Optimal Rotations for Quasi-Orthogonal STBC with Two-Dimensional Constellations

Dũng Ngọc Đào and Chintha Tellambura,
Department of Electrical and Computer Engineering, University of Alberta
Edmonton, Alberta T6G 2V4, Canada
Email: {dndung, chintha}@ece.ualberta.ca Telephone: (1-780) 492 7228, Fax: (1-780) 492 1811

Abstract— **Quasi-orthogonal space-time block codes (QSTBC) achieve full diversity by constellation rotations. Several authors have introduced optimal rotation angles, found either by computer search or by analytical derivation. However, existing analytical methods do not seem general enough to analyze optimal rotations for arbitrary constellations, and some previous results seem to conflict. We present a novel method to exactly *derive the coding gain* of QSTBC as a function of the rotation angle and the minimum Euclidean distance of two-dimensional constellations such as the ones carved from lattices of squares and triangles, and phase-shift keying (PSK) constellations. The upper bound of coding gain for amplitude PSK (APSK) is also obtained. We find the whole range of optimal rotations for maximizing the coding gain of QSTBC. Simulation results confirm the theoretical analysis.**

I. Introduction

Space-time block codes (STBC) have been extensively studied recently. One of the most well-known STBC is the Alamouti code [1] and it belongs to a general class of codes known as orthogonal STBC (OSTBC) [2]. Orthogonality enables single symbol maximum likelihood (ML) detection but results in low code rates. To improve the code rate, quasi-orthogonal STBC (QSTBC) are independently proposed in [3]–[5]. QSTBC use OSTBC as building blocks and the code rate of QSTBC with $2m$ transmit (Tx) antennas equals the code rate of OSTBC with m Tx antennas. Therefore, QSTBC have higher code rates than OSTBC. In the codeword matrix of QSTBC, not all pairs of columns are orthogonal; however, the transmitted symbols can be ML decoded in pairs.

In order to achieve full diversity, half of the input symbols have to be rotated before encoding [6], [7]. However, given a constellation, the optimum rotation angle must be chosen to minimize the symbol error rate (SER) or BER. Authors in references [6]–[11] provide some specific optimal rotation angles for quadrature amplitude modulation (QAM), phase-shift keying (PSK) and hexagonal constellations (or lattices of equilateral triangles) [12]. However, some results are not identical. For example, according to [10], the optimal rotation angle is $\pi/4$ for 4QAM (and also higher order QAM in general), whereas it can be any value in the range from $\pi/6$ to $\pi/3$, the best one should be $\pi/6$ [13], (or approximately $\pi/6$ [7] due to simulation results) or in the vicinity of 35^0 [6]. This problem motivates a general approach to derive optimal rotation angles for an arbitrary constellation.

In this paper, we present a simple method using the basic Euclidean geometry to derive not only the optimal rotations for QSTBC but also the coding gain of QSTBC as a function of rotation angle and the minimum Euclidean distance of two-dimensional constellations carved from lattices of squares (such as QAM), lattice of triangles (TRI) and phase-shift keying (PSK) constellations. An upper bound of coding gain for amplitude PSK (APSK) constellations is also obtained. Since the optimal rotation angle is not unique, we propose an additional criterion to select the rotation angles, in which the number of signal points with minimum coding gain is minimized. Simulation results support this idea. To the best of our knowledge, ours is the first presented method that is general enough to analyze the coding gain of QSTBC with arbitrary constellations.

II. System Model and Preliminary Results

We first set some notations to be used in the paper. Superscripts $(\cdot)^*$ and $(\cdot)^\dagger$ denote conjugate and conjugate transpose operations, respectively. The $m \times m$ identity matrix is I_m. A signal constellation $\mathcal{S}$ is a finite set of possibly complex numbers. The minimum Euclidean distance of $\mathcal{S}$ is $d_{\min} = \min\{|s - \hat{s}| \forall s \neq \hat{s}; s, \hat{s} \in \mathcal{S}\}$ and the order or size of $\mathcal{S}$ is the number of elements of $\mathcal{S}$. The set of positive integer numbers is $\mathbb{N}$. A circularly complex Gaussian random variable x with mean m and variance σ^2 is denoted by $x \sim \mathcal{CN}(m, \sigma^2)$.

We consider data transmission over a quasi-static Rayleigh flat fading channel. The transmitter and receiver are equipped with m Tx and n Rx (receive) antennas. The channel gains $h_{ik} (i = 1, 2, ..., m; k = 1, 2, ..., n)$ between any pair of Tx-Rx antennas are assumed $\mathcal{CN}(0, 1)$. We assume no spatial correlation at either Tx or Rx array. The receiver, but not the transmitter, completely knows the channel gains. Modulation symbols are drawn form a constellation $\mathcal{S}$ with zero mean and unit energy. They are encoded into space-time (ST) codeword C of size $t \times m$, where the entry c_{li} denotes the symbol transmitted from antenna i at time l ($1 \leq l \leq t$). The average transmitted power is constrained such that $\sum_{i=1}^{m} \sum_{l=1}^{t} E[|c_{li}|^2] = t$.

