Joint Distribution Functions of Three or Four Correlated Rayleigh Signals and Their Application in Diversity System Analysis

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***Abstract*— Few theoretical results are known about the joint distribution of three or more correlated Rayleigh random variables (RVs). Consequently, theoretical results for the performance of 3-branch and 4-branch equal gain combining (EGC), selection combining (SC) and generalized SC (GSC) over arbitrarily correlated Rayleigh fading channels are not known. This paper derives new infinite series representations of the joint probability density function (pdf) and the joint cumulative distribution function (cdf) of the tri-variate and a certain class of quadri-variate correlated Rayleigh distribution. The new pdf and cdf expressions are used to derive the outage probability of 3-branch SC and the moments of the 3-branch EGC output signal-to-noise ratio (SNR) over arbitrarily correlated Rayleigh fading. New bounds for the complementary cdf (ccdf) of the L-branch SC output SNR are also derived. These long-standing diversity theory problems which have resisted a solution can now be completely solved.**

***Keywords:* Diversity, equal gain combining, Rayleigh fading, selection combining.**

I. Introduction

The Rayleigh distribution is frequently used to model the received signal amplitudes in urban and suburban areas [1]. The L-dimensional joint pdf of a set of L correlated Rayleigh signals is required for some performance analysis problems, including determining the impact of correlation on diversity systems and modelling fading processes. Schwartz, Bennett and Stein claim that the joint pdf for more than two correlated Rayleigh envelopes cannot be found [2]. Consequently, many published papers dealing with SC and EGC over correlated Rayleigh fading are limited to the dual-branch case [2]–[5]. However, since the envelopes of multiple correlated complex Gaussian RVs are Rayleigh distributed, the underlying joint complex Gaussian pdf can be converted to the polar form to give the joint pdf of amplitudes and phases and, in principle, the phase terms can be integrated out to give the joint amplitude pdf. Following this approach, Mallik [6] derives the joint pdf of multi-variate Rayleigh distributions, which requires L-dimensional integration. However, an infinite series representation for the joint pdf is also required for computation. Miller [7] derives an infinite series of product of modified Bessel functions for the joint pdf of three correlated Rayleigh RVs. While this holds for arbitrary correlation models, it is intractable to derive infinite series representations of the joint pdf for $L > 3$ using this approach. Most of the available series representations therefore deal with restricted correlation models. For example, Blumenson and Miller [8] have derived the joint pdf for any L with a specific constraint that the inverse covariance matrix $\mathbf{\Phi}$ is tri-diagonal (i.e. its element $\phi_{ik} = 0$ if $|i - k| > 1$). The exponential correlation model gives rise to this particular pattern. Using Blumenson and Miller's results [8], Karagiannidis *et al.* [9] derive a joint distribution that holds only for exponentially correlated Nakagami-m fading channels. Notice that since the joint pdf is the inverse Fourier transform of its characteristic function (chf), the joint pdf of L correlated Rayleigh RVs can also be derived directly from the available joint chf [11], [12].

In this paper, we use Miller's result [7] to derive the infinite series representations for the joint pdf, cdf and moments of the tri-variate Rayleigh distribution. For four correlated Rayleigh RVs, we generalize Blumenson and Miller's result [8] (which is limited to tri-diagonal inverse covariance matrices) to the case where the inverse covariance is five-diagonal (i.e. only ϕ_{14} needs to be zero). Our tri-variate and quadri-variate cdf series generalize Tan and Beaulieu's series for the bivariate Rayleigh cdf [13]. The new pdf and cdf expressions enable the performance analysis of 3-branch and 4-branch diversity systems over correlated Rayleigh fading channels. Here, we derive the moments of the 3-branch EGC output SNR, the outage probability of 3-branch SC and the new bounds for the ccdf of the L-branch SC output SNR over arbitrary correlated Rayleigh channels.

This paper is organized as follows. Section 2 derives infinite series representations for the joint pdfs and cdfs of the tri-variate and a certain quadri-variate Rayleigh distributions. Section 3 presents several applications of the new results. Section 4 provides some numerical results and concludes this paper.

II. Tri-Variate and Quadri-Variate Distributions

Infinite series representations for the joint pdf and the joint cdf of the tri-variate Rayleigh distribution and a certain quadri-variate Rayleigh distribution, which is more general than previous results [9], are next derived. The joint moments of the tri-variate Rayleigh distribution are also derived.

