Novel Nakagami-m Parameter Estimator for Noisy Channel Samples

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***Abstract*—A novel moment-based m parameter estimator using noisy channel samples is derived. This estimator is simpler than known estimators. Numerical results are presented to demonstrate that, under some practical fading conditions, it outperforms previous estimators.**

***Index Terms*—Fading channels, Nakagami fading, parameter estimation.**

I. Introduction

The Nakagami-m distribution is one of the most widely used fading channel models in wireless communications. It describes the fast channel amplitude changes which occur in many wireless transmission environments [1]. The probability density function (PDF) of the Nakagami-m distribution is

$$f_R(r) = \frac{2r^{2m-1}}{\Gamma(m)}(\frac{m}{\Omega})^m e^{-\frac{mr^2}{\Omega}}, r \geq 0 \tag{1}$$

where $m = \frac{\Omega^2}{E\{(r^2-\Omega)^2\}}$ is the fading measure with $m \geq 0.5$ and $\Omega = E\{r^2\}$ is the second moment [1]. The parameter, m, indicates various fading conditions. For example, when $m = 0.5$, it represents a deeply fading channel. When $m = 1$, it represents a Rayleigh fading channel. When $m = \infty$, it represents a static channel without any fading. Since the value of m measures the channel quality, it is of great importance to obtain an accurate estimate of m, in advanced receiver implementations and in channel data analyses.

Estimation of m has been studied previously by several researchers [2]- [7]. In [2]- [5], noiseless channel samples (unavailable in a practical system) were used. In [6] and [7], noisy channel samples assuming knowledge of the fading phases were used, and the derived estimators require a sample size as high as 10,000 to achieve reliable performances. In [8], a simpler but better moment-based estimator for m using noisy channel samples was developed. This estimator uses exact expressions for the moments of the noisy samples, which are expressed using the hypergeometric function. As a result, a moment of order as high as four has to be used and the estimator performs poorly when the noise is large. It is well known that, generally speaking, the lower the orders of the moments used, the better the performance of the moment-based estimator will be. Thus, one expects that one can improve the performance of the estimator in [8] by using moments of lower orders. In this letter, we derive a new moment-based estimator for m using the first and the second order moments of the noisy samples. In order to do this, the hypergeometric function is approximated by a polynomial. This approximation was previously used in the context of efficient generation of correlated Rayleigh envelope samples where the moments of the amplitude of the envelope are expressed in terms of the hypergeometric function [9]. Numerical results show that this estimator achieves better performance.

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II. New m Parameter Estimator

We use the same system model as that in [8]. It was derived in [8] that the n-th order moment of the noisy channel sample, z_i, satisfies

$$\mu_n = (2\sigma^2)^{\frac{n}{2}}\Gamma(\frac{n}{2}+1)\left(\frac{m}{\gamma+m}\right)^m \cdot F(m, \frac{n}{2}+1; 1; \frac{\gamma}{\gamma+m}) \tag{2}$$

where $F(\cdot,\cdot;\cdot;\cdot)$ is the hypergeometric function [10, p. 556], $\gamma = \frac{\Omega}{2\sigma^2}$ is the average signal-to-noise ratio (ASNR), $2\sigma^2 = \frac{N_0}{E}$ is the inverse of the transmitted-signal-to-noise ratio (TSNR), E is the transmitted signal energy and N_0 is the noise power spectral density. The value of N_0 can be obtained using an independent estimator for N_0. For example, it can be estimated by using an in-band measurement of the noise when a "zero" signal symbol is sent, an out-of-band measurement of the noise, or a quadrature channel measurement in BPSK systems. Furthermore, we examine here the sensitivity of the new estimator to errors in the estimate of N_0. Thus, we consider the case when the measurement filter or the estimator are well-designed and the measurement or the estimate of N_0 is accurate enough such that the effect of the measurement error or estimation error of N_0 on m parameter estimation is negligible, and therefore, we assume known N_0 [8]. Using the second-order moment, an estimator for Ω can be derived from (2) as

$$\hat{\Omega} = \frac{1}{N}\sum_{i=1}^{N} z_i^2 - 2\sigma^2 \tag{3}$$

where N is the number of independent and identically distributed samples used in the estimation and z_i is the i-th noisy sample. In moment-based estimation, μ_n is usually approximated by $\hat{\mu}_n = \frac{1}{N}\sum_{i=1}^{N} z_i^n$. The moment-based estimators for m can be derived by using (2) and (3). The main difficulty lies in the fact that the hypergeometric function is complicated.

