Performance Analysis of Digital Modulations on Weibull Fading Channels

Julian Cheng, Chintha Tellambura, and Norman C. Beaulieu

Abstract—**The Weibull distribution is a flexible statistical model for describing multipath fading channels in both indoor and outdoor radio propagation environments. A new closed-form expression is derived for the moment generating function of the Weibull distribution, valid when its fading parameter assumes integer values. The performance of digital linear modulations operating on Weibull channels is studied. Expressions for the signal outage and average symbol error rate are derived for single-channel reception and multi-channel diversity reception operating on Weibull fading channels.**

I. Introduction

This paper presents new performance results for digital modulation transmission on Weibull fading channels. A study of Weibull fading is motivated for the following reasons. The Weibull distribution is useful for modeling multipath fading signal amplitude. Recall that the Rayleigh distribution is commonly used to model the multipath fading in an urban environment where there are a large number of received radio wave paths. Hence, the Rayleigh model is derived theoretically by using a central limit theorem (CLT) argument. However, when the number of incoming radio paths is limited, the Rayleigh distribution may not be an appropriate fading model since the conditions for validity of the CLT may not hold. Some evidence indicates that the signal amplitude can be well described by a Weibull distribution in this situation. Experimental data supporting the appropriateness of the Weibull model was reported in [1] and Hashemi considered its use as a model for indoor fading channels in [2]. Recent measurements by Babich and Lombardi have revealed that the Weibull distribution gives the best fit to a path-loss model of a narrow-band digital enhanced cordless telecommunications (DECT) system at reference frequency 1.89 GHz [3]. Based on fading channel data obtained from a recent measurement program at 900 MHz, Tzeremes and Christodoulou also reported that the Weibull distribution can be used to model outdoor multipath fading well in some cases [4].

The probability density function (PDF) of the Weibull distribution is [5]

$$f_X(x) = \frac{mx^{m-1}}{\gamma} \exp\left[-\frac{x^m}{\gamma}\right], m > 0, \quad x \geq 0 \quad (1)$$

where the index m is called the Weibull fading parameter and γ is a positive parameter related to the moments and the fading parameter. The Weibull fading parameter can take values between 0 and ∞. In the special case when $m = 1$, the Weibull distribution becomes an exponential distribution; when $m = 2$, the Weibull distribution specializes to a Rayleigh distribution.

Theoretical analyses are facilitated by the use of the Weibull distribution to model fading signal amplitude. This is because the Weibull distribution has an interesting property, That is, the kth power of a Weibull-distributed random variable (RV) with parameters (m, γ) gives another Weibull-distributed RV with parameters $(m/k, \gamma)$. This property is particularly useful in error rate analysis of digital modulations on Weibull fading channels since the received signal-to-noise (power) ratio (SNR) will also have a Weibull distribution. Despite the importances of this fading model, there exist few results pertaining to digital communications over Weibull fading channels with the exception of a recent paper by Alouini and Simon [6]. In this work, the authors studied the output SNR of generalized selection combining (GSC) diversity over Weibull fading channels. However, a treatment of the error rate performance of digital modulations operating on Weibull channels was not given in [6].

The moment generating function (MGF) method is gaining popularity as an approach to unified error rate analysis of digital communications in fading channels [7],[8]. Both symbol error rates (SER's) and outage probabilities (OP's) can be expressed in terms of the MGF of the SNR in a single-channel receiver system and the MGF of the SNR at the output of a diversity combiner in a multi-channel receiver system. The MGF of the Weibull distribution is known to exist for $m \geq 1$, but its form is not considered useful by statisticians and therefore, is not readily available. In this paper, we use a Mellin transform [9] to derive a compact expression for the MGF of a Weibull distribution. Using this novel MGF expression, we study the signal outage and symbol error rate performances of some important linear modulation schemes for both single-channel reception and multi-channel diversity reception operating on Weibull fading channels.

II. Weibull Moment Generating Function

If X is a random variable (RV) distributed (In this work, we assume that m only takes integer values and this condition is required for our mathematical solution.)

The authors are with the Department of Electrical and Computer Engineering, University of Alberta, Edmonton, Alberta, Canada T6G 2V4. (e-mail: {jcheng, chintha, beaulieu}@ee.ualberta).

