A General Combinatorial Sphere Decoder and its Application

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***Abstract*— The conventional problem of searching the shortest vector $\boldsymbol{z}$ of a N-dimensional lattice $\mathcal{L}(\boldsymbol{H})$ with generating matrix $\boldsymbol{H} \in \mathbb{R}^{N\times M}$ is considered in a more general setting. There are P generating matrices $\boldsymbol{H}_i \in \mathbb{R}^{N\times M} (i = 1, 2, ..., P)$ of the P lattices $\mathcal{L}(\boldsymbol{H}_i)$. For a (bounded) integer vector $\boldsymbol{b} \in \mathbb{Z}^M$, we obtain P lattice points $\boldsymbol{H}_i\boldsymbol{b}$. Let d_i be the Euclidean norm of $\boldsymbol{H}_i\boldsymbol{b}$. The problem of interest is how to search for a vector $\boldsymbol{b}$ so that the maximum of d_i is minimized. We propose a new sphere decoder called combinatorial sphere decoder (CSD) to solve this problem. One of the applications of the new CSD is presented in detail to show its effectiveness.**

***Index Terms*— Shortest vector search, closest point search, sphere decoder, partial transmit sequence.**

I. Introduction

CONSIDER a lattice $\mathcal{L}(\boldsymbol{H}) = \{\boldsymbol{H}\boldsymbol{b} \mid \boldsymbol{b} \in \mathbb{Z}^M\}$, where $\boldsymbol{H} \in \mathbb{R}^{K\times M}$ is the basis matrix and $K \geq M$, and an arbitrary point $\boldsymbol{x} \in \mathbb{R}^K$, the closest point search (CPS) in the Euclidean sense l_2 (CPS-l_2) problem [1] can be formulated as follows:

$$\hat{\boldsymbol{b}} = \arg\min_{\boldsymbol{b}} \{\|\boldsymbol{x} - \boldsymbol{H}\boldsymbol{b}\|_2\}. \tag{1}$$

When $\boldsymbol{x}$ coincides with the origin, the CPS problem becomes the shortest vector search (SVS) [2], [3] and is defined as

$$\hat{\boldsymbol{b}} = \arg\min_{\boldsymbol{b}} \{\|\boldsymbol{H}\boldsymbol{b}\|_2\}. \tag{2}$$

The CPS and SVS problems have many applications in communications (see [1], [4] and references therein).

We now investigate a min-max problem, which can be considered as a generalization of the SVS problem and can be easily extended to CPS problem. Assume that P matrices $\boldsymbol{H}_i \in \mathbb{R}^{N\times M} (i = 1, 2, \ldots, P)$ generate P lattices $\mathcal{L}(\boldsymbol{H}_i) = \{\boldsymbol{H}_i\boldsymbol{b} \mid \boldsymbol{b} \in \mathbb{Z}^M\}$. For a given vector $\boldsymbol{b}$, we obtain P lattice points $\boldsymbol{H}_i\boldsymbol{b}$. Let d_i be the Euclidean norm of $\boldsymbol{H}_i\boldsymbol{b}$. Our problem of interest is to find a vector $\boldsymbol{b}$ so that the maximum of d_i is minimized, or mathematically,

$$\hat{\boldsymbol{b}} = \arg\min_{\boldsymbol{b}} \max_{\boldsymbol{H}_i} \{\|\boldsymbol{H}_i\boldsymbol{b}\|_2\}. \tag{3}$$

If $P = 1$, the min-max search (3) reduces to the SVS problem (2). When $P = K$, $N = 1$, and $\boldsymbol{H}_i$ consists of only one row ith of a K-row matrix $\boldsymbol{H}$, the min-max search (3) becomes the CPS in terms of infinite norm l_∞ (CPS-l_∞) [2]. Since (2) can be solved efficiently by a sphere decoder [1], [3], a natural question is that whether (3) can be solved systematically by a variant of the conventional sphere decoders. More important, how does one efficiently implement this decoder?

We address these questions in this letter by presenting a generalization of the sphere decoder called combinatorial sphere decoder (CSD). The usefulness of CSD is illustrated by an example, in which the partial transmit sequence (PTS) is used to reduce the intercarrier interference (ICI) [5] for orthogonal frequency division multiplexing (OFDM) systems.