The received signals y_{lk} of the kth antenna at time l can be arranged in a matrix Y of size $t \times n$. Thus, one can represent the Tx-Rx signal relation compactly as $Y = \sqrt{\rho} C H + Z$, where $H = [h_{ik}]$ and the $t \times n$ matrix $Z = [z_{ik}]$ with $z_{ik} \sim \mathcal{CN}(0, 1)$. The average receive SNR is ρ, independent of m.

The upper-bound of pair-wire error probability (PEP) derived by Tarokh *et al.* [14] is as follows:

$$P(C \to \hat{C}) \leq \left(\prod_{i=1}^{\Gamma} \lambda_i\right)^{-n} \left(\frac{\rho}{4}\right)^{-\Gamma n} \quad (1)$$

where C and $\hat{C}$ are the transmitted and erroneous codewords, Γ is the minimum rank of a matrix Δ_C ($\Delta_C = C - \hat{C}$) for all $C \neq \hat{C}$, and $\lambda_1, \lambda_2, , ..., \lambda_\Gamma$ are non-zero eigenvalues of a product matrix $P_C = \Delta_C^\dagger \Delta_C$. The diversity gain or diversity order d and coding gain G of ST codes are defined as $d = \Gamma n$ and $G = \left(\prod_{i=1}^{\Gamma} \lambda_i\right)^{1/\Gamma}$, respectively. The maximum achievable diversity order is $d = mn$, and in this case, the coding gain is $G = \left[\det(\Delta_C^\dagger \Delta_C)\right]^{1/m}$.

Quasi-orthogonal STBC are constructed from OSTBC. In particular, a QSTB code for $2m$ Tx antennas employs OSTBC for m antennas as a building element. For example, a code proposed by Tirkkonen *et al.* [6] is as follows:

$$\mathcal{Q} = \sqrt{\lambda} \begin{bmatrix} A & B \\ B & A \end{bmatrix} \quad (2)$$

where A and B are $(p \times m)$ OSTBC, $A = \mathcal{O}(x_1, x_2, ..., x_K)$ and $B = \mathcal{O}(x_{K+1}, x_{K+2}, ..., x_{2K})$. The scale factor λ guarantees the transmit power constraint.

Xu and Xia [10] show that $\mathcal{Q}^\dagger \mathcal{Q} = \lambda \begin{bmatrix} aI_m & bI_m \\ bI_m & aI_m \end{bmatrix}$, where $a = \sum_{k=1}^{2K} |x_k|^2$ and $b = \sum_{k=1}^{K} (x_k x_{K+k}^* + x_k^* x_{K+k})$. With other QSTBC [3], [5], similar results can be obtained.

For two distinct QSTB codewords $Q = \mathcal{Q}(s_1, s_2, ... s_{2K})$ and $\hat{Q} = \mathcal{Q}(\hat{s}_1, \hat{s}_2, ... \hat{s}_{2K})$, the difference matrix is $\Delta_\mathcal{Q} = Q - \hat{Q}$. We have the product matrix $P_\mathcal{Q}$ as

$$P_\mathcal{Q} = \Delta_\mathcal{Q}^\dagger \Delta_\mathcal{Q} = \lambda \begin{bmatrix} (\Delta a) I_m & (\Delta b) I_m \\ (\Delta b) I_m & (\Delta a) I_m \end{bmatrix} \quad (3)$$

where $\Delta a = \sum_{i=1}^{2K} |s_i - \hat{s}_i|^2$ and $\Delta b = \sum_{i=1}^{K} [(s_i - \hat{s}_i)(s_{K+i} - \hat{s}_{K+i})^* + (s_i - \hat{s}_i)^*(s_{K+i} - \hat{s}_{K+i})]$. To achieve full diversity, symbols s_{K+i} ($i = 1, 2, ..., K$) are rotated, i.e., half of the input symbols are drawn from a constellation $\mathcal{S}$ and the other half are drawn from the rotated constellation $\mathcal{R} = e^{j\phi}\mathcal{S}$ ($j^2 = -1$) where ϕ is the rotation angle [10]. Then $\det(P_\mathcal{Q}) \neq 0$ and QSTBC achieve full diversity. The coding gain of QSTBC with constellation rotation is given by [10]

$$\begin{aligned} G_\mathcal{Q} &= \min \left[\det(P_\mathcal{Q})\right]^{1/m_\mathcal{Q}} \\ &= \lambda \min_{\substack{s,\hat{s} \in \mathcal{S}; r,\hat{r} \in \mathcal{R} \\ (s,r) \neq (\hat{s},\hat{r})}} \left|(s - \hat{s}) - (r - \hat{r})\right| \cdot \left|(s - \hat{s}) + (r - \hat{r})\right| \\ &= \lambda \min_{\substack{s,\hat{s} \in \mathcal{S}; r,\hat{r} \in \mathcal{R} \\ (s,r) \neq (\hat{s},\hat{r})}} \left|(s - \hat{s})^2 - (r - \hat{r})^2\right|. \end{aligned} \quad (4)$$

