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An Efficient Generalized Sphere Decoder for Rank-Deficient MIMO Systems

Tao Cui and Chintha Tellambura, *Senior Member, IEEE*

***Abstract*— We derive a generalized sphere decoder (GSD) for rank-deficient Multiple Input Multiple Output (MIMO) systems using N transmit antennas and M receive antennas. This problem arises when $N > M$ or when the channel gains are strongly correlated. The upper triangular factorization of the Grammian yields an under-determined system and the standard sphere decoding (SD) fails. For constant modulus constellations, we modify the maximum likelihood (ML) cost metric so that the equivalent Grammian is rank N. The resulting GSD algorithm has significantly lower complexity than previous algorithms. A method to handle non-constant modulus constellations is also developed.**

***Index Terms*— Sphere decoding, MIMO, rank deficiency.**

I. Introduction

Space-time wireless technology using multiple antennas provides very high data rates with low error probability. In this letter, we consider ML detection for rank-deficient multi-antenna systems with N transmit antennas and M receive antennas, denoted by MIMO (N, M). ML decoding of the N symbols transmitted per symbol period over the MIMO channel is an integer least-squares (LS) problem, which is known to be NP-hard. Common, reduced-complexity algorithms such as V-BLAST [1] perform much worse than the true ML detection. Sphere decoder (SD) algorithms [2]–[4] however achieve ML performance by providing an efficient way for generating all candidate solutions that lie inside a hypersphere defined by the channel matrix and the received signal vector. In the high signal-to-noise ratio (SNR) region, the radius of the sphere can be chosen small enough so that only few candidates are found inside the sphere. This search space is therefore drastically smaller than the ML search space which consists of q^N points for a q-ary signal constellation. Thus, the number of operations (complexity) is roughly polynomial in N in the high SNR region [5].

The QR decomposition of the $M \times N$ channel matrix or, equivalently, the Cholesky decomposition of the grammian is required. When $M \geq N$, these matrices have rank N and the standard SD algorithm is readily applicable. However, for $N > M$ (i.e., the downlink detection in MIMO systems) or for highly correlated MIMO channels [6], the matrix rank is less than the number of symbols to be estimated. A generalized SD (GSD) algorithm for this case has been derived in [4], [7]. The signal vector (to be estimated) and the channel matrix are partitioned and M symbols are estimated using SD for all q^{N-M} combinations of the remaining $N - M$ symbols. The complexity is exponential in $(N-M)$ regardless of the SNR. Recently, [8] also presented a fast GSD for rank-deficient MIMO systems by partitioning the subspace of the remaining $N - M$ symbols for an efficient search. In this letter, we utilize a different concept where the ML metric can be advantageously modified for a constant modulus constellation. We thus derive the new algorithm assuming the transmit symbols to be constant modulus. We later show how MIMO systems with non-constant modulus constellations can be adapted so that our algorithm is applicable.

Notation: Bold symbols denote matrices or vectors. $(\cdot)^T$ and $(\cdot)^H$ denote transpose and conjugate transpose, $(\cdot)^\dagger$ denotes pseudo-inverse. The set of all complex $K \times 1$ vectors is denoted by $\mathcal{C}^K$. For 2^q-ary phase shift keying (PSK), the signal constellation $\mathcal{Q}_{2^q} = \{e^{j2\pi k/2^q}, k = 0, 1, \cdots, 2^q - 1\}$ and all PSK $N \times 1$ vectors are denoted by $\mathcal{Q}_{2^q}^N$. 4-ary PSK is commonly known as quadrature PSK (QPSK). If x and y are Gaussian with $E[x] = \mu_x$, $E[y] = \mu_y$ and $E\left[(x-\mu_x)^2\right] = E\left[(y-\mu_y)^2\right] = \sigma^2/2$, then $z = x + jy$ (where $j = \sqrt{-1}$) is Complex Gaussian. We write $z \sim \mathcal{CN}(\mu_x + j\mu_y, \sigma^2)$ in this case. The $N \times N$ identity matrix is denoted by $\mathbf{I}_N$.

II. System Model

The complex baseband input-output relationship of a MIMO(N, M) system can be described in vector notation as

$$\mathbf{y} = \mathbf{H}\mathbf{x}_t + \mathbf{n} \tag{1}$$

where $\mathbf{x}_t \in \mathcal{Q}^N$ is the transmit vector, $\mathbf{y} \in \mathcal{C}^M$ is the receive vector, $\mathbf{H} = [h_{ij}]$ is the $M \times N$ channel matrix and $h_{ij} \sim \mathcal{CN}(0, 1)$ for $i = 1, \cdots, M$ and $j = 1, \cdots, N$ and $\mathbf{n} \in \mathcal{C}^M$ is the additive white Gaussian noise (AWGN) vector with variance σ_n^2. The receiver has perfect knowledge of the random channel matrix $\mathbf{H}$, which remains constant over one or more symbol periods (slow fading). The transmitted vector over the linear MIMO channel (1) is ML detected as

$$\hat{\mathbf{x}}_t = \arg\min_{\mathbf{x} \in \mathcal{Q}^N} \|\mathbf{y} - \mathbf{H}\mathbf{x}\|^2. \tag{2}$$

