Closed-Form BER Analysis for Antenna Selection Using Orthogonal Space-Time Block Codes

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***Abstract*— Despite significant research efforts, closed-form bit error rate (BER) expressions for multiple-input multiple-output (MIMO) systems employing transmit antenna selection and orthogonal space-time block codes (OSTBCs) are not available. We thus derive exact closed-form expressions for the BER of Gray-coded M-ary one and two-dimensional amplitude modulations when an OSTBC is employed and N transmit antennas out of total L_t antennas are selected for transmission. We also derive tight closed-form approximate BER for M-PSK constellations. Our BER expressions are valid for a frequency-flat Rayleigh fading MIMO channel and can be evaluated without numerical integration methods.**

***Index Terms*— Bit error rate (BER), antenna selection, diversity, multiple-input multiple-output (MIMO), space-time code.**

I. Introduction

MULTIPLE antennas for transmitting and/or receiving data effectively mitigates fading. Obtaining the benefits of multiple transmit antennas requires the use of special space-time signaling schemes such as orthogonal space-time block codes (OSTBCs), a class of easily decoded space-time codes that achieve full diversity order [1], [2]. OSTBCs exist only for certain number of transmit antennas, and this limits their potential application. Decoding of OSTBCs is equivalent to decoding a number of independent single-input single-output data streams.

Transmit antenna selection (TAS), where OSTBC signal matrices are transmitted over a selected subset of transmit antennas, is a practical technique for the realization of full diversity [3]. Although receive antenna selection is a well researched topic in which various channel/correlation models have been comprehensively treated (see [4]–[7] among many others), analogous comprehensive results are limited for TAS; e.g. a general, exact closed-form bit error rate (BER) analysis of TAS is not available to date. Although the symbol error rate (SER) of TAS is derived in [8], the formulas require numerical methods. In [9], the SER of single TAS and receive generalized selection combining (GSC) is derived. Exact BER for selecting only **two** transmit antennas with BPSK signals using the Alamouti code is derived in [10], [11].

In this paper, we provide general, closed-form BER expressions for M-ary Pulse Amplitude Modulation (PAM) and M-ary Quadrature Amplitude Modulation (QAM) constellations, and an approximate BER expression for M-ary Phase Shift Keying (PSK) with arbitrary $N \geq 2$ TAS employing OSTBCs. The MGF of N largest instantaneous signal-to-noise ratios (SNRs) for GSC in Nakagami fading is derived in [12]. Since TAS involves selecting the N columns of the channel matrix with the largest Frobenius norms, the results of [12] can be utilized to the problem in hand. Our BER approximations correctly reveal the full diversity order of the system[1].

II. System Model

We consider a MIMO system in a Rayleigh fading environment with L_t transmit and L_r receive antennas. Channel state information (CSI) is perfectly available at the receiver. N transmit antennas out of L_t are selected and activated for the transmission of OSTBC signal matrices, while the remaining transmit antennas are inactive. Let $\widetilde{\mathbf{H}} \in \mathcal{C}^{L_r \times N}$ be a submatrix of the channel matrix $\mathbf{H} \in \mathcal{C}^{L_r \times L_t}$. $\mathbf{H} = [h_{ij}]$ where $h_{ij} \sim \mathcal{CN}(0,1)$ is the channel gain between the ith transmit and jth receive antenna. $\widetilde{\mathbf{H}}$ consists of the channel gains for the N selected transmit antennas and L_r received antennas. Suppose that $\mathbf{h}_j$ $(j = 1, 2, ..., L_t)$ are columns of the channel matrix $\mathbf{H}$. The columns are sorted according to their norms; Assume that $\|\mathbf{h}_{i_1}\| \geq ... \geq \|\mathbf{h}_{i_{L_t}}\|$ where $i_k \in \{1, 2, ..., L_t\}$. Thus, $\widetilde{\mathbf{H}}$ is defined as

$$\widetilde{\mathbf{H}} = [\mathbf{h}_{i_1}\mathbf{h}_{i_2} \cdots \mathbf{h}_{i_N}]. \tag{1}$$

With this selection criterion, we maximize the total received signal power at the receiver. The received signals are expressed as

$$\mathbf{Y} = \sqrt{\frac{E_s}{N}}\widetilde{\mathbf{H}}\mathbf{X} + \mathbf{V} \tag{2}$$

where $\mathbf{Y} \in \mathcal{C}^{L_r \times T}$ is the complex received signal matrix and $\mathbf{X} \in \mathcal{C}^{N \times T}$ is the complex transmitted signal matrix, which is a member of an OSTBC [2], [13]. $\mathbf{V} \in \mathcal{C}^{L_r \times T}$ is the additive noise matrix with independent and identical distributed entries of $\mathcal{CN}(0, N_0)$. The coefficient $\sqrt{E_s/N}$ ensures that the total transmitted power in each channel use is E_s and independent of number of transmit antennas.

