PAPR Reduction of OFDM Signals Using Partial Transmit Sequence: An Optimal Approach Using Sphere Decoding

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***Abstract*— Partial transmit sequence (PTS) is a promising technique for peak-to-average-power ratio (PAPR) reduction in orthogonal frequency division multiplexing (OFDM) systems. Computation of optimal PTS weight factors via exhaustive search requires exponential complexity in the number of subblocks; consequently, many suboptimal strategies have been developed to date. In this letter, we introduce an efficient algorithm for computing the optimal PTS weights that has lower complexity than exhaustive search.**

***Index Terms*— OFDM, PAPR, PTS.**

I. Introduction

ORTHOGONAL frequency division multiplexing (OFDM) is emerging as a key technology for 4th generation (4G) cellular networks. It is increasingly held that OFDM results in an improved downlink performance for 4G [1]. However, a drawback is the potentially high peak-to-average power ratio (PAPR) of OFDM signals. PAPR reduction techniques include block coding based on Golay sequences (with dual capabilities of error correction and peak reduction) [2] and clipping/filtering methods. Another such technique that is distortion-free and of low redundancy is partial transmit sequences (PTS) [3], [4]. Computation of optimal PTS weight factors via exhaustive search requires exponential complexity in the number of subblocks; consequently, many suboptimal strategies have been developed to date. In this letter, we introduce an efficient algorithm for computing the optimal PTS weights that has lower complexity than exhaustive search.

Suboptimal PTS strategies include the following. The iterative flipping algorithm (FA) [5] has complexity linearly proportional to the number of subblocks, and each phase factor is individually optimized regardless of the optimal value of other phases (it is an example of greedy algorithm). A neighborhood search is proposed in [6] using gradient descent search. Reference [7] uses dual layered phase sequencing to reduce complexity, at the price of PAPR performance degradation. In [4] a suboptimal strategy is developed by modifying the problem into an equivalent problem of minimizing the sum of phase-rotated vectors. An initial set of phase vectors are computed by reducing the peak amplitude of each sample and the best phase vector of the set is chosen as the final solution. Finally, [8] gives an orthogonal projection-based approach for computing PTS phase factors.

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II. PTS Approach and Lattice Problems

Let $\mathbf{X} = [X_1, \cdots, X_N]^T$ be a block of N symbols being transmitted where each symbol is modulated to one of the carrier frequencies $\{f_n, n = 1, \cdots, N\}$. In OFDM, the N subcarriers are chosen to be orthogonal ($f_n = n\Delta f$) where $\Delta f = 1/NT$ and T is the signal period. The complex envelope of the transmitted signal is

$$x(t) = \frac{1}{\sqrt{N}} \sum_{n=1}^{N} X_n e^{j2\pi f_n t} \quad 0 \leq t < NT \tag{1}$$

where $j = \sqrt{-1}$. The PAPR of the OFDM signal $x(t)$ is defined as

$$\mathrm{PAPR} = \frac{\max |x(t)|^2}{E\left[|x(t)|^2\right]}. \tag{2}$$

Note that $E\left[|x(t)|^2\right] = 1$ for unitary signal constellations. To better approximate the PAPR (2), (1) can be oversampled by generating LN samples, where $L > 1$ is the oversampling factor. When $L = 1$, Nyquist-rate sampling is obtained. These samples can be computed by using appropriate zero-padding and using an inverse fast Fourier transform (IFFT).

In the PTS approach, $\mathbf{X}$ is divided into M disjoint subblocks $\mathbf{X}_m$ $(1 \leq m < M)$ of length U where $N = MU$ for some integers M and U. For $m = 1, \cdots, M$, let $[R_{m,1}, \cdots, R_{m,LN}]^T$ be the zero-padded IFFT of $\mathbf{X}_m$. PTS combines phase-rotated versions of these sub-block IFFTs in order to minimize the PAPR. The signal samples at the PTS output can be written as

$$\mathbf{x}' = \underbrace{\begin{bmatrix} R_{1,1} & R_{1,2} & \cdots & R_{1,M} \\ R_{2,1} & R_{2,2} & & \\ \vdots & & \ddots & \\ R_{LN,1} & & & R_{LN,M} \end{bmatrix}}_{\mathbf{R}} \cdot \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_M \end{bmatrix}}_{\mathbf{b}} \tag{3}$$

where $\mathbf{x}' = [x'_1, \cdots, x'_{LN}]$ is the block of optimized signal samples. The optimization problem is to find optimum phases b_i according to

$$\{b_1^*, \cdots, b_M^*\} = \underbrace{\arg\min}_{\{b_1, \cdots, b_M\}} \left(\max_{1 \leq k < LN} \left| \sum_{m=1}^{M} b_m R_{k,m} \right| \right) \quad (4)$$

where $b_i \in P = \{e^{\frac{j2\pi k}{q}}, k = 0, \cdots, q-1\}$. The last phase factor can be fixed ($b_M = 1$) without loss of generality. Therefore, q^{M-1} distinct possible vectors $\mathbf{b}$ should be tested to solve (4). Accordingly, in an exhaustive search approach, the computational complexity increases exponentially with the number of sub-blocks. In the next section, we propose a new optimization algorithm that solves (4) with lower complexity.