III. Sphere Decoding Technique

Solving (2) is equivalent to

$$\hat{\mathbf{x}} = \arg\min_{\mathbf{x} \in \mathcal{Q}^N} \|\mathbf{R}(\mathbf{H}^\dagger\mathbf{y} - \mathbf{x})\|^2 \tag{3}$$

where $\mathbf{R}$ is an upper triangular matrix such that $\mathbf{R}^H\mathbf{R} = \mathbf{H}^H\mathbf{H}$. Sphere decoding [2]–[4] provides an efficient recursive method for computing all $\mathbf{x}$ such that $\|\mathbf{R}(\mathbf{H}^\dagger\mathbf{y} - \mathbf{x})\|^2 < r^2$. If $M \geq N$, $\mathbf{H}^H\mathbf{H}$ is full rank (positive definite). The SD is applicable due to the non-zero diagonal terms of $\mathbf{R}$.

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The authors are with the Department of Electrical and Computer Engineering, University of Alberta, Edmonton, AB, Canada (e-mail: {taocui, chintha}@ece.ualberta.ca).

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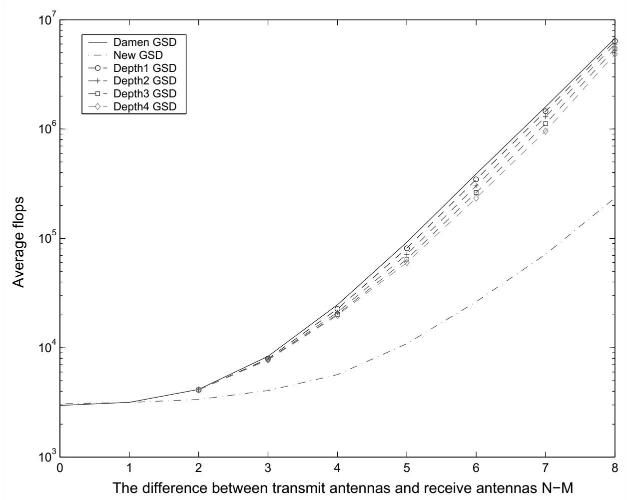

Fig. 1. The average complexity versus $N - M$.

However, if $M < N$, $\mathbf{H}^H\mathbf{H}$ is positive semidefinite. The Cholesky factor $\mathbf{R}$ of $\mathbf{H}^H\mathbf{H}$ is not full rank and only the first M rows of $\mathbf{R}$ are non-zero. The GSD developed in [4], [7] can solve this case. Since the elements of $\mathbf{x}$ are of constant modulus, the product $\alpha\mathbf{x}^H\mathbf{x}$ is equal to αN and we get an equivalent minimization problem as

$$\begin{aligned}\hat{\mathbf{x}} =& \arg\min_{\mathbf{x}\in\mathcal{Q}^N} \|\mathbf{y} - \mathbf{H}\mathbf{x}\|^2 + \alpha\mathbf{x}^H\mathbf{x} \\ =& \arg\min_{\mathbf{x}\in\mathcal{Q}^N} \mathbf{y}^H\mathbf{y} - \mathbf{y}^H\mathbf{H}\mathbf{x} - \mathbf{x}^H\mathbf{H}^H\mathbf{y} + \mathbf{x}^H(\mathbf{H}^H\mathbf{H} + \alpha\mathbf{I}_N)\mathbf{x}.\end{aligned} \tag{4}$$

Let the positive definite matrix $\mathbf{G} = \mathbf{H}^H\mathbf{H} + \alpha\mathbf{I}_N$ be Cholesky factorized as $\mathbf{G} = \mathbf{D}^H\mathbf{D}$, where $\mathbf{D}$ is an upper triangular matrix (i.e., $\mathbf{D} = [d_{ij}]$ and $d_{ij} = 0$ for $i > j$). Define $\rho = \mathbf{G}^{-1}\mathbf{H}^H\mathbf{y}$ and add $\rho^H\mathbf{D}^H\mathbf{D}\rho$ to the both sides of (4). An equivalent optimization problem can be represented as

$$\hat{\mathbf{x}} = \arg\min_{\mathbf{x}\in\mathcal{Q}^N} \|\mathbf{D}(\rho - \mathbf{x})\|^2. \tag{5}$$

Now the diagonal terms of $\mathbf{D}$ (rank N) are all non-zero and the original sphere decoding can be applied to (5). Note (3) and (5) are equivalent only for constant modulus signals. The new radius for (5) becomes

$$r_1^2 = r^2 + \alpha\mathbf{x}^H\mathbf{x} + \mathbf{y}^H\mathbf{H}\mathbf{G}^{-1}\mathbf{H}^H\mathbf{y} - \mathbf{y}^H\mathbf{y}. \tag{6}$$

When $\alpha \to 0$, it can be easily verified $r_1^2 \doteq r^2 - \alpha\mathbf{n}^H\mathbf{H}(\mathbf{H}^H\mathbf{H})^{-2}\mathbf{H}\mathbf{n}$.