III. Analyzing the Coding Gain of QSTBC with Two-Dimensional Constellations

In this section, we approach the problem from a new viewpoint. Different from Su and Xia [10], who work with constellation $\mathcal{S}$, we operate on the constellation generated by taking the differences of all signal points of $\mathcal{S}$.

Let us introduce some basic notations on two-dimensional Euclidean vector space. Let us denote the vector connecting two arbitrary points U and V as $\overrightarrow{UV}$, the modulus of vector $\overrightarrow{UV}$ as $|\overrightarrow{UV}|$. If U overlaps the origin O and the location of V in the two-dimensional complex plane is $v = V_x + jV_y$ where $j^2 = -1$, then $|v| = |\overrightarrow{OV}|$.

We can use the vector notation to represent signal points in the two-dimensional plane. Let s and $\hat{s}$ be two elements of $\mathcal{S}$, which can be labelled as two signal points S and $\hat{S}$ in the complex plane accordingly. Consider a constellation $\mathcal{E}$, which contains the set of differences $(s - \hat{s})$ defined as:

$$\mathcal{E} = \{e \,|\, e = s - \hat{s}\,, \forall\, s, \hat{s} \in \mathcal{S}\}. \quad (5)$$

We define another constellation $\mathcal{F}$ such that

$$\mathcal{F} = \{f \,|\, f = r - \hat{r}\,, \forall\, r\,, \hat{r} \in \mathcal{R}, \mathcal{R} = e^{j\phi}\mathcal{S}\}. \quad (6)$$

Sets $\mathcal{S}$, $\mathcal{E}$ and $\mathcal{F}$ may be called generating constellation, differential constellation and rotated differential constellation, respectively, where no confusion may arise. From (5) and (6), one has $\min_{e \in \mathcal{E},\, e \neq 0} \{|e|\} = \min_{f \in \mathcal{F},\, f \neq 0} \{|f|\} = d_{\min}$. Furthermore, we obtain $\mathcal{F} = e^{j\phi}\mathcal{E}$. In words, constellation $\mathcal{F}$ is a rotated version of the constellation $\mathcal{E}$ with rotation angle ϕ. The coding gain of QSTBC in (4) is rewritten as

$$G_\mathcal{Q} = \min_{E \in \mathcal{E}, F \in \mathcal{F}; (E,F) \neq (O,O)} \left[\left(|e - f| \cdot |e + f|\right)\right]. \quad (7)$$

At this point, we have made no assumption on the structure of constellations $\mathcal{E}$ and $\mathcal{F}$. Let E and F be two arbitrary signal points in the complex plane. Then $|\overrightarrow{OE}| = |e|$, $|\overrightarrow{OF}| = |f|$. Using vector addition and subtraction, we have $\overrightarrow{EF} = \overrightarrow{OF} - \overrightarrow{OE}$ and $\overrightarrow{OZ} = \overrightarrow{OF} + \overrightarrow{OE}$.

We define a quantity ω_{EF} as $\omega_{EF} \triangleq \left(|\overrightarrow{EF}||\overrightarrow{OZ}|\right)^{1/2}$; then

$$\omega_{EF} = \left[\left(|e|^2 + |f|^2\right)^2 - 4|e|^2|f|^2 \cos^2 \varphi\right]^{1/4} \quad (8)$$

where φ is the angle between vectors $\overrightarrow{OE}$ and $\overrightarrow{OF}$; (8) follows from basic trigonometry. We call ω_{EF} the *product distance* because of its geometrical meaning. We further define the *minimum product distance* [1] (MPD) $\Omega_\mathcal{S}$ of constellation $\mathcal{E}$ and $\mathcal{F}$ as [2] $\Omega_\mathcal{S} \triangleq \min_{E \in \mathcal{E}, F \in \mathcal{F}; (E,F) \neq (O,O)} \omega_{EF}$. The subscript $\mathcal{S}$ emphasizes the association of MPD with the generating constellation $\mathcal{S}$. The upper bound for minimum ζ-distance is given in [10] and [13] as $d_{\min,\zeta} \leq d_{\min}$. Equivalently, we can prove the upper bound of MPD. However, the proofs are omitted due to the lack of space. The reader is referred to [16] for full details.

Lemma 1: *The MPD of QSTBC is bounded by* $\Omega_\mathcal{S} \leq d_{\min}$.