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II. Combinatorial Sphere Decoder

We first review the basic ideas of sphere decoder along with the enumerating method proposed by Fincke and Pohst [3]. The SVS problem (2) can be rewritten by

$$\hat{\boldsymbol{b}} = \arg\min_{\boldsymbol{b}} \boldsymbol{b}^\dagger\boldsymbol{H}^\dagger\boldsymbol{H}\boldsymbol{b}. \tag{4}$$

The necessary condition for $\boldsymbol{b}$ to minimize the cost metric (4) is the lattice point $\boldsymbol{H}\boldsymbol{b}$ lies in a hypersphere of a large enough covering radius r. Let $\mathcal{A}(\boldsymbol{b}) = \{\boldsymbol{b} \mid \|\boldsymbol{H}\boldsymbol{b}\|_2 \leq r\}$.

Assume that $\boldsymbol{H}^\dagger\boldsymbol{H}$ is full rank. This assumption is sometimes violated (e.g. when $P = K$, $N = 1$), however, we still can find a way to get around with further constraints. This point will be made clearer in Section III. Using Cholesky factorization, one can find an upper-triangle square matrix $\boldsymbol{R}$ such that $\boldsymbol{R}^\dagger\boldsymbol{R} = \boldsymbol{H}^\dagger\boldsymbol{H}$. Then

$$\begin{aligned} r^2 \geq \boldsymbol{b}^\dagger\boldsymbol{H}^\dagger\boldsymbol{H}\boldsymbol{b} = \boldsymbol{b}^\dagger\boldsymbol{R}^\dagger\boldsymbol{R}\boldsymbol{b} &= \sum_{i=1}^{M}\sum_{k=i}^{M}(R_{i,k}b_k)^2 \\ &= (R_{MM}b_M)^2 + (R_{M-1,M}b_M + R_{M-1,M-1}b_{M-1})^2 + \ldots \end{aligned}$$

The following necessary conditions can be drawn:

$$\begin{aligned} &r^2 \geq (R_{MM}b_M)^2 \\ &r^2 \geq (R_{M-1,M}b_M + R_{M-1,M-1}b_{M-1})^2 \\ &\ldots \end{aligned}$$

This set of inequalities can be used to find admissible b_k $(k = 1, 2, \ldots, M)$ as follows. We first look for possible values of b_M. Since R_{kk} obtained from Cholesky factorization is positive, the admissible integer values of b_M must satisfy

$$-\left\lceil \frac{r}{R_{MM}} \right\rceil \leq b_M \leq \left\lfloor \frac{r}{R_{MM}} \right\rfloor \tag{5}$$

where $\lceil\cdot\rceil$ and $\lfloor\cdot\rfloor$ denote ceiling and floor functions, respectively. Define $L_M = -\left\lceil \frac{r}{R_{MM}} \right\rceil$ and $U_M = \left\lfloor \frac{r}{R_{MM}} \right\rfloor$ as the lower and upper bounds of b_M. Hence $L_M \leq b_M \leq U_M$.

For each value of b_M, define $a_{M-1} = R_{M-1,M} b_M$, $r^2_{M-1} = r^2 - R_{M,M} b_M$, the admissible values of b_{M-1} are

$$\left\lceil \frac{-r_{M-1} - a_{M-1}}{R_{M-1,M-1}} \right\rceil \leq b_{M-1} \leq \left\lfloor \frac{r_{M-1} - a_{M-1}}{R_{M-1,M-1}} \right\rfloor. \quad (6)$$

or $L_{M-1} \leq b_{M-1} \leq U_{M-1}$.

Similarly, all admissible values of b_{M-2} to b_1 can be found. We obtain the set $\mathcal{A}(\boldsymbol{b})$ of candidate vectors $\boldsymbol{b}$. If the initial radius r is too small, after the first search the set $\mathcal{A}(\boldsymbol{b})$ may be empty. The radius r can be increased and the search is restarted until $\mathcal{A}(\boldsymbol{b})$ is nonempty. The solution of the SVS problem (2) can be found in the set $\mathcal{A}(\boldsymbol{b})$ faster than the exhaustive search over all possible vectors $\boldsymbol{b}$.