Our new algorithm is motivated by the shortest vector problem (SVP) in a lattice. An M-dimensional lattice is the set of vectors (lattice points) $\{\mathbf{A}\mathbf{b} \,|\, b_i \in \mathbb{Z}\}$, where $\mathbf{b} = (b_1, \ldots, b_M)'$ and the columns of matrix $\mathbf{A} \in \mathbb{R}^{N \times M}$ are called the *basis* for the lattice. The SVP requires finding the shortest non-zero vector in the lattice, where the length can be measured in any l_p ($p \geq 1$) norm. The l_p norm of a vector $\mathbf{x} = (x_1, x_2, \ldots, x_N)$ is defined to be $||\mathbf{x}||_p = (\sum |x_i|^p)^{\frac{1}{p}}$ and $||\mathbf{x}||_\infty = \max_i |x_i|$. Fincke and Phost [9] develop an efficient algorithm for SVP in l_2 (i.e., Euclidean distance) by enumerating all the lattice points inside a sphere centered at the origin. This is one example of sphere decoding which has wide application in communication problems (see [10] for a detailed survey). The signal vectors (3) can be readily interpreted as lattice points generated by $\mathbf{R}$. However, (4) is equivalent to the SVP in l_∞ norm. As such, the original Fincke Phost sphere decoder (FPSD) cannot be directly applied to our problem at hand. Nevertheless, the basic premise of FPSD - to generate *only* lattice points $\mathbf{x}$ for which $||\mathbf{x}||_2 \leq \mu$ - can be adapted; consequently, only lattice points for which $||\mathbf{x}||_\infty \leq \mu$ are generated, and this is equivalent to $|x_k| \leq \mu$ $\forall k$.

III. New FPSD-Based PTS Optimizer

Let $\mathbf{x}' = [x'_1, \cdots, x'_{LN}]$ be defined as in (3) and let $\mathbf{R}_k$ represent the k-th row of the matrix $\mathbf{R}$. Then each element of $\mathbf{x}'$ can be expressed as $x'_k = \mathbf{R}_k \cdot \mathbf{b}$. To find the PAPR of the OFDM signal, the amplitude of x'_k is computed according to

$$\begin{aligned} |x'_k|^2 &= {x'_k}^H \cdot x'_k \\ &= \mathbf{b}^H \cdot \mathbf{R}_k^H \cdot \mathbf{R}_k \cdot \mathbf{b} \\ &= \mathbf{b}^H \cdot \underbrace{\left[\mathbf{R}_k^H \cdot \mathbf{R}_k + \alpha^2 \mathbf{I}\right]}_{\mathbf{A}_k} \cdot \mathbf{b} - \alpha^2 \mathbf{b}^H \cdot \mathbf{b} \\ &= \mathbf{b}^H \cdot \mathbf{A}_k \cdot \mathbf{b} - \alpha^2 M. \end{aligned} \quad (5)$$

where α is an arbitrary nonzero real number and $(\cdot)^H$ denotes conjugate transpose. The resulting $M \times M$ matrix $\mathbf{A}_k$ is positive-definite due to the addition of $\alpha^2\mathbf{I}$, and therefore can be Cholesky factorized as

$$\mathbf{A}_k = \mathbf{Q}_k^H \cdot \mathbf{Q}_k \quad (6)$$

where $\mathbf{Q}_k$ is an upper-triangular matrix. Substituting $\mathbf{A}_k$ from (6) into (5) gives

$$\begin{aligned} |x'_k|^2 &= \mathbf{b}^H \cdot {\mathbf{Q}_k}^H \cdot \mathbf{Q}_k \cdot \mathbf{b} - \alpha^2 M \\ &= \|\mathbf{Q}_k \cdot \mathbf{b}\|^2 - \alpha^2 M \end{aligned} \quad (7)$$

where the signal sample is now a function of the phase vector $\mathbf{b}$. We wish to limit the PAPR (2) to $\mu^2 E\left[|x(t)|^2\right]$ for some positive number μ. The candidate phase vectors can be generated from (7) subject to the following constraint:

$$\left\| \begin{bmatrix} Q_{1,1}^k & \cdots & Q_{1,M}^k \\ 0 & Q_{2,2}^k & \vdots \\ \vdots & \ddots & \\ 0 & 0 & Q_{M,M}^k \end{bmatrix} \cdot \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_M \end{bmatrix} \right\|^2 < \mu^2 + \alpha^2 M \quad (8)$$

for $1 \leq k \leq LN$. Sphere decoding only searches among those candidates that lie inside the sphere of radius $\mu^2 + \alpha^2 M$ and therefore, reduces the complexity of the search. We rewrite (8) as

$$\sum_{v=1}^{M} \left| \sum_{u=v}^{M} Q_{v,u}^k b_u \right|^2 < \mu^2 + \alpha^2 M \quad , \quad 1 \leq k \leq LN. \quad (9)$$

In order to satisfy (9), the following set of inequalities must be satisfied for $1 \leq k \leq LN$:

$$\left| Q_{M,M}^k b_M \right|^2 < \mu^2 + \alpha^2 M, \quad (10)$$

$$\sum_{v=M-1}^{M} \left| \sum_{u=v}^{M} Q_{v,u}^k b_u \right|^2 \mu^2 + \alpha^2 M, \quad (11)$$

$$\sum_{v=M-2}^{M} \left| \sum_{u=v}^{M} Q_{v,u}^k b_u \right|^2 \mu^2 + \alpha^2 M, \quad (12)$$

$$\vdots$$

$$\sum_{v=1}^{M} \left| \sum_{u=v}^{M} Q_{v,u}^k b_u \right|^2 \mu^2 + \alpha^2 M \quad (13)$$

Note that (10) contains b_M only, (11) contains b_{M-1} and b_M only, and so on. We fix $b_M = 1$ without loss of generality. However, (10) constrains the parameter μ (which specifies the achievable PAPR reduction). We use (11) and $b_M = 1$ to generate candidates for b_{M-1}. These candidates and (12) are used to generate candidates for b_{M-2}. This process is repeated until the candidates for the whole phase vector $\mathbf{b}$ are generated. The resulting number of candidates is substantially smaller than q^{M-1}. Therefore, the search space is reduced, compared to exhaustively searching all q^{M-1} phase vectors, which reduces complexity.

TABLE I
NORMALIZED OPTIMIZATION TIME VERSUS μ FOR 512 SUBCARRIER OFDM WHERE $M = 8$ AND $P = \{-1, +1\}$

$\mu^2 = 7$	$\mu^2 = 8$	$\mu^2 = 9$	$\mu^2 = 10$	$\mu^2 = 11$
0.9427	0.6455	0.8408	0.9251	0.9659

TABLE II
NORMALIZED OPTIMIZATION TIME VERSUS μ FOR 512 SUBCARRIER OFDM WHERE $M = 4$ AND $P = \{\pm 1, \pm j\}$

$\mu^2 = 7$	$\mu^2 = 8$	$\mu^2 = 9$	$\mu^2 = 10$	$\mu^2 = 11$
0.7294	0.5106	0.5576	0.9082	0.9624

IV. SIMULATION RESULTS

Computer simulation is used to compare the performance of our algorithm with that of exhaustive search and the suboptimal FA [5]. We simulate OFDM signals with 512 8-PSK subcarriers and an oversampling factor $L = 4$ is used. We arbitrarily choose $\alpha = \sqrt{\frac{1}{M}}$.

Fig. 1 compares the complementary cumulative density functions (CCDF's) of the PAPR where the number of subblocks in PTS is 8 ($M = 8$). Also, the PTS phase factors are chosen from $P = \{+1, -1\}$. Note also that both our proposed algorithm and exhaustive search perform identically, verifying that our proposed algorithm is optimal, resulting in approximately 1-dB additional reduction compared to the FA.

Tables 1 and 2 show the execution time of the proposed algorithm averaged over 10,000 OFDM symbols and normalized with respect to the exhaustive search time, which is constant in each setup (because all possible values are tested). Simulation was done using MATLAB on a Pentium IV Intel processor with 512MB RAM and 2.4 GHz clock speed. Table 1 shows the normalized optimization time versus the square of the constructed sphere radius (μ^2). The results show that the computational time of the proposed optimizer can be as low as 65% of that of an exhaustive optimizer.

CCDF's for the case that the number of subblocks in PTS is 4 ($M = 4$) is shown in Fig. 2. The PTS phase factors are chosen from $P = \{\pm 1, \pm j\}$. Again, the results show that both the proposed algorithm and exhaustive search have identical performance in PAPR reduction. Note that they both give additional PAPR reduction compared to the FA.