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A. Pdf and Cdf of the Tri-Variate Rayleigh Distribution

Let $\mathbf{g} = \{g_1, g_2, g_3\}$ be jointly complex Gaussian distributed with zero means and positive definite covariance matrix $\mathbf{\Psi}$ whose element is defined as $\psi_{ik} = E(g_i g_k^*)$. We may write g_k in terms of polar coordinates as

$$g_k = r_k \exp(j\theta_k), \quad k = 1, 2, 3, \tag{1}$$

where $j = \sqrt{-1}$ and $r_k = |g_k|$ is the amplitude of g_k. Thus, $\mathbf{r} = \{r_1, r_2, r_3\}$ is a set of Rayleigh RVs and $\theta = \{\theta_1, \theta_2, \theta_3\}$ is a set of jointly distributed phases. The joint pdf $f_{\mathbf{r},\theta}(\mathbf{r}, \theta)$ can be readily related to the density $f_{\mathbf{g}}(\mathbf{g})$ of $\mathbf{g}$. Hence, the marginal density $f_{\mathbf{r}}(\mathbf{r})$ can be obtained by integrating out $f_{\mathbf{r},\theta}(\mathbf{r}, \theta)$ over θ. This approach yields [7]

$$\begin{aligned} f_{\mathbf{r}}(\mathbf{r}) &= 8\det(\mathbf{\Phi}) r_1 r_2 r_3 \exp[-(r_1^2\phi_{11} + r_2^2\phi_{22} + r_3^2\phi_{33})] \\ &\times \sum_{m=0}^{\infty} \varepsilon_m (-1)^m \cos(m\chi) I_m(2r_1 r_2 |\phi_{12}|) \\ &\times I_m(2r_2 r_3|\phi_{23}|) I_m(2r_3 r_1 |\phi_{31}|) \end{aligned} \tag{2}$$

where $\chi = \chi_{12} + \chi_{23} + \chi_{31}$, $\det(\mathbf{X})$ is the determinant of matrix $\mathbf{X}$, ε_m is the Neumann factor ($\varepsilon_0 = 1, \varepsilon_m = 2$ for $m = 1, 2, \cdots$), $I_m(x)$ is the m-th order modified Bessel function of the first kind, and $\mathbf{\Phi}$ is the inverse covariance matrix of the underlying complex Gaussian RVs $\mathbf{g}$

$$\mathbf{\Phi} = \mathbf{\Psi}^{-1} = \begin{bmatrix} \phi_{11} & \phi_{12} & \phi_{13} \\ \phi_{12}^* & \phi_{22} & \phi_{23} \\ \phi_{13}^* & \phi_{23}^* & \phi_{33} \end{bmatrix} \tag{3}$$

where x^* denotes the complex conjugate of x and $\phi_{ik} = |\phi_{ik}| e^{j\chi_{ik}}$ is the (i, k)-th element of $\mathbf{\Phi}$. We use $\mathbf{\Psi}$ and $\mathbf{\Phi}$ in (3) to characterize the correlation among Rayleigh RVs.

Replacing the modified Bessel function by an infinite series [14, Eq. (9.6.10)], we obtain an infinite series representation for the joint pdf as

$$\begin{aligned} f_{\mathbf{r}}(\mathbf{r}) &= 8\det(\mathbf{\Phi}) \exp[-(r_1^2\phi_{11} + r_2^2\phi_{22} + r_3^2\phi_{33})] \\ &\times \sum_{m=0}^{\infty} \varepsilon_m (-1)^m \cos(m\chi) \sum_{i,k,l=0}^{\infty} \frac{|\phi_{12}|^{2i+m} |\phi_{23}|^{2k+m}}{i!k!(i+m)!(k+m)!} \\ &\times \frac{|\phi_{31}|^{2l+m}}{l!(l+m)!} r_1^{2(i+l+m)+1} r_2^{2(i+k+m)+1} r_3^{2(k+l+m)+1}. \end{aligned} \tag{4}$$