The above decoder to solve the SVS problem can be called Fincke-Pohst sphere decoder (FP-SD). We now apply the FP-SD to solve the min-max problem (3). Since there are P lattices, one can select an initial radius $r > 0$, and run P FP-SD's in parallel. Each FP-SD for lattice $\mathcal{L}(\boldsymbol{H}_i)$ will generate a candidate set $\mathcal{A}_i(\boldsymbol{b})$. The solution vector $\hat{\boldsymbol{b}}$ of the min-max problem (3) should be in the set $\overline{\mathcal{A}}(\boldsymbol{b}) = \cap_{i=1}^{P} \mathcal{A}_i(\boldsymbol{b})$. If the initial radius r is too small, $\overline{\mathcal{A}}(\boldsymbol{b})$ may be empty, one needs to increase the radius and run the search again until the set $\overline{\mathcal{A}}(\boldsymbol{b})$ is nonempty.

There are two limitations of the above searching procedure. First, if the number of lattices P is large, the number of FP-SD to be run in parallel will be large; the algorithm may be too complex to implement. Second, the complexity of the algorithm depends largely on the initial radius.

We thus propose the combinatorial sphere decoder to solve these two problems. The main idea of CSD is to combine P-parallel FP-SD's into a single sphere decoder to reduce the complexity of P FP-SD's greatly. Then the lattice-point enumeration method proposed by Schnorr and Euchner [6] is applied to reduce the dependence of the complexity on the initial radius [4].

A large (possibly infinite) initial radius r is used for the CSD to search for the first candidate sequence. At each step of searching admissible values of b_m $(m = 1, 2, \ldots, M)$, the *compound* lower and upper bounds of b_m are calculated by

$$L_m = \max(L_m^1, L_m^2, \ldots, L_m^P),$$
$$U_m = \min(U_m^1, U_m^2, \ldots, U_m^P)$$

where L_m^i and U_m^i $(k = 1, 2, \ldots, P)$ are the lower and upper bounds of symbol b_m^i derived from the corresponding equation of lattice $\boldsymbol{H}_i$.

Whenever a candidate vector $\boldsymbol{b}$ is found, the sphere radius of CSD is updated so that

$$r = \max_{i=1,\ldots,P} (\|\boldsymbol{H}_i \boldsymbol{b}\|_2). \quad (7)$$

Since the candidate vector lies inside the initial sphere, the new updated radius is smaller than the initial one, thus the maximum Euclidean norm of all the lattice vectors reduces. Then the CSD is run again to search for a new candidate inside the new radius in a zig-zag fashion of Schnorr-Euchner enumeration method [6]. The search is ended when no new candidate vector is found. The solution of the min-max search (3) is the last candidate vector found.

Remarks on the complexity: The expected complexity of SD for CPS problem is shown to be exponential in the number of variables for any implementation of sphere decoder, including Schnorr-Euchner enumeration [7], [8]. Thus the complexity of the proposed CSD is also exponential. However, the complexity reduction of CSD compared with the exhaustive search is significant. We will examine a particular application of CSD in the following to show its effectiveness.

III. Optimal Partial Transmit Sequence Search for ICI Reduction in OFDM Systems

In OFDM systems, ICI caused by frequency offset (FO) [9] can be mitigated by using the partial transmit sequence (PTS) [5]. Exhaustive search is used in [5] to find the best PTS weights, which has exponential complexity in the number of the PTS weights. We therefore will apply the CSD to reduce the complexity of the PTS search.

To simplify the analysis, we consider the transmission of OFDM signal with K subscarrier over AWGN channels and binary PTS weights $(-1, 1)$. Define the normalized FO by $\varepsilon = \delta f / \Delta f$, where Δf is the subscarrier spacing, δf is the carrier FO. The baseband received signal y_k at subcarrier kth $(k = 1, 2, \ldots, K)$ can be given by [9]

$$y_k = S_0 c_k + I_k + n_k \quad (8)$$

where c_k is the data symbol sent over kth subcarrier, n_k is the additive white Gaussian noise (AWGN) at the receiver, $I_k = \sum_{i=0, i\neq k}^{K-1} S_{ki} c_i$ is the ICI from the other subcarriers on subcarrier k,

$$S_{ki} = \frac{\sin[\pi(i - k + \varepsilon)]}{K \sin\left[\frac{\pi}{K}(i - k + \varepsilon)\right]} \exp\left[j\pi\left(1 - \frac{1}{K}\right)(i - k + \varepsilon)\right], \quad (9)$$

and $S_0 = S_{kk}$ is independent of the subcarrier index k.

