A General Method to Decode ABBA Quasi-Orthogonal Space-Time Block Codes

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***Abstract*— ABBA codes, a class of quasi-orthogonal space-time block codes (STBC) proposed by Tirkkonen *et al.*, allow low complexity pair-wise complex-symbol decoding. A refined version of ABBA codes with special signal mapping leads to pair-wise real-symbol decoding, i.e. with minimum decoding complexity (MDC) achievable by non-orthogonal STBC. Despite these advantages, a general simple closed-form method to decode ABBA and MDC-ABBA codes for an arbitrary number of transmit/receive antennas is not available. We thus derive an equivalent channel representation to address this issue. The maximum mutual information of ABBA/MDC-ABBA codes is also derived. We found that MDC-ABBA codes perform better than ABBA codes when using two 8QAM constellations.**

***Index Terms*— Quasi-orthogonal space-time block codes.**

I. Introduction

ABBA codes [1], a class of QSTBC, have been proposed to increase the rate of orthogonal space-time block codes (OSTBC) [2]. ABBA QSTBC also have low complexity pair-wise complex-symbol decoding and performs better than OSTBC [3]. ABBA codes have been widely studied for coherent and non-coherent transmissions, beamforming, and others. Recently, Yuen *et al.* (see [4] and references therein) have shown that the ABBA codes enable pair-wise real-symbol (PWRS) decoding; they call such codes minimum decoding complexity (MDC) codes. Thus, not only is their code rate higher than that of OSTBC, but also their decoding complexity is equal to that of OSTBC. In the following, we reserve the term "ABBA" for the QSTBC proposed by Tirkkonen *et al.* [1] with pair-wise complex-symbol decoding [3] and the term "MDC-ABBA" for the ABBA codes with PWRS decoding [4].

Despite extensive research on the ABBA QSTBC, a general decoding method for ABBA codes for arbitrary numbers of transmit (Tx) and receive (Rx) antennas is not available. One reason for this gap is that the equivalent channel for ABBA codes is not known in the most general case. Several decoders for ABBA codes have been proposed, but only for some specific cases, for example with 4 or 6 antennas in [5].

In this letter, we propose a general and closed-form method to decode ABBA/MDC-ABBA codes. We show how the ABBA space-time (ST) channel can be decoupled into parallel independent channels, each of which carries a pair of data symbols. The maximum mutual information (MMI) of QSTBC codes was calculated in [6] for 4 Tx antennas only. Thus, we derive a general MMI expression for ABBA/MDC-ABBA codes. Although the optimal coding gain of ABBA codes is higher than that of MDC-ABBA codes [4], we found that MDC-ABBA codes perform better than ABBA codes for two well-known 8QAM constellations.

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II. Decoders for ABBA and MDC-ABBA Codes

We consider data transmission over a quasi-static Rayleigh flat fading channel. The transmitter and receiver are equipped with M Tx and N Rx antennas. The receiver, but not the transmitter, completely knows the channel gains.

From matrix representation theory, the mapping of a block of K data symbols $(s_1, s_2, \cdots, s_K)$ into a $T \times M$ code matrix of a STBC can be represented in a general form [7] as follows:

$$\mathcal{X}_M = \sum_{k=1}^{K} (s_k A_k + s_k^* B_k) \tag{1}$$

where A_k and B_k, $(k = 1, 2, \cdots, K)$ are $T \times M$ constant basis matrices, superscript * denotes conjugate [1]. The average energy of code matrices $X \in \mathcal{X}_M$ is constrained such that $\mathbb{E}[\|X\|_{\mathsf{F}}^2] = T$. The code rate $\mathsf{R}_{\mathcal{X}_M}$ of a STBC $\mathcal{X}_M$, in symbols per channel use (pcu), is defined by $\mathsf{R}_{\mathcal{X}_M} = K/T$.