Integrating (4) and using the definition of the incomplete gamma function $\gamma(a, x)$ [14, Eq. (6.5.2)], we obtain the corresponding infinite series representation for the joint tri-variate cdf as

$$\begin{aligned} F_{\mathbf{r}}(\lambda) &= \frac{\det(\mathbf{\Phi})}{\phi_{11}\phi_{22}\phi_{33}} \sum_{m=0}^{\infty} \varepsilon_m (-1)^m \cos(m\chi) \\ &\times \sum_{i,k,l=0}^{\infty} \nu_{12}^{i+\frac{m}{2}} \nu_{23}^{k+\frac{m}{2}} \nu_{31}^{l+\frac{m}{2}} \frac{\gamma(i+l+m+1, \lambda_1^2\phi_{11})}{i!\,(l+m)!} \\ &\times \frac{\gamma(i+k+m+1, \lambda_2^2\phi_{22}) \gamma(k+l+m+1, \lambda_3^2\phi_{33})}{k!\,l!\,(i+m)!\,(k+m)!} \end{aligned} \tag{5}$$

where $\lambda = \{\lambda_1, \lambda_2, \lambda_3\}$ and $\nu_{ik} = \frac{|\phi_{ik}|^2}{\phi_{ii}\phi_{kk}}$ will be used throughout this paper. The joint cdf (5) holds for any arbitrary 3×3 correlation matrix. Let us consider two commonly used spatial correlation models.

1. Exponential Correlation Model

The exponential correlation model may be used to describe the correlation among equally-spaced linear antenna arrays. The normalized covariance matrix of this model is described as $\psi_{ik} = \rho^{|i-k|}$, where $0 \le \rho < 1$. It can be shown that $\phi_{31} = \phi_{13} = 0$. Thus the joint cdf (5) can be simplified considerably to

$$\begin{aligned} F_{\mathbf{r}}(\lambda) &= \frac{1-\rho^2}{1+\rho^2} \sum_{i,j=0}^{\infty} \frac{\left(\frac{\rho^2}{1+\rho^2}\right)^{i+j}}{(i!)^2 (j!)^2} \gamma\left(i+1, \frac{\lambda_1^2}{1-\rho^2}\right) \\ &\times \gamma\left(i+j+1, \frac{(1+\rho^2)\lambda_2^2}{1-\rho^2}\right) \gamma\left(j+1, \frac{\lambda_3^2}{1-\rho^2}\right) \end{aligned} \tag{6}$$

which is equivalent to [9, Eq. (6)].

2. Constant Correlation Model

The constant correlation model is valid for a set of closely-placed antennas. The normalized covariance matrix of this model is $\psi_{ik} = \rho$ for $i \ne k$ and $\psi_{ii} = 1$, where $-\frac{1}{2} \le \rho < 1$. It can be shown that $\chi = \chi_{12} + \chi_{23} + \chi_{31} = 3\pi$. Thus, the joint cdf (5) reduces to

$$\begin{aligned} F_{\mathbf{r}}(\lambda) &= \frac{(1-\rho)(1+2\rho)^2}{(1+\rho)^3} \sum_{m=0}^{\infty} \varepsilon_m \sum_{i,k,l=0}^{\infty} \left(\frac{\rho}{1+\rho}\right)^{\beta} \\ &\times \frac{\gamma\left(i+l+m+1, \frac{(1+\rho)\lambda_1^2}{1+\rho-2\rho^2}\right) \gamma\left(i+k+m+1, \frac{(1+\rho)\lambda_2^2}{1+\rho-2\rho^2}\right)}{i!\,k!\,l!\,(i+m)!\,(k+m)!\,(l+m)!} \\ &\times \gamma\left(k+l+m+1, \frac{(1+\rho)\lambda_3^2}{1+\rho-2\rho^2}\right) \end{aligned} \tag{7}$$

where $\beta = 2(i+k+l) + 3m$.

The convergence rate of the cdf series (5) depends on the correlation matrix $\mathbf{\Psi}$. Considering a worst-case situation, we investigate the convergence property of the cdf series for the constant correlation model. For simplicity, we let $\lambda_1 = \lambda_2 = \lambda_3 = \lambda$ in (7). Table 1 lists the number of terms required in each sum of (7) to achieve six significant figure accuracy for different values of ρ and λ. The total number of terms required is equal to $M \times I \times K \times L$, where M, I, K, L denote the number of terms required in the variables m, i, k and l, respectively. We find that the cdf series (7) converges much faster as the correlation ρ or λ decreases.