according to (1), its MGF is given by

$$\mathcal{M}_X(s) = \int_0^\infty e^{-sx} f_X(x)dx. \quad (2)$$

In order to evaluate the integral in (2), it is required to evaluate an integral of the type

$$I = \int_0^\infty x^{p-1} e^{-zx-\alpha x^r} dx \quad \alpha, z, p > 0 \quad (3)$$

where r is a positive integer. In Appendix A, we show that

$$I = (2\pi)^{\frac{1-r}{2}} r^{p-\frac{1}{2}} z^{-p} \times G_{1,r}^{r,1}\left(\frac{z^r}{\alpha r^r} \middle| \begin{array}{c} 1 \\ p/r, \ldots, (p+r-1)/r \end{array} \right) \quad (4)$$

where $G_{p,q}^{m,n}(\cdot)$ is the Meijer's G function [10, (9.301)]. Applying (4) to (2) we obtain the MGF for the Weibull distribution as

$$\mathcal{M}_X(s) = \left(\frac{m}{\gamma}\right) (2\pi)^{\frac{1-m}{2}} m^{(m-\frac{1}{2})} s^{-m} \times G_{1,m}^{m,1}\left(\gamma \left(\frac{s}{m}\right)^m \middle| \begin{array}{c} 1 \\ 1, \ldots, 1+(m-1)/m \end{array} \right). \quad (5)$$

III. System Model

We consider signal transmission over slow, frequency-nonselective Weibull fading channels. The received signal can be written as

$$s_o = \sum_{i=1}^{L} w_i [s_i + n_i] \quad (6)$$

where s_i is the signal component and it is assumed to have unity power, i.e., $E[s_i^2] = 1$. In this paper, two diversity combining methods are considered. When selection combining (SC) is employed, the weighting factors become $w_j = 1,\ w_i = 0,\ \forall i \neq j,\ i,j = 1, \ldots, L$ where $\max\{X_i^2\} = X_j^2$, and when maximal ratio combining (MRC) is employed, the weighting factors become $w_i = X_i e^{-j\alpha_i}, \quad i = 1, \ldots, L$ where X_i is the ith branch fading amplitude and α_i is the ith branch fading phase, and they are both assumed to be perfectly estimated. We further assume that the background noise has unity power, i.e., $E[n_i^2] = 1$. Under these assumptions, the SNR per symbol on the ith diversity branch becomes X_i^2.

IV. SNR and Outage Probability

A. Single Channel Reception

Let Y denote the combiner output SNR. Under the assumptions of our system model, the output SNR at the receiver is $Y = X^2$. One can show that the PDF of Y is

$$f_Y(y) = \frac{(\frac{m}{2}) y^{m/2-1}}{\gamma} \exp\left[-\frac{y^{m/2}}{\gamma}\right] \quad (7)$$

where $E[Y] = E[X^2] = \gamma^{2/m}\Gamma(1+2/m)$. The outage probability of single channel reception can be calculated as

$$\begin{aligned} P_{out}(\xi) &= \int_0^\xi f_Y(y)dy \\ &= 1 - \exp\left[-\frac{\xi^{m/2}}{\gamma}\right]. \end{aligned} \quad (8)$$

B. Selection Combining Diversity Reception

The outage probability for SC is given by

$$P_{out,sc}(\xi) = \left[1 - \exp\left[-\frac{\xi^{m/2}}{\gamma}\right]\right]^L. \quad (9)$$

To obtain the average SNR at the output of the SC, $\overline{SNR_{o,sc}}$, we first note that the outage probability computed in (9) is essentially the cumulative distribution function (CDF) of $SNR_{o,sc}$, i.e, $\Pr(SNR_{o,sc} < \xi) = F(\xi)$. One can then show that the average output SNR is given by

$$\begin{aligned} \overline{SNR_{o,sc}} &= \int_0^\infty \xi dF(\xi) \\ &= L\gamma^{2/m} \sum_{k=0}^{L-1} (-1)^k \binom{L-1}{k} \frac{\Gamma\left(\frac{2}{m}+1\right)}{(k+1)^{\frac{2}{m}+1}}. \end{aligned} \quad (10)$$

In the special case of Rayleigh fading ($m = 2$), it can be shown that (10) reduces to the well-known result $\overline{SNR_{o,sc}} = \sum_{k=1}^{L} 1/k$ [11].

C. Maximal Ratio Combining Diversity Reception

In maximal ratio combining, the output SNR, denoted by $SNR_{o,mrc}$, is equal to the sum of the branch SNR values. Therefore,

$$\overline{SNR_{o,mrc}} = E[X_1^2 + X_2^2 + \ldots + X_L^2] = LE[X_i^2] \quad (11)$$

where

$$E[X_i^2] = \gamma^{2/m}\Gamma(1+2/m). \quad (12)$$

The outage probability for MRC is given by

$$P_{out,mrc}(\xi) = \Pr\left(\sum_{i=1}^{L} SNR_i < \xi\right) \quad (13)$$

which can be calculated using the MGF method. The MGF of the MRC output SNR can be written as

$$\mathcal{M}_Y(s) = [\mathcal{M}_X(s, m/2)]^L. \quad (14)$$

Since the outage probability is just the inverse Laplace transform of $\mathcal{M}_Y(s)/s$ evaluated at a threshold ξ, and one has

$$P_{out,mrc}(\xi) = \frac{1}{2\pi j} \int_{\sigma-j\infty}^{\sigma+j\infty} \frac{[\mathcal{M}_X(s, m/2)]^L}{s} e^{s\xi} ds \quad (15)$$

where σ is a constant properly chosen in the region of convergence in the complex s plane.