The interference-to-carrier power (ICR) ratio of subcarrier kth is given by $\frac{|I_k|^2}{|S_0 c_k|^2}$. Furthermore, the peak ICR (PICR) is defined as [5]

$$\text{PICR} = \max_{1 \leq k \leq K} \left\{ \frac{|I_k|^2}{|S_0 c_k|^2} \right\}. \quad (10)$$

In AWGN channels, at high signal-to-noise ratio (SNR), ICI power dominates AWGN noise. The data symbol with higher ICR will likely experience higher error rate. Therefore, reducing PICR will improve the error rate performance. To do this, in [5], the data vector $\boldsymbol{c} \begin{bmatrix} c_1 & c_2 & \ldots & c_K \end{bmatrix}^T$ is partitioned into M equal-sized disjoint blocks $\boldsymbol{c} = [\boldsymbol{c_1}\, \boldsymbol{c_2} \ldots \boldsymbol{c_M}]^T$, each block consists of $K_0 = K/M$ contiguous subcarriers. The symbols of block $\boldsymbol{c}_m$ are multiplied by weight b_m, where $|b_m| = 1$, for $m = 1, 2, \ldots, M$. Let $\boldsymbol{b} = [b_1\, b_2 \ldots b_M]$. Application of the PTS weights to the data vector $\boldsymbol{c}$ results in vector $\hat{\boldsymbol{c}}$. We can rewrite (8) to include the effect of PTS weights compactly as

$$\boldsymbol{Y} = \boldsymbol{S}\hat{\boldsymbol{c}} + \boldsymbol{N} = S_0 \boldsymbol{I}_K \hat{\boldsymbol{c}} + \hat{\boldsymbol{S}}\hat{\boldsymbol{c}} + \boldsymbol{N} \quad (11)$$

where $\boldsymbol{S} = [S_{i,k}]$, $\boldsymbol{I}_K$ is the K-by-K identity matrix and $\hat{\boldsymbol{S}} = \boldsymbol{S} - S_0 \boldsymbol{I}_K$.

We define a matrix $\boldsymbol{T}$ such that: $T_{ki} = \hat{S}_{ki}c_i$, $(i, k = 1, 2, ..., K)$ and another matrix $\boldsymbol{Q}$ by applying the following rule: $Q_{km} = \sum_{n=(m-1)K_0+1}^{mK_0} T_{kn}$, where $k = 1, 2, ..., K, m = 1, 2, ..., M$. It can be shown that the ICI term I_k becomes $I_k = \boldsymbol{q}_k\boldsymbol{b}$, where $\boldsymbol{q}_k$ is the k-th row of $\boldsymbol{Q}$. The PICR in (10) becomes:

$$\mathrm{PICR} = \max_{1\le k\le K}\left\{\frac{|\boldsymbol{q}_k\boldsymbol{b}|^2}{|S_0 c_k|^2}\right\} = \max_{1\le k\le K}\left\{\boldsymbol{b}^\dagger\hat{\boldsymbol{q}}_k^\dagger\hat{\boldsymbol{q}}_k\boldsymbol{b}\right\} \quad (12)$$

where $\hat{\boldsymbol{q}}_k = \boldsymbol{q}_k/S_0c_k$. Let $\hat{\boldsymbol{Q}} = [\hat{\boldsymbol{q}}_1^\mathsf{T}\ \hat{\boldsymbol{q}}_2^\mathsf{T}\ \ldots\ \hat{\boldsymbol{q}}_K^\mathsf{T}]^\mathsf{T}$.

It is assumed that worst-case FO ε is known at the transmitter [5]. The problem of finding optimal PTS weights to minimize PICR can be mathematically given below:

$$\hat{\boldsymbol{b}} = \arg\min_{\boldsymbol{b}}\max_{1\le k\le K}\left\{\boldsymbol{b}^\dagger\hat{\boldsymbol{q}}_k^\dagger\hat{\boldsymbol{q}}_k\boldsymbol{b}\right\}. \quad (13)$$

The PTS method is initially proposed to reduce the peak of ICR [5]. However, if the total ICI power, denoted by SICR,

$$\mathrm{SICR} = \sum_{k=1}^{K}\|\hat{\boldsymbol{q}}_k\boldsymbol{b}\|_2^2 = \|\hat{\boldsymbol{Q}}\boldsymbol{b}\|_2^2 = \boldsymbol{b}^\dagger\hat{\boldsymbol{Q}}^\dagger\hat{\boldsymbol{Q}}\boldsymbol{b} \quad (14)$$

is minimized, the overall system BER can also be reduced. We formulate a new problem of finding a PTS sequence subject to minimizing the total ICI power as follows:

$$\hat{\boldsymbol{b}} = \arg\min_{\boldsymbol{b}}\|\hat{\boldsymbol{Q}}\boldsymbol{b}\|_2. \quad (15)$$

