_5 \\ u_5 & u_6 & u_1 & u_2 & u_3 & u_4 \\ u_4 & u_5 & u_6 & u_1 & u_2 & u_3 \\ u_3 & u_4 & u_5 & u_6 & u_1 & u_2 \\ u_2 & u_3 & u_4 & u_5 & u_6 & u_1 \end{bmatrix} = \mathcal{C}_6 \quad (35)$$

To construct $\mathcal{D}_{2,3}$, we use the orthogonal basis matrices of OSTBC $\mathcal{O}_3$ [4] by deleting the last columns of $\mathcal{O}_4$.

The main parameters of OEST codes for $M = 6$ are summarized in Table I. We provide simulation results to illustrate the performance the OEST codes for $M = 6$ with rate of 3 bits pcu. However, except for $\mathcal{D}_{1,6}$ or $\mathcal{O}_6$ with symbol rate $\mathsf{R}_{\mathcal{O},6} = 2/3$ symbols pcu, there is no constellation that matches the bit rate of 3 bits pcu. Thus, 16QAM is selected, resulting in the bit rate of $8/3$ bits pcu. The shapes of 8QAM and 8Hex are sketched in Fig. 1. For $M = 6$, with 8QAM, the optimal rotations for circulant ST codes is not available analytically. The rotation angles $\phi = e^{\mathrm{j}\,\pi/4}$ is used.

From Fig. 2, $\mathcal{S}_6$ yields better performance among the investigated OEST codes. $\mathcal{S}_6$ with 8QAM gains about 0.5 and 1.2 dB over $\mathcal{C}_6$ and $\mathcal{O}_6$, respectively. Moreover, $\mathcal{S}_6$ with 8Hex even outperforms OSTBC, which has lower spectral efficiency. The coding gain of SAST and LTAST codes are the same; however, from simulations, SAST codes yield better performance compared with LTAST codes (see also [6]). It means that the geometrical distance spectrum of SAST codes is improved compared with LTAST codes.




(a) 8QAM Square (b) 8Hex

Fig. 1. Geometrical shapes of 8QAM square and 8Hex.

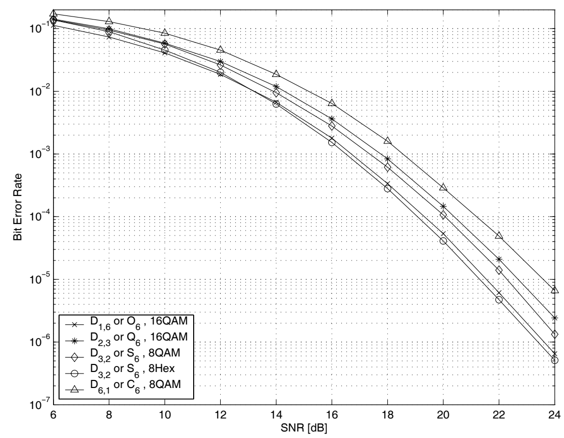



Fig. 2. Performance of four implementations of OEST codes for 6 Tx antennas, 3 bits pcu, except $\mathcal{D}_{1,6}$ (or $\mathcal{O}_6$) with $8/3$ bits pcu.

## V. Conclusion

We have presented new general constructions of OEST codes, which are a class of STBC with embedded orthogonal designs. At the receiver, the transmitted symbols can be decoupled into subgroups to reduce the detection complexity. The OEST codes subsume previously known ST codes as its special cases: OSTBC, QSTBC, SAST and rate-one circulant ST codes. The OEST framework can also be readily extended for differential ST modulation, in which each circulant matrix is replaced by a unitary ST matrix (e.g. [14]). Some other related problems such as channel information feedback can be further developed systematically.

TABLE I

Comparisons of several OEST codes

| $M$ | Codes | Constellation | Coding gain | Bit rate | Decoding |
|---|---|---|---|---|---|
| 6 | $\mathcal{D}_{1,6}/\mathcal{O}_6$ | 16QAM | 0.1 | 8/3 | 1 symbol |
| 6 | $\mathcal{D}_{2,3}/\mathcal{Q}_6$ | 16QAM | 0.089 | 3 | 2 symbols |
| 6 | $\mathcal{D}_{3,2}/\mathcal{S}_6$ | 8QAM/8Hex | 0.111/0.155 | 3 | 3 symbols |
| 6 | $\mathcal{D}_{1,6}/\mathcal{C}_6$ | 8QAM/8Hex | 0.111/0.155 | 3 | 6 symbols |

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