Assume that Q symbols $\{s_1, ..., s_Q\}$ with average energy equal to one, chosen from an M-PAM or M-QAM constellations, are transmitted by the transmission matrix $\mathbf{X}$. Since T symbol periods are necessary to transmit Q symbols, the symbol rate R_s of the STBC is defined as $R_s = Q/T$. When an OSTBC is used, the MIMO system is equivalent to Q independent single input single output (SISO) systems defined as [2], [13]

$$\tilde{s}_q = \sqrt{\frac{E_s}{N}}\left(\frac{1}{R_s}\|\widetilde{\mathbf{H}}\|_F^2\right)s_q + \nu_q, \quad q = 1, ..., Q \tag{3}$$

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[1] *Notation*: The Frobenius norm of matrix $\mathbf{A}$ is denoted by $\|\mathbf{A}\|_F$ and the Euclidean norm for vector $\mathbf{h}$ is $\|\mathbf{h}\| = (h_1^2 + \cdots + h_{L_r}^2)^{1/2}$. A circularly symmetric complex Gaussian variable with mean μ and variance σ^2 is denoted by $z \sim \mathcal{CN}(\mu, \sigma^2)$.

where $\nu_q \sim \mathcal{CN}\left(0, \frac{1}{R_s}\|\widetilde{\mathbf{H}}\|_F^2 N_0\right)$. We conclude that the achievable SNR per bit in M-ary constellation is

$$\gamma_b(\rho) = \frac{E_s}{N_0}\frac{1}{R_s N \log_2 M}\|\widetilde{\mathbf{H}}\|_F^2 = c\rho\|\widetilde{\mathbf{H}}\|_F^2 \tag{4}$$

where $\rho = \frac{E_s}{N_0}$ is the SNR per channel and $c = 1/(R_s N \log_2 M)$. Therefore, the antenna selection criterion in (1), which selects N transmit antennas, maximizes the instantaneous SNR and thereupon minimizes the error rate.

Let $\gamma_k = c\rho\|\mathbf{h}_k\|^2$, $k = 1, 2, ..., L_t$ be the scaled norms of the columns of $\mathbf{H}$. Therefore, γ_k is a chi-squared i.i.d. random variable with the pdf given by

$$p_{\gamma_k}(\gamma_k) = \frac{\gamma_k^{L_r-1}}{(c\rho)^{L_r}(L_r-1)!}e^{-\gamma_k/c\rho}. \tag{5}$$

In transmit antenna selection (1), the best N antennas with the largest γ_k are selected. Thus, the received SNR per bit (4) can be written as

$$\gamma_b = \sum_{k=1}^{N}\gamma_{(k)} \tag{6}$$

where $\gamma_{(k)} = c\rho\|\mathbf{h}_{i_k}\|^2$. The MGF of γ_b is given by [12]

$$\begin{aligned}\Phi_{\gamma_b}(s) =& N\binom{L_t}{N}\frac{(c\rho)^{-L_r N}}{\Gamma(L_r)^N}\sum_{i_1,...,i_{N-1}} a(L_r; i_1, ..., i_{N-1}) \\ & \prod_{k=1}^{N-1}\frac{i_k!}{k^{i_k}}\sum_{j=0}^{L_t-N}\binom{L_t-N}{j}(-1)^j \\ & \times\left\{\sum_{n\in B}\binom{j}{n_0,...,n_{L_r-1}}\frac{(c_{nj}+L_r-1)!}{(c\rho)^{c_{nj}}A_{nj}}\right. \\ & \left.\frac{1}{(\frac{1}{c\rho}+s)^{r+N-1}}\cdot\frac{1}{(\frac{1}{c\rho}(N+j)+Ns)^{c_{nj}+L_r}}\right\}\end{aligned} \tag{7}$$

where $a(L_r; i_1, ..., i_{N-1})$ is the coefficient of $x_1^{i_1}\dots x_{N-1}^{i_{N-1}}$ in expression

$$(x_1 + x_2 + \dots + x_N)^{L_r-1}(x_2 + \dots + x_N)^{L_r-1}\cdots x_{N-1}^{L_r-1}$$

and B is the set of all combinations of nonnegative integers of $n_0, n_1, ..., n_{L_r-1}$ such that $\sum_{k=0}^{L_r-1} n_k = j$, $c_{nj} = \sum_{k=1}^{L_r-1} kn_k$, $A_{nj} = \prod_{k=2}^{L_r-1}(k!)^{n_k}$ and $r = \sum_{k=1}^{N-1} i_k$.