V. BER and SER of Linear Modulations

A. CBPSK, CBFSK, and M-PAM

The conditional BER or SER for coherent binary phase shift keying (CBPSK), coherent binary frequency shift keying (CBFSK), and M-ary pulse amplitude modulation (M-PAM) can be written in a compact form as [12]

$$P_e(\gamma_t) = aQ(\sqrt{2b\gamma_t}) \tag{16}$$

where the Gaussian Q-function is defined as

$$P_e(\gamma_t) = \frac{a}{\pi}\int_0^{\pi/2} \exp\left(-\frac{b\gamma_t}{\sin^2\phi}\right) d\phi. \tag{17}$$

and where γ_t is the total output SNR per symbol. Eqn. (16) is valid for CBPSK when $a = 1$ and $b = 1$; for CBFSK when $a = 1$, and $b = 1/2$; and for symmetric M-PAM when $a = 2(M-1)/M$ and $b = 3/(M^2-1)$. Using an alternative Q-function representation [7], the conditional error rate can be written as

$$P_e(\gamma_t) = \frac{a}{\pi}\int_0^{\pi/2} \exp\left(-\frac{b\gamma_t}{\sin^2\phi}\right) d\phi. \tag{18}$$

Averaging over the PDF of the received SNR, the average error rate becomes

$$\overline{P_e} = \frac{a}{\pi}\int_0^{\pi/2} \mathcal{M}_{\gamma_t}\left(\frac{b}{\sin^2\phi}\right) d\phi \tag{19}$$

where in the case of single channel reception, $\mathcal{M}_{\gamma_t}(s) = \mathcal{M}_X(s, m/2)$, which is the MGF of a Weibull with fading parameter $m/2$. In the case of SC, in order to calculate the MGF of γ_t, the SNR at the output of the SC, we can utilize the derivative property of the Laplace transform and relate the MGF of a RV to its CDF as

$$\mathcal{M}_Z(s) = s\int_0^{\infty} \exp(-sz) F_Z(z) dz. \tag{20}$$

For our problem, the CDF of the SNR at the SC output can be shown to be

$$\begin{aligned} F_{\gamma_t}(\gamma_t) &= \left[1 - \exp\left(-\frac{\gamma_t^{m/2}}{\gamma}\right)\right]^L \\ &= \sum_{n=0}^{L} (-1)^n \binom{L}{n} \exp\left[-\frac{n\gamma_t^{m/2}}{\gamma}\right]. \end{aligned} \tag{21}$$

Using (20) and (21), one can show that the MGF of the SC output SNR is

$$\begin{aligned} \mathcal{M}_{\gamma_t}(s) = 1 + (2\pi)^{\frac{1}{2}-\frac{m}{4}} \sqrt{\frac{m}{2}} \sum_{n=1}^{L} \binom{L}{n} (-1)^n \\ \times G_{1,m/2}^{m/2,1}\left(\left(\frac{\gamma}{n}\right)\left(\frac{2s}{m}\right)^{m/2} \middle| \begin{matrix} 1 \\ 2/m, 4/m, \ldots, 1 \end{matrix}\right). \end{aligned} \tag{22}$$

In the case of MRC, the total output SNR is equal to the sum of the independent branch SNR's. Therefore, the MGF of γ_t becomes

$$\mathcal{M}_{\gamma_t}(s) = \left[\mathcal{M}_X(s, m/2)\right]^L, \tag{23}$$

and the error rate is given by

$$\overline{P_{e,mrc}} = \frac{a}{\pi}\int_0^{\pi/2} \left[\mathcal{M}_X\left(\frac{b}{\sin^2\phi}, m/2\right)\right]^L d\phi. \tag{24}$$

B. M-PSK

The average SER of M-PSK for L-branch MRC diversity reception can be derived as

$$\overline{P_{e,mrc}} = \frac{1}{\pi}\int_0^{(M-1)\pi/M} \left[\mathcal{M}_X\left(\frac{g_{PSK}}{\sin^2\phi}, m/2\right)\right]^L d\phi \tag{25}$$

where $g_{PSK} = \sin^2(\pi/M)$. For single channel reception, substituting $L = 1$ into (25), we obtain

$$\overline{P_e} = \frac{1}{\pi}\int_0^{(M-1)\pi/M} \mathcal{M}_X\left(\frac{g_{PSK}}{\sin^2\phi}, m/2\right) d\phi. \tag{26}$$

When SC combining is used, the SER of MPSK is given by

$$\overline{P_{e,sc}} = \frac{1}{\pi}\int_0^{(M-1)\pi/M} \mathcal{M}_{\gamma_t}\left(\frac{g_{PSK}}{\sin^2\phi}\right) d\phi \tag{27}$$

where $\mathcal{M}_{\gamma_t}(s)$ is the MGF of the SC output SNR given in (22).