Table 2 depicts the normalized optimization time required by the proposed algorithm. The results show that the complexity of the proposed optimizer can be reduced with an appropriate choice of μ^2. Note that nearly 50% complexity saving is obtained in this case when $\mu^2 = 8$.

V. CONCLUSION

In this paper, we proposed a novel algorithm for computing the optimal PTS phase factors. Only those phase vectors that guarantee that the PAPR is bounded are searched by our algorithm. It is based on the Fincke and Phost SD. Computer simulation results show that our algorithm provides optimal PAPR reduction and with lower complexity compared to exhaustive search.

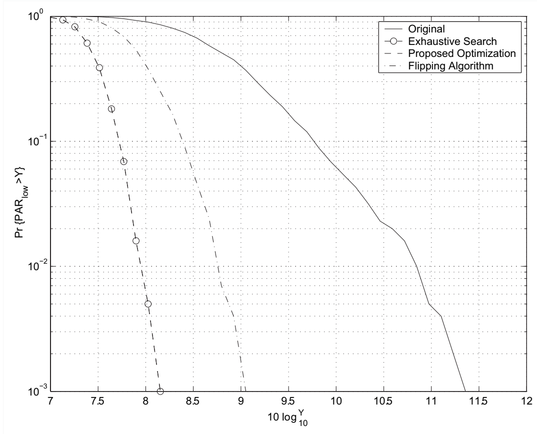

Fig. 1. CCDF of the PAPR for exhaustive search, proposed optimizer and flipping algorithm. $M = 8$.

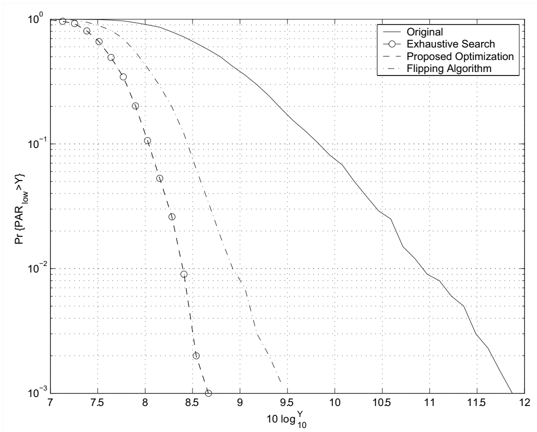

Fig. 2. CCDF of the PAPR for exhaustive search, proposed optimizer and flipping algorithm.

REFERENCES

[1] M. Quan and P. Mannion. (2000, Sept.) AT&T, Nortel plot 4G wireless nets. [Online]. Available: http://www.eet.com/story/OEG20000915S0034/
[2] J. A. Davis and J. Jedwab, "Peak-to-mean power control in OFDM, Golay complementary sequences, and Reed-Muller codes," *IEEE Trans. Inform. Theory*, vol. 45, pp. 2397–2417, Nov. 1999.
[3] S. H. Müller and J. B. Huber, "OFDM with reduced peak-to-average power ratio by optimum combination of partial transmit sequences," *IEE Elect. Lett.*, vol. 33, pp. 368–369, Feb. 1997.
[4] C. Tellambura, "Improved phase factor computation for the PAR reduction of an OFDM signal using PTS," *IEEE Commun. Lett.*, vol. 5, pp. 135–137, Apr. 2001.
[5] L. J. Cimini and N. R. Sollenberger, "Peak-to-average power ratio reduction of an OFDM signal using partial transmit sequences," *IEEE Commun. Lett.*, vol. 4, pp. 86–88, Mar. 2000.
[6] S. H. Han and J. H. Lee, "PAPR reduction of OFDM signals using a reduced complexity PTS technique," *IEEE Sig. Process. Lett.*, vol. 11, pp. 887–890, Nov. 2004.
[7] W. S. Ho, A. S. Madhukumar, and F. Chin, "Peak-to-average power reduction using partial transmit sequences: a suboptimal approach based on dual layered phase sequencing," *IEEE Trans. Veh. Technol.*, pp. 1268–1272, Apr. 2003.
[8] H. Chen and G. J. Pottie, "An orthogonal projection-based approach for PAR reduction in OFDM," *IEEE Commun. Lett.*, vol. 6, pp. 169–171, May 2002.
[9] U. Fincke and M. Pohst, "Improved methods for calculating vectors of short length in a lattice, inlcuding a complexity analysis," *Math Computation*, vol. 44, pp. 463–471, Apr. 1985.
[10] W. H. Mow, "Universal lattice decoding: principle and recent advances," *Wireless Commun. and Mobile Computing*, vol. 3, pp. 553–569, Aug. 2003.