We now review the main properties of OSTBC $\mathcal{O}_M$ to be used later. The basis matrices of OSTBC satisfy [2]:

$$A_i^\dagger A_i + B_i^\dagger B_i = \boldsymbol{I}_M, \qquad i = 1, 2, \cdots, K \tag{2a}$$

$$A_i^\dagger A_j + B_j^\dagger B_i = \mathbf{0}_M, \qquad 1 \le i < j \le K \tag{2b}$$

$$A_i^\dagger B_j + A_j^\dagger B_i = \mathbf{0}_M, \qquad i, j = 1, 2, \cdots, K. \tag{2c}$$

Let the data symbols are drawn from a constellation with unit average power. To guarantee the average power constraint, the OSTBC matrices are multiplied by a constant $\kappa = \frac{1}{M \mathsf{R}_{\mathcal{O}_M}}$.

A. General Decoder of ABBA Codes

Let A_k and B_k $(k = 1, 2, \cdots, K)$ be the $t \times m$ basis matrices of an OSTBC $\mathcal{O}_m$. Two blocks of data, each of K symbols, are mapped into two code matrices $\mathcal{A}$ and $\mathcal{B}$ of $\mathcal{O}_m$ as $\mathcal{A} = \sum_{k=1}^{K} (s_k A_k + s_k^* B_k), \mathcal{B} = \sum_{k=1}^{K} (s_{k+K} A_k + s_{k+K}^* B_k)$.

The ABBA code matrices for $M = 2m$ Tx antennas are constructed from $\mathcal{O}_m$ as $\mathcal{Q}_M = \begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{B} & \mathcal{A} \end{bmatrix}$, or

$$\mathcal{Q}_M = \sum_{k=1}^{K} (\mathcal{C}_k \otimes A_k + \mathcal{C}_k^\dagger \otimes B_k) \tag{3}$$

[1]From now on, superscripts $^\mathsf{T}$ and † denote matrix transpose and transpose conjugate. The $n \times n$ identity and all-zero matrices are denoted by $\boldsymbol{I}_n$ and $\mathbf{0}_n$, respectively. $\|X\|_{\mathsf{F}}$ denotes Frobenius norm of matrix X and $\otimes$ denotes Kronecker product. $\mathbb{E}[\cdot]$ denotes average. A mean-m and variance-σ^2 circularly complex Gaussian random variable is written by $\mathcal{CN}(m, \sigma^2)$.

where $\mathcal{C}_k = \begin{bmatrix} s_k & s_{k+K} \\ s_{k+K} & s_k \end{bmatrix}$. Let $\Pi = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, then $\Pi = \Pi^{-1}, \Pi^2 = \boldsymbol{I}_2$, and

$$\mathcal{C}_k = (s_k \Pi^0 + s_{k+K}\Pi). \tag{4}$$

For the sake of simplicity, we consider one Rx antenna in the following. Generalization for $N > 1$ is straightforward.

Let $\boldsymbol{h} = [h_1\, h_2\, \cdots\, h_M]^\mathsf{T}$ denote the channel vector with $h_i \sim \mathcal{CN}(0,1)$. Let $Q \in \mathcal{Q}_M$ be a transmitted code matrix, the Rx signal vector is $\boldsymbol{y} = \sqrt{\rho\kappa/2}\, Q\boldsymbol{h} + \boldsymbol{n}$, where $\boldsymbol{n}$ is noise vector with independently, identically distributed (i.i.d.) entries $\sim \mathcal{CN}(0,1)$; ρ is the average Rx signal-to-noise ratio (SNR).