Remarks

- A tree of N levels and $\frac{q^{N+1}-1}{q-1}$ nodes can represent the entire solution space $\mathcal{Q}^N$. The basic SD approach is to prune as many nodes of the tree as possible without eliminating the optimal solution. The key difference between the GSD (5) and that of [7] is as follows. First, the derived gram matrix $\mathbf{D}$ is positive definite. Second, the original GSD [7] cannot prune any nodes in the first $N - M$ levels while the proposed GSD (5) can prune from the first level. For large $N - M$, the new GSD can prune more nodes and hence has lower complexity.
- The complexity of the new GSD depends on the choice of α. If $\alpha = 0$, the new GSD reverts to the GSD in [7]. There exists an optimal α that leads to the lowest complexity. It appears impossible to derive a closed-form expression for the optimal α, which depends on the variance of the additive noise and $N - M$. In simulation, we test $\alpha = 1$ and $\alpha = \sigma_n^2$, respectively. We find even with these arbitrary choices of α the new GSD has less complexity than the previous GSDs [7], [8].
- Recently, Hassibi and Vikalo [5] showed that SD has polynomial complexity in the high SNR region with $M \geq N$. This is not true for any previous GSDs [7], [8]. The GSD (5) also has complexity exponential in $N - M$.
- The proposed GSD does not depend on the rank of matrix $\mathbf{H}$. Thus, the original SD algorithm or its derivatives can be used regardless of $N \leq M$ or $N > M$.
- As the proposed GSD exploits the constant modulus property of the transmitted signals and other algorithms do not make use of this information, we expect the proposed GSD to have low complexity.
- Like the GSD in [7], the proposed GSD can also be used for vector quantization and multiuser detection.

Next, we show how the GSD (5) can also detect non-constant modulus. When N is very large, $\mathbf{x}^H\mathbf{x}$ becomes the time average of x_i times N using the weak law of large numbers and is still a constant. The proposed constant modulus GSD can also be used. If N is small, we note that any M-ary Quadrature Amplitude Modulation (M-QAM) ($M = 2^n$) constellation can be represented as a weighted sum of $n/2$ QPSK constellations when n is an even number [9]. That is, for $z \in M$-QAM and $z_i \in$ QPSK, $0 \leq i < n/2$, we have

$$z = \sum_{i=0}^{\frac{n}{2}-1} 2^i \left(\frac{\sqrt{2}}{2}\right) z_i. \tag{7}$$

For brevity, we only show how to decompose 16QAM so that our algorithm is applicable. Other M-QAM cases can be derived similarly. Using (7), the 16QAM transmit vector $\mathbf{x}$ can be expressed as

$$\mathbf{x} = \sqrt{2}\mathbf{x}_1 + \frac{\sqrt{2}}{2}\mathbf{x}_2 \tag{8}$$

where $\mathbf{x}_1, \mathbf{x}_2 \in \mathcal{Q}_4^N$. Eq. (1) can then be represented as

$$\begin{aligned}\mathbf{y} =& \begin{bmatrix} \sqrt{2}\mathbf{H} & \frac{\sqrt{2}}{2}\mathbf{H} \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix} + \mathbf{n} \\ =& \tilde{\mathbf{H}}\tilde{\mathbf{x}} + \mathbf{n}.\end{aligned} \tag{9}$$

That is, an MIMO(N, M) system with 16QAM is equivalent to MIMO($2N, M$) with QPSK. The new GSD (5) can be used with (8).

