Approximate ML Detection for MIMO Systems Using Multistage Sphere Decoding

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Abstract— **We derive a new multistage sphere decoding (MSD) algorithm, a generalization of the conventional sphere decoder (SD). This new MSD exploits that many higher-order signal constellations can naturally be decomposed into several lower-order constellations. We develop a 2-stage SD for a 16-ary quadrature amplitude modulation (16QAM) MIMO system by decomposing 16QAM into two 4QAM constellations. The first stage generates a list of 4QAM vectors. For each of these, the second stage computes an optimal 4QAM vector. In the low SNR region, our MSD performs close to the original (single-stage) SD, but has lower complexity. In the high SNR region, our MSD is not suitable for reaching near ML performance.**

I. Introduction

Space-time processing and multiple input multiple output (MIMO) systems have emerged as promising high-capacity communication techniques. The sphere decoder (SD) [1], a computationally efficient decoding algorithm with maximum likelihood (ML) performance, has received considerable interest. The original SD has been used to decode lattice codes [2] and space-time codes [3]. For a MIMO system with m transmit and n receive antennas ($n \geq m$), the input and output relationship is $\mathbf{y} = \mathbf{H}\mathbf{s}^* + \mathbf{n}$, where $\mathbf{s}^*$ is the transmitted signal vector, $\mathbf{H} \in \mathcal{C}^{n\times m}$ is the channel perfectly known to the receiver, $\mathbf{y} \in \mathcal{C}^n$ and $\mathbf{n}$ is the additive noise vector. The basic detection problem is the discrete least-squares optimization

$$\hat{\mathbf{s}}_{\text{ml}} = \arg\min_{\mathbf{s}\in\mathcal{Q}^m} \|\mathbf{y} - \mathbf{H}\mathbf{s}\|^2 \tag{1}$$

where $\|\cdot\|$ denotes the Euclidean norm, $\mathcal{Q}$ is a finite complex set called the signal constellation, $s_k \in \mathcal{Q}$ and $\mathbf{s} = [s_1, s_2, \ldots, s_m]^T$. Typically, the computational complexity for solving $\hat{\mathbf{s}}_{\text{ml}}$ grows exponentially with m and the size of $\mathcal{Q}$. Remarkably, the SD achieves polynomial complexity in the high SNR region (however its complexity is exponential in the low SNR region). Nevertheless, the complexity of the SD may be too high for certain applications. Suboptimal detectors for (1) include the zero-forcing (ZF) detector and the vertical Bell Labs layered space time (V-BLAST) detector (nulling and cancelling) [4]. The complexity of the SD is substantially higher than that of V-BLAST in the low SNR region, but the SD performs much better than V-BLAST. Attempts have been made to develop other suboptimal algorithms that have performance/complexity advantages over SD and V-BLAST (see [5], [6]).

Multilevel codes and multistage decoding allow the use of component codes and their sequential decoding [7]. This approach has been widely used to construct complicated codes. Each stage uses the decisions from the previous stages and decodes one component code. Multistage sequential decoding facilitates this approach since ML decoding of large codes can be prohibitively complex. To the best of our knowledge, such an approach has not been developed for uncoded MIMO systems. Spectrally-efficient, higher-order constellations such as 16QAM, popular with MIMO systems, can naturally be decomposed into several lower-order constellations [8]. Thus, the uncoded MIMO detection problem can be viewed as a multistage decoding problem. This paper develops a multistage sphere decoder using such decompositions.

Notation: Bold symbols denote matrices or vectors. $(\cdot)^T$ and $(\cdot)^H$ denote transpose and conjugate transpose. The set of all complex $K \times 1$ vectors is denoted by $\mathcal{C}^K$ and $j = \sqrt{-1}$. We use the terms SSD(M, m) and LSD(M, m) to denote a conventional, single-stage SD and a list SD over the $\mathcal{Q}_M^m$ space. For $\mathbf{u} \in \mathcal{Q}_{16}^m$, we write $\mathbf{u} = \sqrt{2}\mathbf{u}_1 + \sqrt{2}/2\mathbf{u}_2$ where $\mathbf{u}_1, \mathbf{u}_2 \in \mathcal{Q}_4^m$.

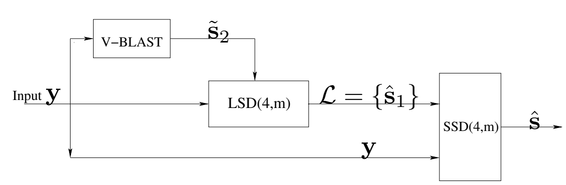

Fig. 1. The block diagram of the multistage sphere decoder for 16QAM systems.