[1]The term *minimum product distance* is also defined for full modulation diversity signals and DAST codes (see [15] and references therein).

[2]This parameter is called as *minimum* ζ*-distance* in [10]. In our geometric approach, the name *minimum product distance* is more meaningful.

This full text paper was peer reviewed at the direction of IEEE Communications Society subject matter experts for publication in the IEEE GLOBECOM 2005 proceedings.

The coding gain of QSTBC is now proportional to the MPD Ω of two constellations $\mathcal{E}$ and $\mathcal{F}$. $G_{\mathcal{Q}} = \lambda \Omega_{\mathcal{S}}^2$. Furthermore, it is bounded as $G_{\mathcal{Q}} \leq \lambda\, d_{\min}^2$.

We distinguish two cases: constellation $\mathcal{S}$ has either lattice or non-lattice structure.

A. QSTBC with Lattice-Based Constellations

In general, any point S belonging to a two-dimensional lattice $\mathcal{S}$ can be represented in the complex plane as $s = s_0 + k_1\nu_1 + k_2\nu_2$, where k_1 and k_2 are integers, and ν_1 and ν_2 are complex numbers that form a basis for the lattice [17].

Proposition 1: *If $\mathcal{S}$ has lattice structure, the constellation $\mathcal{E}$ defined in* (5) *has the following properties:*

- *Lattices $\mathcal{S}$ and $\mathcal{E}$ have the same basis.*
- *Lattice $\mathcal{E}$ includes the origin and is symmetric via the origin.*
- *If lattice $\mathcal{S}$ is finite, so is the lattice $\mathcal{E}$.*

The proof is straightforward and is not given here. Proposition 1 shows the advantages of using the differential lattice instead of the original one. First, the lattice structure is preserved. Second, although some signal constellations may not be symmetric around the origin, their differential constellations are always include the origin and are symmetric about the origin.

We now examine two important lattices: Lattices of squares (including square and rectangular QAM) and lattices of equilateral triangles (TRI) (including hexagonal constellations) [12]. The optimal rotation angles for these constellations are discussed in [10]. However, we will show that the results of [10] are just special cases of our results.

1. Square Lattices

For square lattices, $\nu_1 = d_{\min}$ and $\nu_2 = e^{j\pi/2} d_{\min}$. We first study the basic 4-point square lattices, in particular, 4QAM plotted in Fig. 1. Note that there is no restriction on the locations of the square lattices on the two-dimensional plane.

We have $|\overrightarrow{OA}| = |\overrightarrow{OB}| = d_{\min}$ and $|\overrightarrow{OC}| = \sqrt{2} d_{\min}$, and because of the symmetry of $\mathcal{E}$, we first examine the case $\phi \in (0, \pi/2)$. One obtains

$$\begin{aligned}\Omega_{4\,\mathrm{QAM}} &= \min\{\omega_{BA}, \omega_{BC}, \omega_{BO}\} \\ &= \min\{(2\sin\phi)^{1/2} d_{\min}, (2\cos\phi)^{1/2} d_{\min}, d_{\min}\}.\end{aligned}$$

Therefore, the maximal value of $\Omega_{4\,\mathrm{QAM}}$ is $d_{\min}$, and is achieved if and only if the rotation angle satisfies $\pi/6 \leq \phi \leq \pi/3$. Because of the symmetry of constellation $\mathcal{R}$, any rotation angle ϕ such that $\frac{\pi}{6} + \frac{n\pi}{2} \leq \phi \leq \frac{\pi}{3} + \frac{n\pi}{2}, n = 0, 1, 2, 3$ also maximizes the MPD and hence maximizes the coding gain.

For higher order constellations, the lattices $\mathcal{E}$ and $\mathcal{F}$ are expanded outward. We have the following facts about their MPD.

Proposition 2: *Let E and F be the signal points, each drawn from lattices $\mathcal{E}$ and $\mathcal{F}$, where $\mathcal{E}$ and $\mathcal{F}$ are generated from a square lattice $\mathcal{S}$ by rules* (5) *and* (6)*.*

- *If $|e| \neq |f|$, the product distance of E and F is lower bounded by $d_{\min}$.*

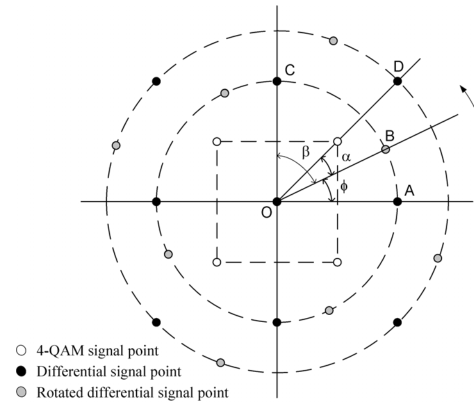

Fig. 1. 4QAM, its differential and rotated differential constellations.