IV. Numerical Results

We consider a frequency flat fading MIMO channel defined by $\mathbf{H} = [h_{ij}]$. We incorporate the so-called Schnorr-Euchner strategy [3], [4] into the GSD in [7] and the new GSD with $\alpha = 1$. Fig. 1 compares the computational complexity

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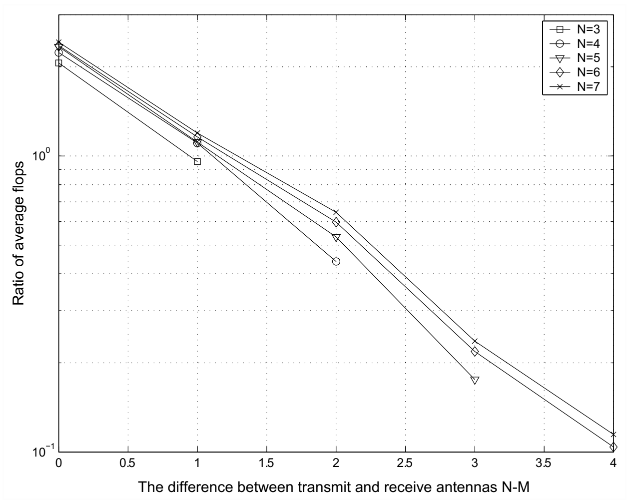

Fig. 2. The complexity ratio $\frac{\text{Average flops of New GSD}}{\text{Average flops of Damen GSD}}$ for 16QAM with $N = 3, \ldots, 7$.

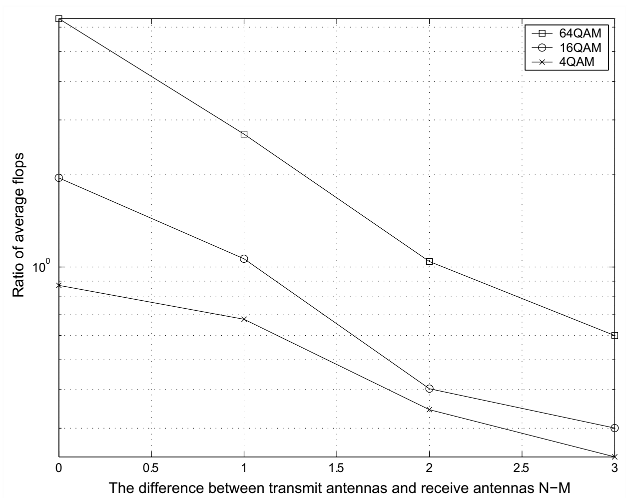

Fig. 3. The complexity ratio $\frac{\text{Average flops of New GSD}}{\text{Average flops of Damen GSD}}$ for 64QAM, 16QAM and 4QAM with $N = 4$.

of the proposed sphere decoder with those of [7] and [8] with different depths for fixed $N = 15$ and QPSK at an SNR of 20dB. We simulate this system using MATLAB V5.3 on a PC with an Intel Pentium-4 processor at 1.8GHz. The matlab command "flops" is used to count the number of flops. Only the flops of the search algorithm are counted without accounting for the prepossessing stage. The Depth$i$ GSD in Fig. 1 denotes the GSD in [8]. The new sphere decoder has significantly lower decoding complexity than the GSD in [7] and [8], especially when the difference between N and M is large. For $N - M = 8$, the average number of flops are 6.76×10^6 for the GSD in [7] and 2.35×10^5 for GSD (5).

Fig. 2 shows the complexity ratio between the new GSD and the Damen GSD with $\alpha = \sigma_n^2$ for 16QAM, $N = 3, \ldots, 7$ and $M = N - 1, \ldots, 2$ at an SNR of 20dB. We apply the V-BLAST reordering [4] to the new GSD and the complexity of reordering is not counted. Note that this reordering is not possible with the Damen GSD. Since the new GSD enlarges the dimension (9) for non-constant modulus constellations such as 16QAM, when the order of rank deficiency $N - M$ is small, the complexity of new GSD is larger than that of the Damen GSD, e.g., $N - M < 2$ (Fig. 2). However when $N - M$ increases, the new GSD reduces the complexity significantly and is more efficient.

The complexity ratio between the new GSD and the Damen GSD with $\alpha = \sigma_n^2$ for 4QAM, 16QAM and 64QAM with fixed $N = 4$ and $M = 1, \ldots, N$ at an SNR of 20dB is given in Fig. 3. For 4QAM, 16QAM and 64QAM, our new GSD is more efficient when $N - M$ is equal or larger than 1, 2 and 3, respectively. Since our GSD enlarges the matrix dimensions when handling non-constant modulations, it increases the complexity. So when $N - M$ is small, this complexity increase outweighs the savings achieved by avoiding exhaustively searching of the $N - M$ complementary dimensions (e.g., the Damen GSD).

V. Conclusion

For rank-deficient multiple antenna systems, we have developed a new sphere decoder assuming constant modulus signalling alphabets. This assumption is not limiting because a QAM constellation can be represented as a linear sum of QPSK points. The new GSD is ML and has significantly lower decoding complexity than the Damen GSD [4], [7] and the improved GSD [8] for constant modulus constellations. For higher-order constellations such as 16QAM, our GSD is computationally more efficient when $N - M$ is sufficiently large. The proposed GSD can also be extended to frequency selective MIMO channels.

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