TABLE I
THE DERIVED POLYNOMIAL COEFFICIENTS.

C_0^0	C_0^1	C_0^2
0.288300	-0.129765	-0.000005
C_1^0	C_1^1	C_1^2
-0.776352	1.124807	0.000012
C_2^0	C_2^1	C_2^2
0.000304	0.000066	-0.000000025

In [8], this was solved by using recurrence relationships of the hypergeometric function. As a result, a moment of order as high as four has to be used. Here, we approximate the hypergeometric function enabling us to obtain estimators using lower orders of moments.

Denote two real numbers as p and q . From (2), one has

$$\begin{aligned} f_{p,q}(m,\gamma) &= \frac{F(m,\frac{p}{2}+1;1;\frac{\gamma}{\gamma+m})}{F(m,\frac{q}{2}+1;1;\frac{\gamma}{\gamma+m})} h_1(m)h_2(\gamma) \\ &= (2\sigma^2)^{\frac{q-p}{2}} \frac{\mu_p}{\mu_q} \cdot \frac{\Gamma(\frac{q}{2}+1)}{\Gamma(\frac{p}{2}+1)} h_1(m)h_2(\gamma) \quad (4) \end{aligned}$$

where $h_1(m)$ and $h_2(\gamma)$ are polynomials in m and γ, respectively. We have found by numerical experiments that, for most values of p and q, there exist some $h_1(m)$ and $h_2(\gamma)$ such that $f_{p,q}(m,\gamma)$ can be closely approximated by a polynomial in both m and γ, $g_{p,q}(m,\gamma)$. This suggests that one can derive moment-based estimators for m by approximating $f_{p,q}(m,\gamma)$ with $g_{p,q}(m,\gamma)$ and solving the resulting equation for m. To show how the approximation method can be applied to obtain moment-based estimators for m, we discuss the case when $p=1$ and $q=2$ in the following.

First, we need to determine the polynomials of $h_1(m)$ and $h_2(\gamma)$ such that $f_{p,q}(m,\gamma)$ can be well approximated by a polynomial. Using [11, eq. (9.121.1)], [11, eq. (9.137.11)] and [11, eq. (9.131.1)], one has

$$F(m,2;1;\frac{\gamma}{\gamma+m}) = (\gamma+1)\left(\frac{m}{\gamma+m}\right)^{-m} \quad (5)$$

$$F(m,\frac{3}{2};1;\frac{\gamma}{\gamma+m}) = \left(\frac{m}{\gamma+m}\right)^{-m} F(m,-\frac{1}{2};1;-\frac{\gamma}{m}).$$

By applying the series expansion of the hypergeometric funtion in $F(m,-\frac{1}{2};1;-\frac{\gamma}{m})$, one also has [11, eq. (9.100)]

$$\frac{F(m,\frac{3}{2};1;\frac{\gamma}{\gamma+m})}{F(m,2;1;\frac{\gamma}{\gamma+m})} = \sum_{i=0}^{\infty} \frac{a_i}{m^i} \quad (6)$$

where a_i $(i = 0,1,\cdots,\infty)$ are constants independent of m. For large values of m, $\frac{F(m,1.5;1;\frac{\gamma}{\gamma+m})}{F(m,2;1;\frac{\gamma}{\gamma+m})} \approx a_0 + a_1\frac{1}{m}$, and therefore, $h_1(m) = m$ can be chosen. Also, numerical experiments show that $h_2(\gamma) = \gamma^{\frac{3}{2}}$ will make $f_{p,q}(m,\gamma)$ an approximately linear function of γ for fixed values of m.

Second, we approximate the function

$$f_{1,2}(m,\gamma) = \frac{m\gamma^{\frac{3}{2}} F(m,\frac{3}{2};1;\frac{\gamma}{\gamma+m})}{F(m,2;1;\frac{\gamma}{\gamma+m})} \quad (7)$$

with a polynomial. Since both m and γ are unknown, this is actually a two-dimensional surface-fitting problem. We use a

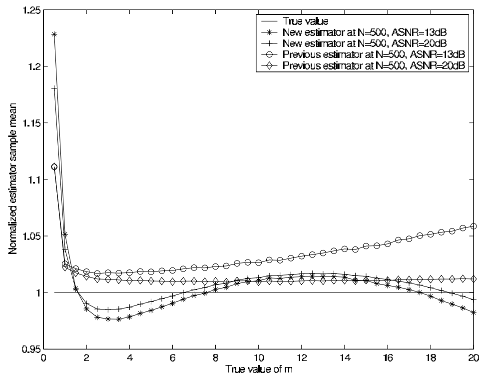

Fig. 1. Normalized sample mean of the new estimator in a noisy Nakagami-m fading channel for $N = 500$.