II. Single Stage Sphere Decoding Algorithm

The single stage SD algorithm by Fincke and Phost [1] (FPSD hereafter) is proposed to solve the shortest vector problem (SVP). SSD restricts the search space to the lattice points within a sphere instead of searching all the lattice points. Each time a valid lattice point is found, the search space is restricted further by updating the radius. Although its worst case complexity was shown to be exponential, SSD has been widely used in many communication problems.

The primary difficulty with the optimization in (1) is that $\mathbf{s}$ is a *discrete* vector, and hence widely-available multidimensional continuous optimization strategies cannot be applied. A brute force algorithm must search all $|\mathcal{Q}|^M$ possible values of $\mathbf{s}$. Next we we briefly describe the basic idea of FPSD and for brevity we consider the variables in (1) to be real without loss of generality. Minimizing (1) is equivalent to

$$\begin{aligned}\hat{\mathbf{s}} &= \arg\min_{\mathbf{s}\in\mathcal{Q}^M}\left\|\mathbf{y}-[\mathbf{Q}_1,\mathbf{Q}_2]\begin{bmatrix}\mathbf{R}\\ \mathbf{0}\end{bmatrix}\mathbf{s}\right\|^2\\ &= \arg\min_{\mathbf{s}\in\mathcal{Q}^M}\left\|\begin{bmatrix}\mathbf{Q}_1^H\\ \mathbf{Q}_2^H\end{bmatrix}\mathbf{y}-\begin{bmatrix}\mathbf{R}\\ \mathbf{0}\end{bmatrix}\mathbf{s}\right\|^2\\ &= \arg\min_{\mathbf{s}\in\mathcal{Q}^m}\arg\min_{\mathbf{s}\in\mathcal{Q}^m}\|\mathbf{y}'-\mathbf{R}\mathbf{s}\|^2\end{aligned} \tag{2}$$

where $\mathbf{y}' = \mathbf{Q}_1^H\mathbf{y}$, the $m\times m$ upper triangular matrix $\mathbf{R}$ and the $n\times n$ orthogonal matrix $\mathbf{Q}=[\mathbf{Q}_1,\mathbf{Q}_2]$ are the QR factorization of $\mathbf{H}$. The matrices $\mathbf{Q}_1$ and $\mathbf{Q}_2$ represent the first m and last $n-m$ orthonormal columns of $\mathbf{Q}$. The lattice point lies inside a hypersphere with radius r when

$$r^2 \geq \|\mathbf{y}'-\mathbf{R}\mathbf{s}\|^2 \tag{3}$$

We assume the initial radius r is large enough so that the hypersphere (3) contains the ML solution. Let the entries of $\mathbf{R}$ be denoted by r_{ij}, $i\leq j$. To this end, we assume that $n\geq m$ so that $\mathbf{R}$ has full rank (positive definite). The diagonal terms of $\mathbf{R}$ are non-zero ($r_{ii}\neq 0$). Eq. (3) can be rewritten as

$$\begin{aligned}&\|\mathbf{y}'-\mathbf{R}\mathbf{s}\|^2\\ &=\sum_{i=1}^{m}\left(y_i'-\sum_{j=i}^{m}r_{ij}s_j\right)^2\\ &=\sum_{i=1}^{m}r_{ii}^2\left(s_i-\rho_i\right)^2\leq r^2\end{aligned} \tag{4}$$

where

$$\rho_i=\frac{y_i'}{r_{ii}}-\sum_{j=i+1}^{m}\frac{r_{ij}}{r_{ii}}s_j \tag{5}$$

Since each term in the above equation is nonnegative, a necessary condition for $\mathbf{s}$ to lie inside the hypersphere is that

$$\left\lceil\rho_m-\frac{r}{r_{mm}}\right\rceil\leq s_m\leq\left\lfloor\rho_m+\frac{r}{r_{mm}}\right\rfloor \tag{6}$$

where $\lceil\cdot\rceil$ denotes the smallest integer greater than or equal to its argument. $\lfloor\cdot\rfloor$ denotes the largest integer less than or equal to its argument. For s_{m-1}, we have

$$\begin{aligned}&r_{m-1,m-1}^2\left[s_{m-1}-\rho_{m-1}\right]^2\\ &\quad+r_{mm}^2\left[s_m-\rho_m\right]^2\leq r^2\end{aligned} \tag{7}$$

which leads to the following bound

$$\begin{aligned}&\left\lceil-\frac{\sqrt{r^2-r_{mm}^2\left[s_m-\rho_m\right]^2}}{r_{m-1,m-1}}+\rho_{m-1}\right\rceil\leq s_{m-1}\\ &\leq\left\lfloor\frac{\sqrt{r^2-r_{mm}^2\left[s_m-\rho_m\right]^2}}{r_{m-1,m-1}}+\rho_{m-1}\right\rfloor\end{aligned} \tag{8}$$