From (3) and (4), we have

$$\boldsymbol{y} = \sqrt{\rho\kappa/2} \sum_{k=1}^{K} \sum_{i=1}^{2} \Big[\left(\Pi^{i-1} \otimes A_k\right) \boldsymbol{h} s_{k+(i-1)K} + \left(\Pi^{1-i} \otimes B_k\right) \boldsymbol{h} s^*_{k+(i-1)K} \Big] + \boldsymbol{n}. \tag{5}$$

Let $\boldsymbol{e}_{ki} = \left(\Pi^{i-1} \otimes A_k\right)\boldsymbol{h}, E_k = [\boldsymbol{e}_{k1} \quad \boldsymbol{e}_{k2}], \boldsymbol{f}_{ki} = \left(\Pi^{1-i} \otimes B_k\right)\boldsymbol{h}, F_k = [\boldsymbol{f}_{k1} \quad \boldsymbol{f}_{k2}]$, and $\boldsymbol{s}_k = [s_k \quad s_{k+K}]^\mathsf{T}$, (5) can be rewritten as

$$\boldsymbol{y} = \sqrt{\rho\kappa/2}\, [E_1 \quad F_1 \quad E_2 \quad F_2 \quad \cdots \quad E_K \quad F_K] \times \left[\boldsymbol{s}_1^\mathsf{T} \quad \boldsymbol{s}_1^\dagger \quad \boldsymbol{s}_2^\mathsf{T} \quad \boldsymbol{s}_2^\dagger \quad \cdots \quad \boldsymbol{s}_K^\mathsf{T} \quad \boldsymbol{s}_K^\dagger\right]^\mathsf{T} + \boldsymbol{n}. \tag{6}$$

Using a trick in [8], (6) is written equivalently as

$$\begin{bmatrix} \boldsymbol{y} \\ \boldsymbol{y}^* \end{bmatrix} = \sqrt{\rho\kappa/2}\, \underbrace{\begin{bmatrix} E_1 & F_1 & \cdots & E_K & F_K \\ F_1^* & E_1^* & \cdots & F_K^* & E_K^* \end{bmatrix}}_{W} \times \left[\boldsymbol{s}_1^\mathsf{T} \quad \boldsymbol{s}_1^\dagger \cdots \quad \boldsymbol{s}_K^\mathsf{T} \quad \boldsymbol{s}_K^\dagger\right]^\mathsf{T} + \begin{bmatrix} \boldsymbol{n} \\ \boldsymbol{n}^* \end{bmatrix}. \tag{7}$$

It can be shown that the columns of matrix W are orthogonal.

Proof: We will show that the following equations hold:

$$\begin{bmatrix} E_k \\ F_k^* \end{bmatrix}^\dagger \begin{bmatrix} E_l \\ F_l^* \end{bmatrix} = E_k^\dagger E_l + F_k^\mathsf{T} F_l^* = \mathbf{0}_2 \quad \text{for } k \neq l, \tag{8a}$$

$$\begin{bmatrix} E_k \\ F_k^* \end{bmatrix}^\dagger \begin{bmatrix} F_l \\ E_l^* \end{bmatrix} = E_k^\dagger F_l + F_k^\mathsf{T} E_l^* = \mathbf{0}_2 \qquad \forall k, l. \tag{8b}$$

We just provide the proof for (8a); (8b) can be shown similarly. Let $Z_{kl} = (E_k^\dagger E_l + F_k^\mathsf{T} F_l^*)$, its element can be calculated as

$$\begin{aligned} [Z_{kl}]_{ij} &= \boldsymbol{e}_{ki}^\dagger \boldsymbol{e}_{lj} + \boldsymbol{f}_{ki}^\mathsf{T} \boldsymbol{f}_{lj}^* \\ &= \boldsymbol{h}^\dagger [(\Pi^{j-i}) \otimes (A_k^\dagger A_l)]\boldsymbol{h} + \boldsymbol{h}^\mathsf{T} [(\Pi^{i-j}) \otimes (B_k^\mathsf{T} B_l^*)]\boldsymbol{h}^* \\ &= \boldsymbol{h}^\dagger [(\Pi^{j-i}) \otimes (A_k^\dagger A_l + B_k^\dagger B_l)]\boldsymbol{h} \\ &= \begin{cases} 0, & k \neq l; \\ \boldsymbol{h}^\dagger (\Pi^{j-i} \otimes \boldsymbol{I}_m)\boldsymbol{h}, & k = l. \end{cases} \end{aligned} \tag{9}$$