Generalization of the minimization problems subject to peak ICI power (13) and sum of ICI power (15) can be stated as follows. *Among K subcarriers, picking $\bar{K}$ subcarriers such that their total ICI noise power $\mathrm{SICR}_{\bar{K}}$ is largest. Find the optimal PTS sequence so that $\mathrm{SICR}_{\bar{K}}$ is minimized.* It is well matched with the min-max search framework (3). When $\bar{K} = 1$, minimizing PICR (13) becomes the min-max problem (3). Additionally, minimization of total ICI noise power (15) is equivalent to the SVS problem (2).

Using the CSD to solve (13) and (15) requires some modifications. First, our CSD works with real basis matrices. We thus need to convert the the complex $\hat{\boldsymbol{q}}_i$ of (13) and $\hat{\boldsymbol{Q}}$ of (15) into real matrices by transformations $\bar{\boldsymbol{q}}_i = \left[\Re(\boldsymbol{q}_i)^\mathsf{T} \quad \Im(\boldsymbol{q}_i)^\mathsf{T}\right]^\mathsf{T}$ and $\bar{\boldsymbol{Q}} = \left[\Re(\boldsymbol{Q})^\mathsf{T} \quad \Im(\boldsymbol{Q})^\mathsf{T}\right]^\mathsf{T}$. Second, $\bar{\boldsymbol{q}}_i^\dagger\bar{\boldsymbol{q}}_i$ is of size $M \times M$ with rank $2 \le M$, Cholesky factorization does not work. We can solve the rank deficiency problem by using a trick proposed in [10]. The main idea in [10] is that if b_k has constant modulus (which is true for the PTS weights b_k [5]), then $\mu\boldsymbol{b}^\dagger\boldsymbol{b} = \mu M$, where μ is a constant. Thus $|\bar{\boldsymbol{q}}_i\boldsymbol{b}|^2 + \mu M = \boldsymbol{b}^\dagger(\bar{\boldsymbol{q}}_i^\dagger\bar{\boldsymbol{q}}_i + \mu\boldsymbol{I}_M)\boldsymbol{b}$. The new positive-definite matrix $\boldsymbol{X}_i = \bar{\boldsymbol{q}}_i^\dagger\bar{\boldsymbol{q}}_i + \mu\boldsymbol{I}_M$ can be Cholesky factorized. $\boldsymbol{X}_i$ and $\bar{\boldsymbol{Q}}$ can be used for our CSD.

We present the simulation results using the CSD to search for binary PTS weights with $M = 8, 16$. For binary weights, the value of b_M can be set to 1 without loss of generality. The OFDM system with 64 subcarriers and 4-QAM modulation. For brevity, the searches (13) and (15) are called MinMax and MinSum searches accordingly. We implement the CSD with Schnorr-Euchner enumeration based on the improved algorithm in [4, Algorithms II]. The initial radius is equal to the PICR or SICR before optimization so that at least one candidate vector $\boldsymbol{b} = [1\ 1 \ldots 1]^\mathsf{T}$ exists.

The average flops of the proposed algorithms for a normalized FO of 10% are reported as a measure of complexity. For $M = 8$, the average flops counted are $270.5 \times 10^3, 150.1 \times 10^3, 12.9 \times 10^3$ for exhaustive search, MinMax and MinSum, respectively. For $M = 16$, these numbers are accordingly $136.35 \times 10^6, 12.1 \times 10^6$ and 0.417×10^6. These results show the efficiency of the CSD in reducing the complexity of the PTS search.

IV. Conclusion

We have presented the combinatorial sphere decoder to solve a min-max search problem, a generalization the shortest vector search problem [3]. The settings of the presented min-max problem can be easily extended for the closest point search problem in the Euclidean sense. An example of utilizing the CSD in searching optimal PTS weights to reduce the ICI in OFDM has been presented. Recently, the problem CPS-l_∞ has been applied to find the optimal PTS weights to reduce the peak-to-average-power ratio in OFDM systems [11], [12]. The proposed CSD can well be applied for this problem, that is similar to the PICR minimization in our example. There may be more problems that can be solved by the proposed CSD. This fact motivates further studies on the low complexity implementations of CSD. During the review process, one of the reviewers raised question "why do we study the min-max problem (3), but not a min-min problem?" Thus the min-min problem could be a topic for another study.

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