- *Otherwise, their MPD is given by $\omega_{EF} = (2\sin\alpha)^{1/2}|e|$, where α is the angle between two vectors $\overrightarrow{OE}$ and $\overrightarrow{OF}$.*

Thus, for square lattices, MPD is the product distance of the two signal points lying on the innermost circle, where $|e|$ is minimum. Thus the MPD of any square lattices is upper-bounded by the MPD of 4QAM. For 4QAM, we state the result as follows.

Lemma 2: *For 4-point square lattices with minimum Euclidean distance $d_{\min}$, the coding gain of QSTBC with rotation angle $\phi \in [\frac{k\pi}{2}, \frac{(k+1)\pi}{2}]$, $k = 0, 1, 2, 3$, is given by $G_{\mathcal{Q}} = \lambda \min\{2|\sin(\phi - \frac{k\pi}{2})|\, d_{\min}^2, 2|\cos(\phi - \frac{k\pi}{2})|\, d_{\min}^2, d_{\min}^2\}$. The coding gain is maximized, $G_{\mathcal{Q}}^{\max} = \lambda d_{\min}^2$, if and only if the rotation angle ϕ satisfies $\frac{(3k+1)\pi}{6} \leq \phi \leq \frac{(3k+2)\pi}{6}$.*

This result agrees with that found in [13]. For a higher order of QAM, we need to solve an additional constraint, which is more difficult for 32QAM or higher order. We can show analytically that the rotation $\phi = \pi/4$ meets this additional constraint [16]. We have not shown analytically that $\phi = \pi/3, \pi/6$ meet the constraint. Nevertheless, we can easily verify numerically that $\phi = \pi/3, \pi/6$ are also optimal for square QAM constellations with at least up to 2^{20} points.

2. Lattices of Equilateral Triangles

For the TRI lattice, $\nu_1 = d_{\min}$ and $\nu_2 = e^{j\pi/3} d_{\min}$. We present the main results for TRI lattices for brevity; however the derivations are similar to the case of square lattices.

Theorem 1: *For lattices of equilateral triangles with minimum Euclidean distance $d_{\min}$, the coding gain of QSTBC with rotation angle $\phi \in [\frac{k\pi}{3}, \frac{(k+1)\pi}{3}]$, $k = 0, 1, ..., 5$, is given by $G_{\mathcal{Q}} = \lambda \min\{2|\sin(\phi - \frac{k\pi}{3})|\, d_{\min}^2, 2|\sin[\frac{(k+1)\pi}{3} - \phi]| d_{\min}^2, d_{\min}^2\}$. The coding gain is maximized, $G_{\mathcal{Q}}^{\max} = \lambda d_{\min}^2$, if and only if the rotation angle ϕ takes one of the values $\phi = \frac{(2k+1)\pi}{6}$.*

Note that our Theorem 1 subsumes [10, Theorem 2], where only rotation angle $\phi = \pi/6$ is provided. Using the same approach, with minimal effort, one can show that for PAM (one-dimensional lattice), the optimum rotation is $\pi/2$. This simple case is also briefly mentioned in [10].

B. QSTBC with Non-Lattice Constellations

1. *MPSK Constellations*

The coding gain and optimal rotations for MPSK constellations are stated in the following theorems.

Theorem 2: *For any $MPSK$ constellation (M is an even integer larger than 4) with minimum Euclidean distance $d_{\min} = 2\sin\frac{\pi}{M}$, the coding gain of QSTBC with rotation angle $\phi \in [\frac{2k\pi}{M}, \frac{2(k+1)\pi}{M}]$, $k = 0, 1, ..., M-1$ is given by $G_{\mathcal{Q}} = \lambda \min\{2|\sin(\phi - \frac{2k\pi}{M})|\, d_{\min}, 2|\sin(\frac{2(k+1)\pi}{M} - \phi)| d_{\min}\}$. The coding gain is maximized, $G_{\mathcal{Q}}^{\max} = 2\lambda \sin\frac{\pi}{M} d_{\min}^2$, if and only if the rotation angle ϕ satisfies $\phi = \frac{(2k+1)\pi}{M}$.*

Theorem 3: *For any $MPSK$ constellation (M is an odd integer greater than 1) with minimum Euclidean distance $d_{\min} = 2\sin\frac{\pi}{M}$, the coding gain of QSTBC with rotation angle $\phi \in [\frac{k\pi}{M}, \frac{(k+1)\pi}{M}]$, $k = 0, 1, ..., 2M-1$, is given by $G_{\mathcal{Q}} = \lambda \min\{2|\sin(\phi - \frac{k\pi}{M})|\, d_{\min}, 2|\sin(\frac{(k+1)\pi}{M} - \phi)| d_{\min}\}$. The coding gain is maximized, $G_{\mathcal{Q}}^{\max} = 2\lambda \sin\frac{\pi}{2M} d_{\min}^2$, if and only if the rotation angle ϕ satisfies $\phi = \frac{(2k+1)\pi}{M}$.*

Theorem 2 agrees with the results of [13] and Theorem 3 subsumes the results of [11]. The MPD's of MPSK constellations for $M = 3, 4, ..., 16$ are provided in Table I. Note that for $M = 3, 6$, signal points of $\mathcal{S}$ belong to TRI lattices, and for $M = 4$, $\mathcal{S}$ is actually 4QAM. Hence the coding gain of MPSK with these values of M achieves the upper bound $d_{\min}^2$. For other values of M, the coding gain cannot achieve the upper bound.