Thus, $Z_{kl} = \mathbf{0}_2$ if $k \neq l$. □

Since for $k = l$, the matrices $Z_{kk} = Z\, \forall k$, where the entries of Z are $z_{ij} = \boldsymbol{h}^\dagger(\Pi^{j-i} \otimes \boldsymbol{I}_m)\boldsymbol{h}$. In particular, $z_{1,1} = z_{2,2} = \|\boldsymbol{h}\|_\mathsf{F}^2, z_{1,2} = z_{2,1} = \sum_{k=1}^{K}(h_k h_{k+K}^* + h_k^* h_{k+K})$. Therefore, Z is also a circulant real matrix and can be represented as

$$Z = \sum_{i=1}^{m} H_i^\dagger H_i \tag{10}$$

where $H_i = \begin{bmatrix} h_i & h_{i+m} \\ h_{i+m} & h_i \end{bmatrix}$. To separate the transmitted vector $\boldsymbol{s}_k$ $(k = 1, 2, \ldots K)$ at the receiver, we can multiply the two sides of (7) with $\left[E_k^\dagger \quad F_k^\mathsf{T}\right]$ to get

$$E_k^\dagger \boldsymbol{y} + F_k^\mathsf{T} \boldsymbol{y}^* = \sqrt{\rho\kappa/2}\, Z \boldsymbol{s}_k + \underbrace{(E_k^\dagger \boldsymbol{n} + F_k^\mathsf{T} \boldsymbol{n}^*)}_{\bar{\boldsymbol{n}}_k}. \tag{11}$$

The vector $\left[E_k^\dagger \quad F_k^\mathsf{T}\right]$ plays the role of the spatial signature of data vector $\boldsymbol{s}_k$. However, the noise $\bar{\boldsymbol{n}}_k$ is color with covariance matrix $V = \mathbb{E}[\bar{\boldsymbol{n}}_k \bar{\boldsymbol{n}}_k^\dagger] = Z \neq \boldsymbol{I}_M$. This color noise can be whitened by multiplying the two sides of (11) with a whitening matrix $Z^{-\frac{1}{2}}$. The received signal with whitened noise is

$$\underbrace{Z^{-\frac{1}{2}}(E_k^\dagger \boldsymbol{y} + F_k^\mathsf{T} \boldsymbol{y}^*)}_{\hat{\boldsymbol{y}}_k} = \sqrt{\rho\kappa/2}\, \underbrace{Z^{\frac{1}{2}}}_{\hat{H}} \boldsymbol{s}_k + \underbrace{Z^{-\frac{1}{2}} \bar{\boldsymbol{n}}_k}_{\hat{\boldsymbol{n}}_k} \tag{12}$$

Thus, (12) *is the general and compact detection equation for ABBA codes*. Since this decoder decouples the ABBA ST channels into K parallel channels, one needs to apply K times (12) to decode ABBA codes for any $M = 2m$ Tx antennas.

To achieve full diversity, K data symbols s_{k+K} $(k = 1, 2, \ldots, K)$ must be rotated by an angle α [3]. So *the general detection equation of ABBA codes with complex symbol rotation* is:

$$\hat{\boldsymbol{y}}_k = \sqrt{\rho\kappa/2}\hat{H}\, \mathrm{diag}(1, e^{\mathrm{j}\,\alpha})\boldsymbol{s}_k + \hat{\boldsymbol{n}}_k, \qquad \mathrm{j}^2 = -1. \tag{13}$$

B. General Decoder of MDC-ABBA Codes

Since Z is a 2×2 normal circulant matrix, its two eigenvalues λ_1 and λ_2 are non-negative; Z can be diagonalized by a 2×2 (real) Fourier transform matrix $F_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ as $Z = F_2^\dagger\, \mathrm{diag}(\lambda_1, \quad \lambda_2)F_2$. If $\hat{H}^2 = Z$, then $\hat{H} = F_2^\dagger\, \mathrm{diag}(\sqrt{\lambda_1}, \sqrt{\lambda_2})F_2$. Thus $\hat{H}$ is real.