fourth-order polynomial

$$\begin{aligned} g_{1,2}(m,\gamma) &= C_0^0 + C_0^1\gamma + C_0^2\gamma^2 + (C_1^0 + C_1^1\gamma + C_1^2\gamma^2)m \\ &\quad +(C_2^0 + C_2^1\gamma + C_2^2\gamma^2)m^2 \quad (8) \end{aligned}$$

where C_0^0, C_0^1, C_0^2, C_1^0, C_1^1, C_1^2, C_2^0, C_2^1, C_2^2 are coefficients to be determined. We have found that polynomials with higher or lower orders won't provide simpler estimators with better performances in all examples tested. By applying the least squares method, one determines the coefficients shown in Table 1.

Finally, we solve the equation to obtain moment-based estimators for m. One has from the preceding results that

$$\frac{2\hat{\gamma}^{\frac{3}{2}}\sqrt{2\sigma^2}\hat{\mu}_1}{\sqrt{\pi}\hat{\mu}_2} m \approx g_{1,2}(m,\hat{\gamma}) \quad (9)$$

where the true value of γ is replaced by its estimate, $\hat{\gamma}$. Solving the equation for m, a moment-based estimator for m can be derived as

$$\hat{m}_1 = \frac{-b+\sqrt{b^2-4ac}}{2a} \quad (10)$$

where $a = C_2^0 + C_2^1\hat{\gamma} + C_2^2\hat{\gamma}^2$, $b = C_1^0 + C_1^1\hat{\gamma} + C_1^2\hat{\gamma}^2 - \frac{2\hat{\gamma}^{\frac{3}{2}}\sqrt{2\sigma^2}\hat{\mu}_1}{\sqrt{\pi}\hat{\mu}_2}$, $c = C_0^0 + C_0^1\hat{\gamma} + C_0^2\hat{\gamma}^2$, $\hat{\gamma} = \frac{\hat{\Omega}}{2\sigma^2}$ and $\hat{\Omega}$ is obtained from (3).

Following similar procedures as previously, m parameter estimators using other values of p and q can also be derived. It seems not possible to derive simple estimators using moments of orders lower than $p = 1$ and $q = 2$, since both Ω and m are unknown . Note that the use of $h_1(m)$ and $h_2(\gamma)$ makes $f_{p,q}(m,\gamma)$ as linear as possible before approximation, and therefore, eases the approximation. Note further that $f_{p,q}(m,\gamma)$ can also be approximated by an exponential or a ratio of polynomials. However, an exponential or rational approximation is much more complex than a polynomial approximation. In addition, these approximations usually don't provide estimators with explicit forms. The moment-based estimator for m derived in [8] is

$$\hat{m}_2 = \frac{a_2(b_1c_2 - b_2c_1) + b_2(a_2c_1 - a_1c_2)}{c_2(b_2c_1 - b_1c_2)} \quad (11)$$

where $a_1 = \hat{\mu}_2 - 2\sigma^2$, $b_1 = 6\sigma^2\hat{\mu}_2 - 4\sigma^4 - \hat{\mu}_4$, $c_1 = \hat{\mu}_2$, $a_2 = \hat{\mu}_1 - \sigma^2\hat{\mu}_{-1}$, $b_2 = 8\sigma^2\hat{\mu}_1 - 2\hat{\mu}_3 - 2\sigma^4\hat{\mu}_{-1}$ and $c_2 = 2\hat{\mu}_1$.

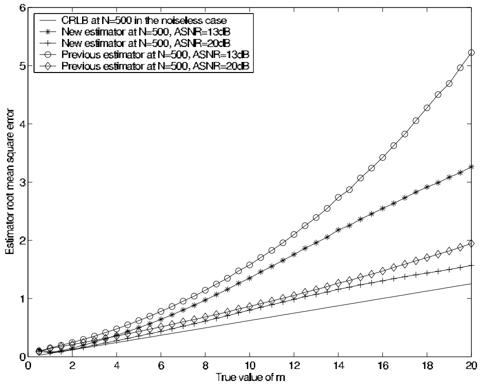

Fig. 2. Root-mean-square-error of the new estimator in a noisy Nakagami-m fading channel for $N = 500$.