Optimal rotation angles and MPD of M-point constellations for $M = 4, 8, 16$ belonging to QAM, TRI and PSK are summarized in Table II (see Fig. 2 for illustrations of 8-point constellations).

2. *Amplitude Phase-Shift Key Constellations*

An amplitude phase-shift key (APSK) constellation consists of signal points lying on several rings. The notation $(p_1, p_2, p_3, ...)$ can be used to denote an APSK constellation, where p_i represents the number of points on the ith ring, counting from the origin outward. Since APSK is the union of several PSK constellations, its MPD is bounded by the smallest MPD of the PSK subconstellations. Although the upper bound is rather loose, it is useful because analyzing the MPD of an APSK constellation with arbitrary geometric shape is complicated.

Considering (1, p) constellations ($p \in \mathbb{N}$), which consists of the origin as a signal point and the other p points lie equally-spaced on the circle of radius $R = 1$. We found that $\Omega_{(1,3)} = 1$, whereas $\Omega_{4\,\mathrm{PSK}} = 1.4142$, and $\Omega_{(1,7)} < \Omega_{7\,\mathrm{PSK}} = 0.5789$ is well below $\Omega_{8\,\mathrm{PSK}}$. Therefore, further calculation of the exact MPD for (1, p) constellations is not fruitful for practical applications.

We found that with constellation (4, 4) [18], [19] its exact MPD $\Omega_{(4,\,4)} = (2\sin\frac{\pi}{12})^{1/2} d_{\min} \approx 0.6615$ is even worse than that of 8PSK. We examine four 16PSK constellations including the best (8, 8) constellation [18] and the other three constellations investigated in [20]. Their MPD upper bounds (summarized in Table III) are well below the exact MPD of 16QAM in Table II. Further investigation on the MPD upper bounds of different 32APSK and 64APSK constellations presented in [19], [20] also shows that no examined APSK constellation has an MPD upper bound better than QAM square with the same spectral efficiency.While we do not comprehensively analyze optimal APSK constellations for QSTBC, the available results show that the most well-known APSK constellations are inferior to square QAM.

Before concluding this section, we note that two optimal rotations for TRI lattices are $\phi = \pm\pi/2$ ($k = 1, 4$), and they do not require explicit multiplications; hence the encoding complexity is minimized without any loss of performance.

TABLE I

MINIMUM PRODUCT DISTANCE OF MPSK CONSTELLATIONS

M	3	5	7	9	11	13	15
$\Omega_{M\,\mathrm{PSK}}$	1.732	0.924	0.579	0.403	0.300	0.235	0.190
M	4	6	8	10	12	14	16
$\Omega_{M\,\mathrm{PSK}}$	1.414	1	0.670	0.486	0.372	0.297	0.244

TABLE II

SUMMARY OF OPTIMAL ROTATION ANGLES

Constellation	Optimal ϕ	$\Omega_{\mathcal{S}}$	$d_{\min}$
4QAM	$\left[\frac{\pi}{6} + \frac{k\pi}{2}, \frac{\pi}{3} + \frac{k\pi}{2}\right]$, $k = 0, 1, 2, 3$	1.414	1.414
4TRI	$\frac{\pi}{6} + \frac{k\pi}{3}$, $k = 0, 1, ..., 5$	1.414	1.414
(4, 4)	$\frac{\pi}{12} + \frac{k\pi}{6}$, $k = 0, 1, ..., 11$	0.662	0.919
8PSK	$\frac{\pi}{8} + \frac{k\pi}{4}$, $k = 0, 1, ..., 7$	0.670	0.765
8QAM square	$(\frac{\pi}{3}, \frac{\pi}{4}, \frac{\pi}{6}) + \frac{k\pi}{2}$, $k = 0, 1, 2, 3$	0.817	0.817
8QAM rect.	as 8QAM square	0.817	0.817
8QAM rotated	as 8QAM square	0.906	0.906
8TRI-a	as 4TRI	0.943	0.943
8TRI-b	as 4TRI	0.963	0.963
16PSK	$\frac{\pi}{16} + \frac{k\pi}{8}$, $k = 0, 1, ..., 15$	0.244	0.390
16QAM square	as 8QAM square	0.633	0.633
16TRI	as 4TRI	0.676	0.676
32QAM square	as 8QAM square	0.447	0.447
32TRI	as 4TRI	0.477	0.477
64QAM square	as 8QAM square	0.309	0.309
64TRI	as 4TRI	0.337	0.337

TABLE III

UPPER BOUND OF MPD OF APSK CONSTELLATIONS

Constellations	(1, 5, 10)	(5, 11)	(4, 12)	(8, 8)
Upper bound of MPD	0.5336	0.3438	0.4207	0.5049

IV. SIMULATION RESULTS

This section verifies theoretical analysis by the simulations performed for a MIMO system with 4 Tx and 1 Rx antennas.