We can rewrite (12) by decoupling the real and imaginary parts of the two sides of (12) as

$$\begin{bmatrix} \Re(\hat{\boldsymbol{y}}_k) \\ \Im(\hat{\boldsymbol{y}}_k) \end{bmatrix} = \sqrt{\rho\kappa/2}\, \underbrace{\begin{bmatrix} \hat{H} & \mathbf{0}_2 \\ \mathbf{0}_2 & \hat{H} \end{bmatrix}}_{\widetilde{H}} \begin{bmatrix} \Re(\hat{\boldsymbol{s}}_k) \\ \Im(\hat{\boldsymbol{s}}_k) \end{bmatrix} + \begin{bmatrix} \Re(\hat{\boldsymbol{n}}_k) \\ \Im(\hat{\boldsymbol{n}}_k) \end{bmatrix}. \tag{14}$$

The real and imaginary parts of the data vectors (of two complex symbols) can be separately detected as below:

$$\Re(\hat{\boldsymbol{y}}_k) = \sqrt{\rho\kappa/2}\hat{H}\Re(\hat{\boldsymbol{s}}_k) + \Re(\hat{\boldsymbol{n}}_k), \tag{15a}$$

$$\Im(\hat{\boldsymbol{y}}_k) = \sqrt{\rho\kappa/2}\hat{H}\Im(\hat{\boldsymbol{s}}_k) + \Im(\hat{\boldsymbol{n}}_k). \tag{15b}$$

There are only two real symbols in (15) to be jointly detected, thus (15a) *and* (15b) *are the general detection equations for the MDC-ABBA codes*.

To achieve full diversity, a real linear signal transformation R is required for the real data vector $[\Re(\hat{\boldsymbol{s}}_k), \Im(\hat{\boldsymbol{s}}_k)]^\mathsf{T}$. Moreover, R must be designed so that the equivalent channel matrix (either in the form $\widetilde{H}R$ or $R^\dagger \widetilde{H} R$) is still block diagonal. For example, transformation R in [4] meets this requirement.

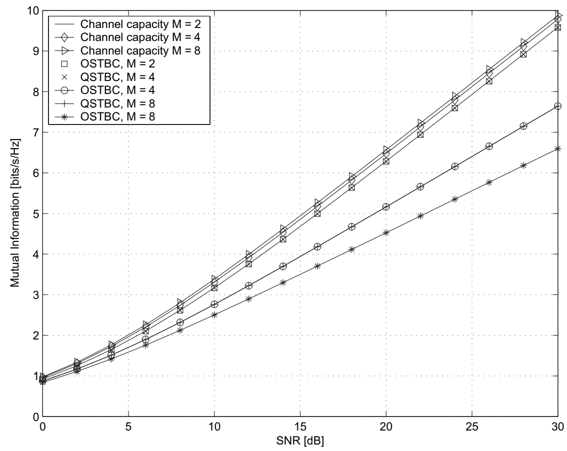

Fig. 1. MMI of ABBA/MDC-ABBA codes and OSTBC over MISO channels.

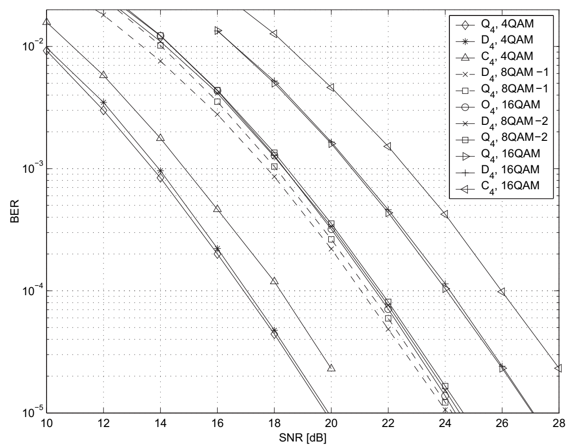

Fig. 2. Performance of MDC-ABBA codes for 4 Tx antennas compared with ABBA codes and OSTBC.