TABLE I

THE NUMBER OF TERMS NEEDED IN (7) TO ACHIEVE SIX SIGNIFICANT FIGURE ACCURACY.

	$\lambda = 1$	$\lambda = 2$
$\rho = 0.2$	$M = 2, I = K = L = 3$	$M = I = K = L = 4$
$\rho = 0.5$	$M = 3, I = K = L = 4$	$M = 5, I = K = L = 7$
$\rho = 0.8$	$M = 5, I = K = L = 7$	$M = 12, I = K = L = 13$

B. Pdf and Cdf of the Quadri-Variate Rayleigh Distribution

Blumenson and Miller [8] derive the joint pdf and the joint cdf for multi-variate Rayleigh distribution. However, their expression is only valid for a distribution with a banded inverse covariance matrix $\mathbf{\Phi}$ in which $\phi_{ik} = 0$ for $|i-k| > 1$ (i.e. for the matrix in (8), both ϕ_{13} and ϕ_{24} would be zero). For quadri-variate Rayleigh distribution, we consider a more general case in which the inverse covariance matrix is given by

$$\mathbf{\Phi} = \mathbf{\Psi}^{-1} = \begin{bmatrix} \phi_{11} & \phi_{12} & \phi_{13} & 0 \\ \phi_{12}^* & \phi_{22} & \phi_{23} & \phi_{24} \\ \phi_{13}^* & \phi_{23}^* & \phi_{33} & \phi_{34} \\ 0 & \phi_{24}^* & \phi_{34}^* & \phi_{44} \end{bmatrix} \tag{8}$$

where $\phi_{ik} = |\phi_{ik}|e^{j\chi_{ik}}$ is the (i,k)-th element of $\mathbf{\Phi}$. It can be shown that the joint pdf of the quadri-variate Rayleigh distributed RVs $\mathbf{r} = \{r_1, r_2, r_3, r_4\}$ with positive definite covariance matrix $\mathbf{\Psi}$ and its inverse covariance matrix $\mathbf{\Phi}$ satisfying (8) is given by

$$\begin{aligned} f_{\mathbf{r}}(\mathbf{r}) &= 16\det(\mathbf{\Phi}) r_1 r_2 r_3 r_4 \exp[-h(\mathbf{r})] \sum_{m=0}^{\infty} \sum_{n=-\infty}^{\infty} (-1)^{m+n} \\ &\times \varepsilon_m \cos\vartheta I_m(2r_1r_2|\phi_{12}|) I_m(2r_1r_3|\phi_{13}|) \\ &\times I_n(2r_2r_4|\phi_{24}|) I_n(2r_3r_4|\phi_{34}|) I_{m+n}(2r_2r_3|\phi_{23}|) \end{aligned} \tag{9}$$

where $h(\mathbf{r}) = r_1^2\phi_{11} + r_2^2\phi_{22} + r_3^2\phi_{33} + r_4^2\phi_{44}$ and $\vartheta = m(\chi_{12} + \chi_{23} + \chi_{31}) + n(\chi_{23} + \chi_{34} + \chi_{42})$. We omitted detail derivations here. To the best of our knowledge, (9) is a novel result which allows evaluation of several 4-branch diversity systems over correlated fading. Eq. (9) reduces to previous results for two special cases. Notice that since the inverse covariance matrix of the constant correlation model does not satisfies (8), the joint pdf (9) is not valid for the constant correlation case.

Case 1: Independent fading

The covariance matrix $\mathbf{\Psi}$ for independent distribution is diagonal. Thus, the inverse covariance matrix $\mathbf{\Phi}$ is also diagonal where $\phi_{ik} = 0$ for $i \neq k$ and $\phi_{ii} = 1/\psi_{ii}$, $(i, k = 1, \cdots, 4)$. Therefore, our new expression (9) can be simplified to

$$f_{\mathbf{r}}(\mathbf{r}) = \frac{16 r_1 r_2 r_3 r_4}{\psi_{11}\psi_{2}\psi_{33}\psi_{44}} \, e^{-\left(\frac{r_1^2}{\psi_{11}} + \frac{r_2^2}{\psi_{22}} + \frac{r_3^2}{\psi_{33}} + \frac{r_4^2}{\psi_{44}}\right)}. \tag{10}$$

As expected, (10) is the product of four independent Rayleigh pdfs [16, Eq. (2-1-128)].