We verify the performance of different constellations with the same spectrum efficiency. Seven 8-point constellations with MPDs given in Table II are compared. Fig. 3 shows that the constellation with higher MPD has better SER performance at high SNR. Let $N_{\mathcal{S}}$ denote the average number of neighbors at distance $d_{\min}$ of constellations $\mathcal{S}$. The performance of 8TRI-a and -b is only slightly better than that of the 8QAM rotated square. The reason is that $N_{8\,\mathrm{TRI}-a} = 3.25$ and $N_{8\,\mathrm{TRI}-b} = 3.5$ are higher than $N_{8\,\mathrm{QAM\,rotated\,square}} = 2.5$.

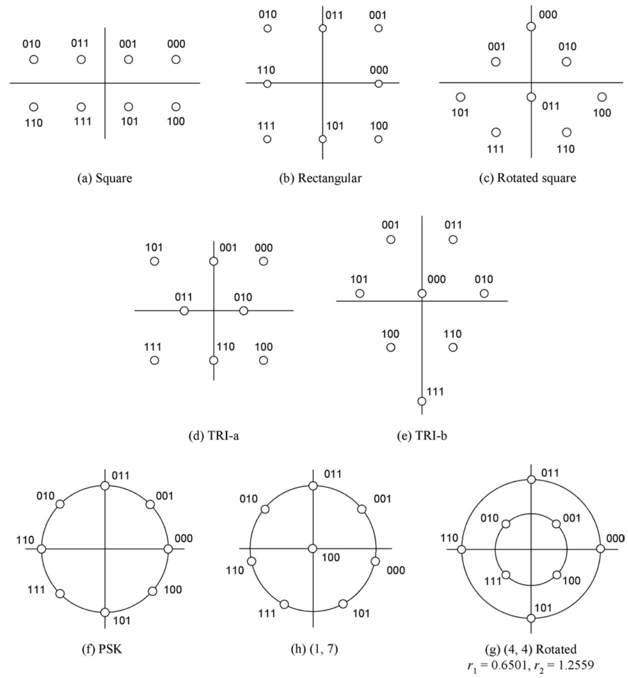

Fig. 2. 8-point constellations

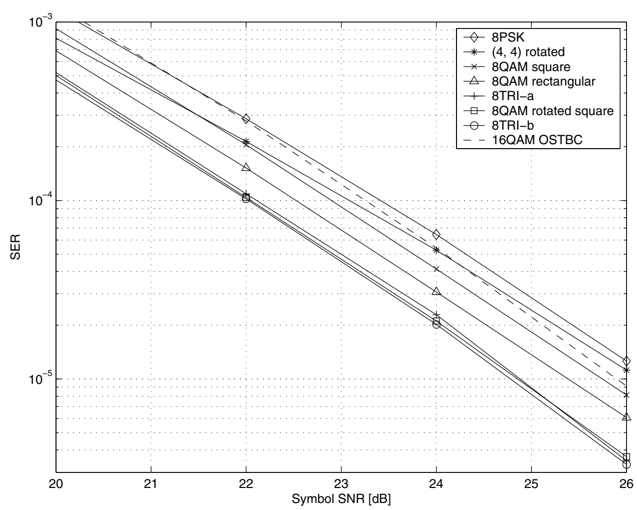

Fig. 3. SER of QSTBC with different types of 8-point modulations, 4 Tx and 1 Rx antennas.

This side effect also causes the performance of 8QAM square to be worse than that of 8QAM rectangular ($N_{8\,\mathrm{QAM\,square}} = 2.5, N_{8\,\mathrm{QAM\,rect.}} = 2$). Note that the constellation (4, 4) has the smallest MPD among the 8-point constellations (Table II); its SER should therefore be the worst. However, this prediction is not corroborated in Fig. 3 because the SNR is not high enough

V. Conclusion

We have introduced a general method to analyze the coding gain and optimal rotation angles for QSTBC with arbitrary two-dimensional signal constellations. Our new framework is general, unifies the existing results and delivers some new results. For lattices of equilateral triangles, we proposed optimal rotations of $\phi = \pm\pi/2$ to minimize the encoding complexity by eliminating the multiplications. Optimal rotations for PSK and various constellations are also presented. More important, the coding gain (or its upper bound) of QSTBC can be derived for any constellation. Thus, the performance of the codes can be accurately predicted.