III. BER Analysis of M-ary Constellations

A. Exact BER for M-ary PAM

We first derive the BER for M-ary PAM with antenna selection and OSTBCs using Gray mapping. In an AWGN channel, the exact BER of the n-th bit is given by [14]

$$P_M^{\text{AWGN}}(n;\rho) = \frac{2}{M}\sum_{i=0}^{k_n}B_i\mathcal{Q}\Big(D_i\sqrt{\gamma_b(\rho)}\Big) \tag{8}$$

where

$$k_n = (1-\frac{1}{2^n})M - 1 \tag{9}$$

$$B_i = (-1)^{\left\lfloor\frac{i.2^{n-1}}{M}\right\rfloor}\left(2^{n-1} - \left\lfloor\frac{i.2^{n-1}}{M}+\frac{1}{2}\right\rfloor\right) \tag{10}$$

$$D_i = (2i+1)\sqrt{\frac{6\log_2 M}{M^2-1}}. \tag{11}$$

Thus, to obtain the average BER, we take the expectation with respect to the channel statistics:

$$\begin{aligned}P_M(n;\rho) &= \frac{2}{M}\sum_{i=0}^{k_n}B_i\mathcal{E}_{\widetilde{\mathbf{H}}}\left[\mathcal{Q}\left(D_i\sqrt{\gamma_b(\rho)}\right)\right] \\ &= \frac{2}{M}\sum_{i=0}^{k_n}B_i\frac{1}{\pi}\int_0^\infty\frac{\Phi_{\gamma_b}\left(\frac{D_i^2(1+t^2)}{2}\right)}{1+t^2}dt\end{aligned} \tag{12}$$

where $\mathcal{Q}(x) = \frac{1}{\pi}\int_0^\infty e^{-x^2(1+t^2)/2}/(1+t^2)dt$.

The exact average BER of an OSTBC with M-PAM is given by

$$P_M(\rho) = \frac{1}{\log_2 M}\sum_{n=1}^{\log_2 M}P_M(n;\rho). \tag{13}$$

In order to obtain the exact BER using (7), for the antenna selection scheme, we must compute the integral form of

$$W(a,b;\alpha,\beta) = \int_0^\infty\underbrace{\frac{1}{1+x^2}\cdot\frac{1}{(a^2+x^2)^\alpha}\cdot\frac{1}{(b^2+x^2)^\beta}}_{f(x)}dx. \tag{14}$$

We consider complex function $f(z)$ and use the residue theorem to solve (14). This integral can be thought of as an integral over a part of a contour C_R consisting of a line segment along the real axis between $-R$ and R. We close the contour by using the upper semi-circle with radius R centered at the origin. We can show that the integral over the added part of C_R asymptotically vanishes as $R \to \infty$. Thus, the resulting integral would be

$$\int_0^\infty f(x)dx = \pi i\left[\operatorname*{Res}_{z=ia}f(z) + \operatorname*{Res}_{z=ib}f(z) + \operatorname*{Res}_{z=i}f(z)\right] \tag{15}$$

where $\text{Res}_{z=z_0}f(z)$ is the residue of function $f(z)$ evaluated at $z = z_0$. For brevity, we omit the details. Therefore, the BER can be obtained as

$$\begin{aligned}P_M(\rho) =& \frac{2N}{M\log_2 M}\binom{L_t}{N}\frac{(c\rho)^{-L_r N}}{[(L_r-1)!]^N}\sum_{n=1}^{\log_2 M}\sum_{i=0}^{k_n}B_i \\ & \sum_{i_1,...,i_{N-1}}a(L_r;i_1,...,i_{N-1})\prod_{k=1}^{N-1}\frac{i_k!}{k^{i_k}}\sum_{j=0}^{L_t-N}\binom{L_t-N}{j}(-1)^j \\ & \times\left\{\sum_{n\in B}\binom{j}{n_0,...,n_{L_r-1}}\frac{(c_{nj}+L_r-1)!}{(c\rho)^{c_{nj}}A_{nj}}\right. \\ & \left.\times W\left(\sqrt{\frac{2}{cD_i^2\rho}+1},\sqrt{\frac{2(N+j)}{NcD_i^2\rho}+1};r+N-1,c_{nj}+L_r\right)\right\}.\end{aligned} \tag{16}$$

B. Exact BER for M-ary QAM

Note that a rectangular or square QAM constellations can be composed to two independent PAM constellations: I-ary PAM for the in-phase component and J-ary PAM for the quadrature component, where $M = I \times J$. Thus, the exact average BER of M-QAM is given by

$$P_M(\rho) = \frac{1}{\log_2(I.J)}\left(\sum_{n=1}^{\log_2 I}P_I(n;\rho) + \sum_{m=1}^{\log_2 J}P_J(m;\rho)\right). \tag{17}$$