Case 2: Exponentially correlated fading

The inverse covariance matrix of the exponential correlation model is tri-diagonal [9]. Substituting $\phi_{24} = \phi_{13} = 0$ into (9), we obtain the joint pdf for the exponentially correlated quadri-variate Rayleigh distribution

$$\begin{aligned} f_{\mathbf{r}}(\mathbf{r}) &= \frac{16 r_1 r_2 r_3 r_4}{(1-\rho^2)^3} \exp\left[-\left(\frac{r_1^2 + r_4^2}{1-\rho^2} + \frac{(1+\rho^2)(r_2^2 + r_3^2)}{1-\rho^2}\right)\right] \\ &\times I_0\left(\frac{2r_1r_2\rho}{1-\rho^2}\right) I_0\left(\frac{2r_3r_4\rho}{1-\rho^2}\right) I_0\left(\frac{2r_2r_3\rho}{1-\rho^2}\right) \end{aligned} \tag{11}$$

where $\rho = \psi_{12} = \psi_{23}$ is the correlation between two adjacent elements. Eq. (11) is equivalent to the previous result [9, Eq. (3)].

Expanding the $I_m(x)$ in series and integrating (9) yield an infinite series for the joint cdf of this kind of quadri-variant Rayleigh distribution

$$\begin{aligned} F_{\mathbf{r}}(\lambda) &= \frac{\det(\mathbf{\Phi})}{\prod_{i=1}^4 \phi_{ii}} \sum_{m=0}^{\infty} \sum_{n=-\infty}^{\infty} (-1)^{m+n} \varepsilon_m \cos\vartheta \\ &\times \sum_{i,k,l,u=0}^{\infty} \nu_{12}^{i+\frac{m}{2}} \nu_{13}^{k+\frac{m}{2}} \nu_{24}^{l+\frac{|n|}{2}} \nu_{34}^{u+\frac{|n|}{2}} \frac{\gamma(i+m+k+1, \lambda_1^2\phi_{11})}{i!\,k!\,(i+m)!\,(k+m)!} \\ &\times \frac{\gamma(u+|n|+l+1, \lambda_4^2\phi_{44})}{l!\,u!\,(u+|n|)!\,(l+|n|)!} \sum_{v=0}^{\infty} \frac{\nu_{23}^{v+\frac{|m+n|}{2}}}{v!\,(v+\frac{|m+n|}{2})!} \\ &\times \gamma(i+l+v+\tau+1, \lambda_2^2\phi_{22}) \gamma(k+u+v+\tau+1, \lambda_3^2\phi_{33}) \end{aligned} \tag{12}$$

where $\lambda = \{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$ and $\tau = (|m+n| + |n| + m)/2$. We observe that both (4) and (9) are series of the product of several modified Bessel functions. For brevity, we only discuss the tri-variate Rayleigh distribution and its applications in this paper. Similar results can be obtained for the special quadri-variate Rayleigh distribution.

C. Joint Moments

Moments can also be used to characterize the RVs. Using the infinite series representation for the joint tri-variate pdf (4), we derive the joint moments of $\mathbf{r} = \{r_1, r_2, r_3\}$. If $\alpha, \beta, \theta > -2$,

$$\begin{aligned} E(r_1^\alpha r_2^\beta r_3^\theta) &= \frac{\det(\mathbf{\Phi})}{\phi_{11}^{1+\frac{\alpha}{2}} \phi_{22}^{1+\frac{\beta}{2}} \phi_{33}^{1+\frac{\theta}{2}}} \sum_{m=0}^{\infty} \varepsilon_m (-1)^m \cos(m\chi) \\ &\times \sum_{i,k,l=0}^{\infty} \nu_i^{i+\frac{m}{2}} \nu_{23}^{k+\frac{m}{2}} \nu_{31}^{l+\frac{m}{2}} \frac{\Gamma\left(i+l+m+\frac{\alpha}{2}+1\right)}{i!\ (l+m)!} \\ &\times \frac{\Gamma\left(i+k+m+\frac{\beta}{2}+1\right) \Gamma\left(j+l+m+\frac{\theta}{2}+1\right)}{k!\ l!\ (i+m)!\ (k+m)!} \end{aligned} \tag{13}$$

where $\Gamma(x)$ is the gamma function and $E(x)$ denotes the average of x.