References

[1] S. M. Alamouti, "A simple transmitter diversity scheme for wireless communication," *IEEE J. Select. Areas. Commun.*, vol. 16, pp. 1451–1458, Oct. 1998.
[2] V. Tarokh, H. Jafarkhani, and A. R.Calderbank, "Space-time block codes from orthogonal designs," *IEEE Trans. Inform. Theory*, vol. 45, pp. 1456–1466, July 1999.
[3] H. Jafarkhani, "A quasi-orthogonal space-time block code," *IEEE Trans. Commun.*, vol. 49, pp. 1–4, Jan. 2001.
[4] O. Tirkkonen, A. Boariu, and A. Hottinen, "Minimal nonorthogonality rate 1 space-time block code for 3+ Tx antennas," in *Proc. IEEE 6th Int. Symp. Spread-Spectrum Techniques and Applications (ISSSTA 2000)*, Sept. 2000, pp. 429–432.
[5] C. B. Papadias and G. J. Foschini, "Capacity-approaching space-time codes for systems employing four transmitter antennas," *IEEE Trans. Inform. Theory*, vol. 49, pp. 726–732, Mar. 2003.
[6] O. Tirkkonen, "Optimizing space-time block codes by constellation rotations," in *Proc. Finnish Wireless Communications Workshop FWCW'01*, Oct. 2001, pp. 59–60.
[7] N. Sharma and C. Papadias, "Improved quasi-orthogonal codes through constellation rotation," *IEEE Trans. Commun.*, vol. 51, pp. 332–335, Mar. 2003.
[8] A. Sezgin and E. Jorswieck, "On optimal constellations for quasi-orthogonal space-time codes," in *Proc. IEEE Int. Conf. Acoustics, Speech, and Signal Processing (ICASSP)*, April 2003, pp. IV – 345–8 vol.4.
[9] C. Yuen, Y. L. Guan and T. T. Tjhung, "Full-rate full-diversity STBC with constellation rotation," in *Proc. IEEE Vehicular Technology Conf. (VTC)*, April 22-25 2003, pp. 296 – 300.
[10] W. Su and X.-G. Xia, "Signal constellations for quasi-orthogonal space-time block codes with full diversity," *IEEE Trans. Inform. Theory*, vol. 50, pp. 2331 – 2347, Oct. 2004.
[11] D. Wang and X.-G. Xia, "Optimal diversity product rotations for quasi-orthogonal STBC with MPSK Symbols," *IEEE Commun. Lett.*, vol. 9, pp. 420 – 422, May 2005.
[12] G. D. Forney Jr., R. G. Gallager, G. R. Lang, F. M. Longstaff, and S. U. Qureshi, "Efficient modulation for band-limited channels," *IEEE J. Select. Areas. Commun.*, vol. 2, pp. 632–647, Sep. 1984.
[13] A. Sezgin, E. A. Jorswieck and H. Boche, "Performance criteria analysis and further performance results for quasi-orthogonal space-time block codes," in *Proc. 3rd IEEE Int. Symposium on Signal Processing and Information Technology (ISSPIT)*, Dec. 2003, pp. 102 – 105.
[14] V. Tarokh, N. Seshadri, and A. R. Calderbank, "Space-time codes for high data rate wireless communication: Performance analysis and code construction," *IEEE Trans. Inform. Theory*, vol. 44, pp. 744–765, Mar. 1998.
[15] M. O. Damen, K. Abed-Meraim and J. -C. Belfiore, "Diagonal algebraic space-time block codes," *IEEE Trans. Inform. Theory*, vol. 48, pp. 628 – 636, March 2002.
[16] D. N. Dao and C. Tellambura, "Analyzing the Coding Gain of Quasi-Orthogonal Space-Time Block Codes with Two-Dimensional Constellations," *manuscript in preparation*.
[17] J. H. Conway and N. J. A. Sloane, *Sphere Packings, Lattices and Groups*, 3rd ed. NY: Springer-Verlag, 1998.
[18] R. Lucky and J. Hancock, "On the optimum performance of N-ary systems having two degrees of freedom," *IRE Trans. Commun. Syst.*, vol. 10, pp. 185–192, June 1962.
[19] L. Hanzo, W. Webb, and T. Keller, *Single- and multi-carrier quadrature amplitude modulation : principles and applications for personal communications, WLANs and broadcasting*. New York: Wiley, 2000.
[20] C. Thomas, M. Weidner, and S. Durrani, "Digital Amplitude-Phase Keying with M-Ary Alphabets," *IEEE Trans. Commun.*, vol. 22, pp. 168–180, Feb. 1974.