The result for transmit diversity and one receiver antenna ($L_r = 1$) can be simplified as

$$\Phi_{\gamma_b}(s)=\frac{L_t!}{(N-1)!}\frac{1}{(1+c\rho s)^{N-1}}\prod_{j=0}^{L_t-N}\frac{1}{N(1+c\rho s)+j} \qquad (18)$$

where we have used $\sum_{j=0}^{n}\binom{n}{j}\frac{(-1)^j}{p+j} = \frac{n!}{p(p+1)\cdots(p+n)}$ to convert the sum to the product (18). Considering the limit for high SNR, we can show that the approximate BER is

$$P_M(\rho) \approx \frac{2}{M\log_2 M}\left(\sum_{n=1}^{\log_2 M}\sum_{i=0}^{k_n}\frac{B_i}{D_i^{2L_t}}\right)\frac{1}{N^{L_t-N+1}c^{L_t}} \times \frac{(2L_t-1)!}{2^{L_t-1}(L_t-1)!(N-1)!}\left(\frac{1}{\rho}\right)^{L_t}, \rho \gg 1 \qquad (19)$$

which clearly indicates a full diversity order of L_t at high SNRs for N transmit antenna selection of an $L_t \times 1$ system.

Fig. 1 compares the exact expression (16), the approximation (19), and the simulation results for $N = 3$ TAS out of $3 \leq L_t \leq 6$ and $L_r = 1$ using the OSTBC with rate $\frac{3}{4}$ as defined in [2], [13] for 16-QAM. Note that (16) asymptotically approaches (19), which is a tight bound of (16) at high SNRs.

C. Approximate BER for M-ary PSK

A tight approximation for the BER of the coherent M-ary PSK in AWGN channels is given by [15, eq. (12)]. Again, using the expression for Q-function and same derivation steps from (12) to (16), an approximate expression for the average BER of OSTBC can be found similarly as

$$P_M(\rho) = \frac{N\binom{L_t}{N}}{\max(\log_2 M, 2)}\frac{(c\rho)^{-L_r N}}{[(L_r-1)!]^N}\sum_{i=1}^{\max(M/4,1)} \sum_{i_1,\ldots,i_{N-1}} a(L_r; i_1, \ldots, i_{N-1})\prod_{k=1}^{N-1}\frac{i_k!}{k^{i_k}}\sum_{j=0}^{L_t-N}\binom{L_t-N}{j}(-1)^j \times \left\{\sum_{n\in B}\binom{j}{n_0,\ldots,n_{L_r-1}}\frac{(c_{nj}+L_r-1)!}{(c\rho)^{c_{nj}}A_{nj}} \times W\left(\sqrt{\frac{1}{c\delta_i^2\rho}+1}, \sqrt{\frac{(N+j)}{Nc\delta_i^2\rho}+1}; r+N-1, c_{nj}+L_r\right)\right\}. \qquad (20)$$

where $\delta_i = \sin\frac{(2i-1)\pi}{M}$.

IV. Conclusion

In this paper, we have investigated the performance of TAS and OSTBCs. The exact BER for M-PAM and M-QAM and an approximate BER for M-PSK were derived. Our results are sufficiently general to handle an arbitrary number of antennas, unlike the previous results. Moreover, we directly derived the BER, not via the symbol error probability. As expected, we find that this scheme achieves full diversity order asymptotically (i.e., L_t not N), as if all the transmit antennas were used.

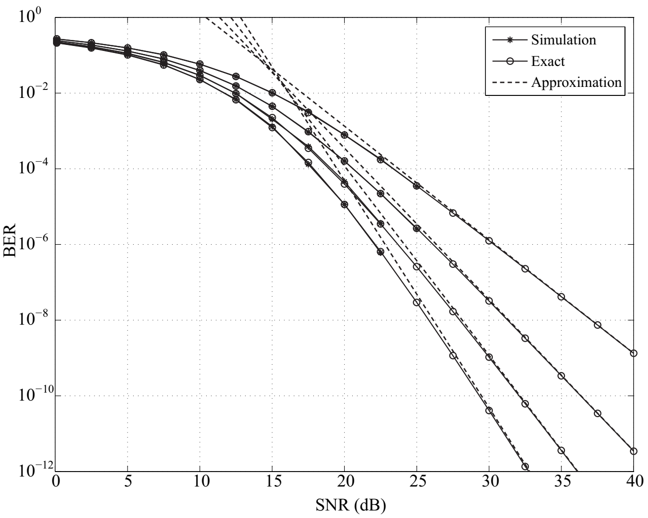

Fig. 1. Comparison between the exact expression, approximation and simulation for $N = 3$ TAS out of $3 \leq L_t \leq 6$ with $L_r = 1$, 16-QAM.

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