III. Applications

The new results developed in Section 2 enable the performance analysis of several 3-branch and 4-branch diversity systems over correlated Rayleigh fading channels. Due to length limitation, we here present only three applications.

A. Moments of the 3-Branch EGC Output SNR

As an alternative to the conventional error-rate analysis, the moments of a combiner output can be used as a performance measure. However, a single moment such as average SNR is not sufficiently informative and the higher order moments can furnish additional information for system design [17]. For example, the Chebyshev inequality yields $\Pr(|X-\mu| > t) < \sigma_X^2/t^2$ where $E(X) = \mu$. Thus, if X is taken to be output of a diversity combiner, the variability of the outputs is indicated by the variance. The new expression (13) enables us to evaluate the moments of the 3-branch EGC output SNR over Rayleigh fading channels. The EGC output SNR can be written as

$\gamma_{egc} = \frac{(r_1+r_2+r_3)^2 E_s}{3N_0}$ where E_s is the energy of transmitted signal, $\frac{N_0}{2}$ is the power spectral density (PSD) of the additive white Gaussian noise (AWGN) per dimension at each branch and r_k's are the amplitudes of the received signals whose joint pdf is given by (4). We assume that the noise components at different branches are independent of the signal components and uncorrelated with each other. The moments of output SNR can be obtained as

$$E(\gamma_{egc}^n) = \left(\frac{E_s}{3N_0}\right)^n E[(r_1+r_2+r_3)^{2n}] = \left(\frac{\bar{\gamma}_1}{3\psi_{11}}\right)^n \sum_{\substack{k_1,k_2,k_3=0 \\ k_1+k_2+k_3=2n}}^{2n} \frac{(2n)!\; E(r_1^{k_1} r_2^{k_2} r_3^{k_3})}{k_1!\,k_2!\,k_3!} \quad (14)$$

where $E(r_1^{k_1} r_2^{k_2} r_3^{k_3})$ can be computed using (13) and $\bar{\gamma}_1$ is the average SNR at the first branch given by $\bar{\gamma}_1 = \frac{E_s}{N_0} E(r_1^2) = \frac{\psi_{11} E_s}{N_0}$. The average output SNR can be simplified as

$$\begin{aligned}\bar{\gamma}_{egc} = \frac{\bar{\gamma}_1}{3\psi_{11}} \Bigg\{ & \sum_{i=1}^{3} \psi_{ii} + \frac{2\det(\mathbf{\Phi})}{\phi_{11}\phi_{22}\phi_{33}} \sum_{m=0}^{\infty} \varepsilon_m (-1)^m \cos(m\chi) \\ & \times \sum_{i,k,l=0}^{\infty} \frac{\nu_{12}^{i+\frac{m}{2}} \nu_{23}^{k+\frac{m}{2}} \nu_{31}^{l+\frac{m}{2}}}{i!\;k!\;l!\;(i+m)!\;(k+m)!\;(l+m)!} \Bigg[\frac{(k+l+m)!}{\sqrt{\phi_{11}\phi_{22}}} \\ & \times \Gamma\left(i+l+m+\frac{3}{2}\right) \Gamma\left(i+k+m+\frac{3}{2}\right) + \frac{(i+k+m)!}{\sqrt{\phi_{11}\phi_{33}}} \\ & \times \Gamma\left(i+l+m+\frac{3}{2}\right) \Gamma\left(k+l+m+\frac{3}{2}\right) + \frac{(i+l+m)!}{\sqrt{\phi_{22}\phi_{33}}} \\ & \times \Gamma\left(i+k+m+\frac{3}{2}\right) \Gamma\left(k+l+m+\frac{3}{2}\right) \Bigg] \Bigg\}. \end{aligned} \quad (15)$$

This new result allows the average output SNR to be computed for any correlation pattern among the three receiving antennas.

B. Outage Probability of 3-Branch SC

Outage probability is a standard performance measure of diversity systems. It is defined as the probability that the output instantaneous SNR γ falls below a certain given threshold γ_{th}. Here, we use the joint cdf (5) of the tri-variate Rayleigh distribution to evaluate the outage probability of 3-branch SC over correlated fading channels.