The sphere decoder generates candidates for s_{m-1} from this. We can continue in the same process for s_{m-2} and so on. The bounds for s_j are

$$\lceil\rho_j-\xi_j\rceil\leq s_j\leq\lfloor\rho_j+\xi_j\rfloor \tag{9}$$

where $\xi_j=[r^2-\sum_{i=j+1}^{m}r_{ii}^2(s_i-\rho_i)^2]^{1/2}/r_{jj}$. If there is no valid candidate for, say, s_j, the SD reverts to s_{j+1} and choose another candidate for s_{j+1}. If SD reaches s_1, a valid $\mathbf{s}'$ for (3) are generated. If the value of $\|\mathbf{y}'-\mathbf{R}\mathbf{s}'\|^2$ is less than r, we replace the radius r. This process continues until no further lattice points are found. The lattice point achieves the smallest value of (1) is ML. If no point in the sphere is found, the initial search radius r must be increased and the search can be restarted.

III. Multistage Sphere Decoding Algorithm

For brevity, we only show how to apply the MSD to 16QAM. An arbitrary 16QAM vector $\mathbf{s}$ can be uniquely expressed as $\mathbf{s}=\sqrt{2}\mathbf{s}_1+\sqrt{2}/2\mathbf{s}_2$, where $\mathbf{s}_1,\mathbf{s}_2\in\mathcal{Q}_4^m$. Similarly, let the true transmit vector be $\mathbf{s}^*=\sqrt{2}\mathbf{s}_1^*+\sqrt{2}/2\mathbf{s}_2^*$. The problem of detecting $\hat{\mathbf{s}}_{\text{ml}}$ (2) is equivalent to detecting two 4QAM component vectors as follows:

$$[\hat{\mathbf{s}}_1,\hat{\mathbf{s}}_2]=\arg\min_{\mathbf{s}_1,\mathbf{s}_2\in\mathcal{Q}_4^m}\|\mathbf{y}'-\mathbf{R}(\sqrt{2}\mathbf{s}_1+\frac{\sqrt{2}}{2}\mathbf{s}_2)\|^2. \tag{10}$$

To begin, we need an initial approximation to the true signal $\mathbf{s}^*$. Let this be $\tilde{\mathbf{s}}=\sqrt{2}\tilde{\mathbf{s}}_1+\sqrt{2}/2\tilde{\mathbf{s}}_2$. Using this, we do a partial interference cancellation as $\mathbf{y}_2=\mathbf{y}'-\sqrt{2}/2\mathbf{R}\tilde{\mathbf{s}}_2$ in the first stage[1]. If $\mathbf{s}_2=\mathbf{s}_2^*$, $\mathbf{y}_2$ is clearly sufficient to detect $\mathbf{s}_1^*$. We search $\mathbf{s}_1$ that minimizes

$$\|\mathbf{y}_2-\sqrt{2}\mathbf{R}\mathbf{s}_1\|^2. \tag{11}$$

However, $\tilde{\mathbf{s}}_2\neq\mathbf{s}_2^*$ in general, and minimizing (11) will likely give a wrong estimate. Therefore, we use a LSD [9] to generate a list $\mathcal{L}$ of the N_{cand} candidates $\hat{\mathbf{s}}_1$ that make (11) smallest. The list size is between 4^m and 1, and is proportional to the probability that the true solution $\mathbf{s}_1^*$ falls in the list. With a properly chosen radius r, we can obtain $\mathcal{L}$ with N_{cand} candidates on average. To obtain a typical value of r, we note that for true $\mathbf{s}_1^*$

$$\|\mathbf{y}_2-\sqrt{2}\mathbf{R}\mathbf{s}_1^*\|^2=\|\frac{\sqrt{2}}{2}\mathbf{R}(\mathbf{s}_2^*-\tilde{\mathbf{s}}_2)+\mathbf{n}\|^2 \tag{12}$$

where $\mathbf{n}$ is the additive Gaussian noise vector with variance σ_n^2. Since $\tilde{\mathbf{s}}_2$ is correlated with $\mathbf{R}$ and $\mathbf{n}$, (12) cannot be treated as a chi-square random variable with $2m$ degrees of freedom. The expected value of this random variable is denoted by E, which can be obtained via simulation. As in [9], one possible choice of radius is $r^2=kE$, where k is chosen so that the average length of the list is N_{cand}. For typical values of σ_n^2 and σ_h^2, r^2 corresponding to N_{cand} can be obtained from simulation and can be stored in memory for practical use.