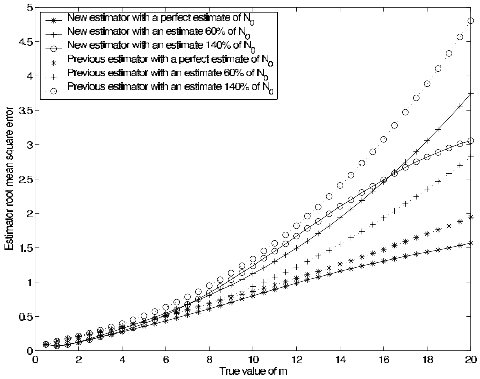

Fig. 3. Root-mean-square-error of the new estimator in a noisy Nakagami-m fading channel at $N = 500$ and $ASNR = 20\,dB$ when N_0 estimates of different accuracies are used.

III. Simulation Results and Discussion

In this section, the performance of $\hat{m}_1$ is examined and compared with that of $\hat{m}_2$. The true value of Ω is set equal to 20. The values of ASNR considered are 20 dB ($\sigma^2 = 0.1$) and 13 dB ($\sigma^2 = 0.5$). A sample size of $N = 500$ is used. Numerical trials suggest that smaller sample sizes won't provide reliable estimator performances in this case. The true value of m varies from 0.5 to 20 in increments of 0.5, as in [8].

Fig. 1 shows the normalized sample mean of $\hat{m}_1$ at $N = 500$. The new estimator has a bias between +5.0% and -2.0% when $ASNR = 13\,dB$, and between +4.0% and -1.5% when $ASNR = 20\,dB$, for $1.0 \leq m \leq 20.0$. Therefore, the estimator bias performance improves little as the ASNR increases in these cases. Comparing $\hat{m}_1$ with $\hat{m}_2$, one sees that their bias performances are similar for $1.0 \leq m \leq 20.0$. Calculation shows that their relative difference (the absolute value of the difference between the sample means of $\hat{m}_1$ and $\hat{m}_2$ divided by the corresponding true value) is less than 7% in all the cases considered, quantitively demonstrating that the biases are of the same order. However, $\hat{m}_2$ outperforms $\hat{m}_1$ for $m < 1.0$, owing to large approximation errors in $\hat{m}_1$ at small values of m.

Fig. 2 shows the root-mean-square-error (RMSE) of $\hat{m}_1$. Since a Cramér-Rao lower bound (CRLB) for the noisy channel is not available, the CRLB for the noiseless channel is used as a benchmark. One sees that, at $m = 20$, the root-mean-square-error of the estimator is about 3.2 when $ASNR = 13\,dB$ and about 1.6 when $ASNR = 20\,dB$. Therefore, the root-mean-square-error of $\hat{m}_1$ decreases as the ASNR increases. Comparing $\hat{m}_1$ with $\hat{m}_2$, one sees that $\hat{m}_1$ has a much smaller root-mean-square-error than $\hat{m}_2$. The difference increases as m increases. This is expected, as the approximation error in $\hat{m}_1$ decreases when m increases. At large values of m, the approximation error will be negligible, and the highest order of moment used in the estimation will become dominant. One also sees from Figs. 1 and 2 that a sample size of 500 and a SNR of $20\,dB$ are enough to achieve good performance in this case. Fig. 3 shows the effect of imperfect estimates of N_0 on $\hat{m}_1$ and $\hat{m}_2$. The RMSE performances of both $\hat{m}_1$ and $\hat{m}_2$ degrade as estimation errors occur in the estimation of N_0. In particular, $\hat{m}_1$ is more sensitive to a negative bias in the estimation of N_0 than $\hat{m}_2$, while $\hat{m}_2$ is more sensitive to a positive bias in the estimation of N_0 than $\hat{m}_1$. Generally, the estimators have comparable sensitivities to errors in the estimation of N_0.

It is concluded that $\hat{m}_1$ outperforms $\hat{m}_2$ when $1.0 \leq m \leq 20.0$. Moreover, $\hat{m}_1$ is computationally simpler than $\hat{m}_2$, as $\hat{m}_1$ only needs $\hat{\mu}_1$ and $\hat{\mu}_2$ while $\hat{m}_2$ needs $\hat{\mu}_{-1}$, $\hat{\mu}_3$ and $\hat{\mu}_4$ additionally, which are sums of N elements. Therefore, when $m \geq 1.0$ or when $0.5 \leq m < 1.0$ but a simpler estimator is perferred, $\hat{m}_1$ should be used. This result is valid when a good estimate of N_0 is available.

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