III. Maximum Mutual Information

The MMI of ABBA (and also MDC-ABBA) codes can be calculated using the equivalent channel in (12) [7]. Omitting the details for brevity, we can show that

$$\mathsf{C}_{\mathcal{Q}_{2m}} = \mathsf{R}_{\mathcal{O}_m}\,\mathbb{E}\left\{\log_2 \det\left[1 + \frac{\rho}{m\mathsf{R}_{\mathcal{O}_m}}\|\bar{H}\|_{\mathsf{F}}^2\right]\right\} = \mathsf{C}_{\mathcal{O}_m} \tag{16}$$

where the entries of $\bar{H} \in \mathbb{C}^{M\times N}$ are $\mathcal{CN}(m,\sigma^2)$, $\mathsf{C}_{\mathcal{O},m}$ is the MMI of the underlying OSTBC $\mathcal{O}_m$ [7], which is used to construct ABBA codes. Therefore,

1) The MMI of ABBA/MDC-ABBA codes for $M = 2m$ Tx antennas equals to that of OSTBC for m Tx antennas; i.e., by doubling number of Tx antennas and replacing OSBTC by ABBA/MDC-ABBA codes, one can get higher diversity benefit but not the capacity benefit.
2) Compared with OSTBC, MDC-ABBA codes attain larger portion of channel capacity.

The MMI of ABBA/MDC-ABBA codes and OSTBC (maximal rates [2]), and channel capacity of multiple-input single-output (MISO) illustrated in Fig. 1 (for $M = 2,4,8$ and $N = 1$) agree with the above analysis.

IV. Simulation Results

Performances of the ABBA code $\mathcal{Q}_4$ [3] and the MDC-ABBA code $\mathcal{D}_4$ (with signal rotation in [4]) using the proposed decoders are presented in Fig. 2 for a system with $M = 4, N = 1$. Performances of OSTBC $\mathcal{O}_4$ [2] and rate-one linear TAST code $\mathcal{C}_4$ [9] are also presented for comparison.

With 4QAM and 16QAM (2 and 4 bits pcu, respectively), the performance of $\mathcal{D}_4$ closely approaches to that of $\mathcal{Q}_4$. $\mathcal{D}_4$ also performs much better than $\mathcal{C}_4$ (with higher decoding complexity, joint decoding of 4 complex symbols).

Using an 8QAM-1 constellation $\{\pm 1, \pm \mathsf{j}, \pm 1 \pm \mathsf{j}\}$ (3 bits pcu), surprisingly, $\mathcal{D}_4$ outperforms both $\mathcal{Q}_4$ and $\mathcal{O}_4$ (with 16QAM), about 0.25 and 0.5 dB, respectively. With another 8QAM-2 $\{\pm 3 \pm \mathsf{j}, \pm 1 \pm \mathsf{j}\}$, $\mathcal{D}_4$ also performs better than $\mathcal{Q}_4$ but slightly worse than $\mathcal{O}_4$.

V. Conclusion

We have presented a general and simple method to decode ABBA and MDC-ABBA QSTBC by deriving the equivalent channel representation. It has also been used to derive maximum mutual information of these codes. Simulations show that MDC-ABBA codes closely approach the performance of ABBA code with 4- and 16QAM, and even perform better with two 8QAM constellations. Compared with OSTBC, the MDC-ABBA codes attain higher portion of channel capacity and perform better. Therefore, MDC-ABBA codes may be a better choice than OSTBC when there are more than 2 Tx antennas. The results of our paper can be developed in different aspects. For example, when a certain form of channel state information is available at the transmitter, the equivalent channel can be used to determine the optimal channel parameters to be fed back to improve the performance of QSTBC.

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