Let γ_k and $\bar{\gamma}_k$ denote the instantaneous and average SNR at the k-th branch ($k = 1, 2, 3$). In SC, the branch with the largest instantaneous SNR is selected as the output, i.e., $\gamma_{sc} = \max(\gamma_1, \gamma_2, \gamma_3)$. Using the relationship $\gamma_k = \frac{\bar{\gamma}_k}{E(r_k^2)} r_k^2 = \frac{\bar{\gamma}_k}{\psi_{kk}} r_k^2$, where r_k's are the branch amplitudes whose joint cdf is given by (5), we may obtain the outage probability as

$$P_{out} = \Pr(0 \le \gamma_{sc} \le \gamma_{th}) = F_{\mathbf{r}}\left(\sqrt{\frac{\gamma_{th}\psi_{11}}{\bar{\gamma}_1}}, \sqrt{\frac{\gamma_{th}\psi_{22}}{\bar{\gamma}_2}}, \sqrt{\frac{\gamma_{th}\psi_{33}}{\bar{\gamma}_3}}\right) \quad (16)$$

where $F_{\mathbf{r}}(\lambda_1, \lambda_2, \lambda_3)$ is given by (5). Note that the covariance matrix $\mathbf{\Psi}$ specifies the correlation (fading correlation) between two complex Gaussian samples. The relationship between the envelope correlation (i.e. the correlation between the two Rayleigh samples) and the fading correlation can be found [18, Eq. (1.5-26)]. Thus, the outage can be evaluated in terms of envelope correlation and average branch SNRs.

C. Bounds for the Output Ccdf of L-Branch SC

Performance of L-branch SC is completely known for independent fading branches. If, however, branch signals are allowed to be correlated (which is a much realistic assumption), known theoretical results are few and far between. In [10] and [11], the performance of L-branch SC over correlated Rayleigh fading channel is analyzed. However, their results are fairly complicated for large $L(>3)$. From a both practical and theoretical standpoint, performance bounds for the L-branch SC output $\gamma_{sc} = \max(\gamma_1, \gamma_2, \cdots, \gamma_L)$, where γ_u's ($u = 1, \ldots, L$) are correlated branch SNRs, are therefore desirable. We thus use (4) to derive new bounds for the ccdf of the L-branch SC output SNR over arbitrarily correlated Rayleigh fading channels.

Since the SC output SNR γ_{sc} is the maximum value of all the branch SNRs, when at least one branch SNR exceeds x, so does the SC output, i.e., $\Pr(\gamma_{sc} > x) = \Pr\left(\bigcup_{u=1}^{L} A_u\right)$, where A_u denotes the event that the u-th branch SNR exceeds x, i.e., $A_u = \{\gamma_u > x\}$. Using the Bonferroni inequalities [19], we can derive the bounds for the output ccdf of L-branch SC as

$$S_1 - S_2 \le \Pr(\gamma_{sc} > x) \le S_1 - S_2 + S_3 \quad (17)$$

where

$$S_1 = \sum_{u=1}^{L} \Pr(A_u) = \sum_{u=1}^{L} \Pr(\gamma_u > x) \quad (18)$$

$$S_2 = \sum_{\substack{u,v=1 \\ u<v}}^{L} \Pr(A_u \cap A_v) = \sum_{\substack{u,v=1 \\ u<v}}^{L} \Pr(\gamma_u > x, \gamma_v > x) \quad (19)$$

can be readily evaluated using [2, Eqs. (10-4-8, A-7-1)]. Using (4), we can readily evaluate S_3, which is given by

$$S_3 = \sum_{\substack{u,v,w=1 \\ u<v<w}}^{L} \Pr(A_u \cap A_v \cap A_w) \quad (20)$$