[1]Note that we cancel $\tilde{\mathbf{s}}_2$ first since any errors in $\tilde{\mathbf{s}}_2$ will be attenuated by $\sqrt{2}$ (see Eq. (10)). Whereas any errors in $\tilde{\mathbf{s}}_1$ will be magnified by $\sqrt{2}$.

In the second stage, for each candidate $\hat{\mathbf{s}}_1 \in \mathcal{L}$, the SSD$(4, m)$ solves

$$\hat{\mathbf{s}}_2 = \arg \min_{\mathbf{s}_2 \in \mathcal{Q}_4^m} \|\mathbf{y}_1 - \frac{\sqrt{2}}{2}\mathbf{R}\mathbf{s}_2\|^2 \tag{13}$$

where $\mathbf{y}_1 = \mathbf{y}' - \sqrt{2}\mathbf{R}\hat{\mathbf{s}}_1$. This process provides N_{cand} pairs of $[\hat{\mathbf{s}}_1, \hat{\mathbf{s}}_2]$ and the best among them is selected as the output. Each time a $\hat{\mathbf{s}}_2$ for (13) is found with a $\hat{\mathbf{s}}_1$, the search radius of the second stage is updated if $\|\mathbf{y}_1 - \sqrt{2}/2\mathbf{R}\mathbf{s}_2\|$ is less than the current radius.

IV. Simulation results

We now compare the MSD with the SSD for a 16QAM, uncoded MIMO system with 4 transmit and 4 receive antennas over a flat Rayleigh fading channel. The initial radius r is chosen according to the noise variance. Both the SSD and the MSD use the Schnorr-Euchner variant of SD [10]. The initial detection uses the ZF-VBLAST.

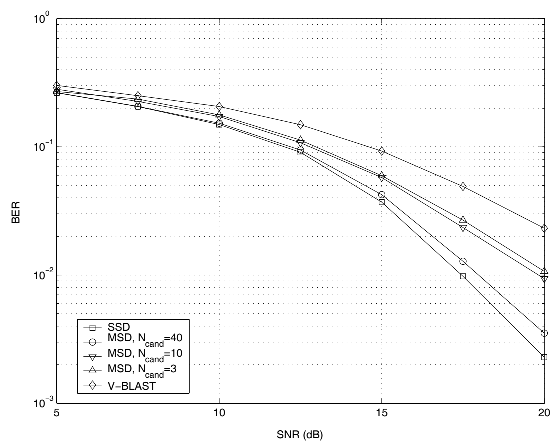

Fig. 2. BER comparison with different N_{cand} for a 16QAM MIMO system with 4 transmit and 4 receive antennas.

Fig. 2 compares the BER of the SSD with that of the MSD as a function of the number of candidates in the first stage N_{cand}. As N_{cand} increases, the MSD performs close to the SSD. As N_{cand} varies, its performance varies between those of V-BLAST and SSD. The complexity of the MSD increases as N_{cand} increases (Fig. 3) and it is lower than that of the SSD's when the SNR is below a threshold. For instance, when $N_{\text{cand}} = 10$, this complexity crossover point is 17dB. Fig. 3 also shows that the complexity of the MSD is almost constant with specific N_{cand}. The major drawback of our MSD is that the complexity needed to achieve near ML performance increases with increasing SNR. Thus our MSD is suitable for the low SNR region, where it can be combined with an outer code to achieve low BER.

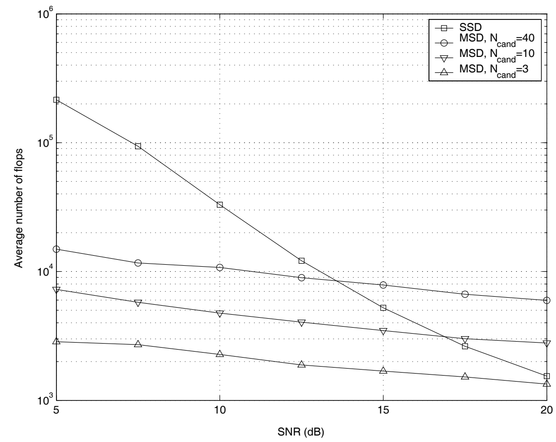

Fig. 3. Complexity comparison with different N_{cand} for a 16QAM MIMO system with 4 transmit and 4 receive antennas.

V. Conclusion

In this paper, we developed a multistage SD by decomposing a large constellation into a sum of smaller constellations. This is particularly easy for QAM where a higher-order QAM constellation can be readily resolved into lower-order QAM components. A series of conventional LSDs and SSDs are used to search over the smaller constellation spaces. As a specific example, 16QAM is resolved into two 4QAM constellations. An LSD and an SSD are used to search over the 4QAM constellations. Simulation results show that in the low SNR region, our MSD performs close to the SSD and reduces complexity.

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