where

$$\begin{aligned} & \Pr(A_u \cap A_v \cap A_w) = \Pr(\gamma_u > x, \gamma_v > x, \gamma_w > x) \\ & = \frac{\det(\mathbf{\Phi})}{\phi_{uu}\phi_{vv}\phi_{ww}} \sum_{m=0}^{\infty} \varepsilon_m (-1)^m \cos m(\chi_{uv} + \chi_{vw} + \chi_{wu}) \\ & \times \sum_{i,k,l=0}^{\infty} \nu_{uv}^{i+\frac{m}{2}} \nu_{vw}^{k+\frac{m}{2}} \nu_{wu}^{l+\frac{m}{2}} \frac{\Gamma\left(i+j+m+1, \frac{x\phi_{uu}\psi_{uu}}{\bar{\gamma}_u}\right)}{i!\,(l+m)!} \\ & \times \frac{\Gamma\left(i+k+m+1, \frac{x\phi_{vv}\psi_{vv}}{\bar{\gamma}_v}\right) \Gamma\left(k+l+m+1, \frac{x\phi_{ww}\psi_{ww}}{\bar{\gamma}_w}\right)}{k!\,l!\,(i+m)!\,(k+m)!} \end{aligned} \quad (21)$$

where $\Gamma(a, x)$ is the complementary incomplete gamma function defined as [14, Eq. (6.5.3)].

IV. Numerical Results and Conclusion

For brevity, few numerical results are provided here. Consider an antenna array with the normalized covariance matrix

$$\mathbf{\Psi} = \begin{pmatrix} 1.0000 & 0.2920 & 0.2998 & 0.1121 \\ 0.2920 & 0.6602 & 0.2031 & 0.1585 \\ 0.2998 & 0.2031 & 0.7625 & 0.1888 \\ 0.1121 & 0.1585 & 0.1888 & 0.6431 \end{pmatrix}. \qquad (22)$$

The inverse covariance matrix $\mathbf{\Phi}$ satisfies (8). Thus, using (12), we may evaluate the outage probability of such 4-branch SC as is shown in Fig. 1. Our numerical results agree with the simulation results.

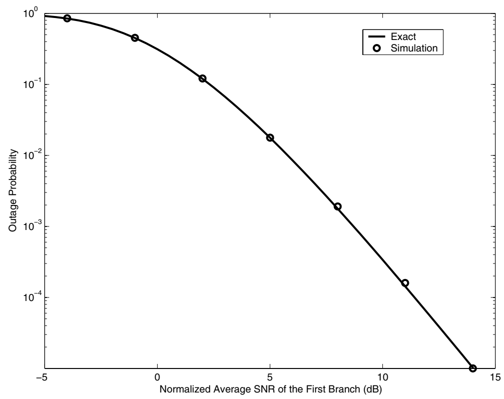

Fig. 1. Outage probability P_{out} of 4-branch SC versus the normalized average SNR of the first branch $\bar{\gamma}_1/\gamma_{th}$ over correlated Rayleigh fading channel.

Fig. 2 shows the effect of fading correlation ρ on the normalized average output SNR $\hat{\gamma}_{egc} = \gamma_{egc}/\bar{\gamma}$ of 3-branch EGC in equally correlated Rayleigh fading channels. As ρ increases, the average output SNR also increases. This contradicts the conventional wisdom as the performance is expected to degrade with the increasing correlation. However, the EGC performance depends not only on the first moment of the output but also on the higher moments. In fact, all the moments appear to increase with correlation. This suggests that the average output SNR by itself is not a comprehensive metric for the EGC performance.

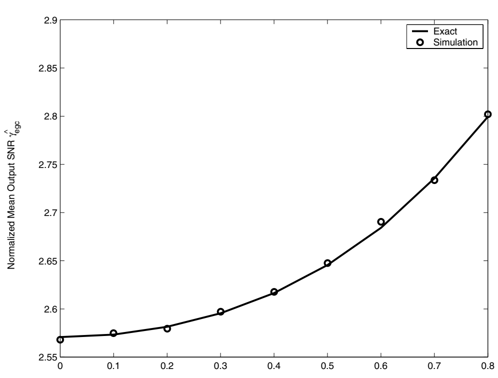

Fig. 2. The average output SNR of 3-branch EGC normalized by the first branch SNR.

In conclusion, we have derived infinite series representations for the joint pdf and the joint cdf of the tri-variate Rayleigh distribution, which can accommodate any arbitrary 3×3 correlation matrix. We have also derived the joint pdf and the joint cdf of a certain quadri-variate Rayleigh distribution and this appears to be the most general result available for the quadri-variate case, other than using a 4-dimensional integral [6]. These representations pave the way for solving certain long-standing diversity problems. For example, the performance of 3-branch and 4-branch SC, EGC and GSC over correlated Rayleigh fading can now be evaluated analytically.

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