C. M-QAM

When M-QAM is employed in Weibull channels, using the MGF method, it can be shown that in the case of MRC reception, the average SER is given by

$$\begin{aligned} \overline{P_{e,mrc}} = \frac{4}{\pi}\left(1 - \frac{1}{\sqrt{M}}\right) \\ \times \int_0^{\pi/2} \left[\mathcal{M}_X\left(\frac{g_{QAM}}{\sin^2\phi}, m/2\right)\right]^L d\phi \\ - \frac{4}{\pi}\left(1 - \frac{1}{\sqrt{M}}\right) \\ \times \int_0^{\pi/4} \left[\mathcal{M}_X\left(\frac{g_{QAM}}{\sin^2\phi}, m/2\right)\right]^L d\phi; \end{aligned}$$

where $g_{QAM} = 3/(2(M-1))$; and in the case of SC reception, the average SER is given by

$$\begin{aligned} \overline{P_{e,sc}} = \frac{4}{\pi}\left(1 - \frac{1}{\sqrt{M}}\right)\int_0^{\pi/2} \mathcal{M}_{\gamma_t}\left(\frac{g_{QAM}}{\sin^2\phi}\right) d\phi \\ - \frac{4}{\pi}\left(1 - \frac{1}{\sqrt{M}}\right)\int_0^{\pi/4} \mathcal{M}_{\gamma_t}\left(\frac{g_{QAM}}{\sin^2\phi}\right) d\phi \end{aligned}$$

where $\mathcal{M}_{\gamma_t}(s)$ is given in (22).

VI. Numerical Results

Fig. 1 shows the outage probability versus average SNR per branch for a single channel reception and dual diversity ($L = 2$) reception with outage threshold $\xi = 10\ dB$ for two different Weibull fading parameters. As expected, the outage probability performance improves when the diversity order increases, and the outage performance of MRC reception outperforms the SC reception with the same level of fading severity. As the Weibull fading parameter increases, the outage probability curves become sharper and the performance approaches that of a nonfading situation. For example, when the $SNR = 4\ dB$, the system can achieve 99% outage performance for MRC reception with dual diversity branches while operating in a Weibull channel with $m = 4$.

Fig. 2 shows the average BER performance for coherent binary frequency shift keying with single channel reception and dual diversity reception. As we expect, the BER performance of diversity reception outperforms that of single channel reception, and a lightly faded Weibull fading channel has a better BER performance than another Weibull fading channel with stronger fading. Fig. 2 also shows that when the value of SNR is greater than $8\ dB$, CBFSK can achieves BER less than 10^{-3} with dual branch MRC when $m = 4$. Similar observations can be made for the symbol error rate performance of 8-PSK (Fig. 3) and 16-QAM (Fig. 4) operating on Weibull channels with multi-channel reception.

VII. Conclusion

In this paper, we have analyzed the performances of digital communication systems on Weibull fading channels. In doing so, we have derived a novel closed-form expression for the MGF of the Weibull distribution when the fading parameters taking integer values. Outage probabilities and average output SNR expressions have been derived for both single channel and multi-channel SC and MRC reception. Using the MGF method, we have also obtained exact single-integral BER and SER expressions for some important linear modulation schemes operating on Weibull channels. Our performance results are valid for arbitrary numbers of independent diversity branches.

Appendix

I. Derivation of (4)

In this appendix, we derive the integral identity in (4). The Mellin transform of $f(x)$ with respect to the complex parameter s is defined as [9]

$$M(s) = \int_0^\infty x^{s-1} f(x) dx \tag{28}$$

and the inverse Mellin transform is given by

$$f(x) = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} M(s) x^{-s} ds \tag{29}$$

where c is a suitably chosen number. Putting $f(x) = e^{-\alpha x^r}$, the Mellin transform of $f(x)$ becomes

$$\begin{aligned} M(s) &= \int_0^\infty x^{s-1} e^{-\alpha x^r} dx \\ &= \frac{1}{r} \alpha^{-s/r} \Gamma\left(\frac{s}{r}\right). \end{aligned} \tag{30}$$

Therefore, from the definition of inverse Mellin transform, we have

$$e^{-\alpha x^r} = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} \alpha^{-s} \Gamma(s) x^{-rs} ds \tag{31}$$

where c is a suitably chosen number. Substituting (31) into (3) and integrating over x, we get

$$I = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} \alpha^{-s} z^{rs-p} \Gamma(s) \Gamma(p - rs) ds. \tag{32}$$

Since r is assumed to be integer, the Gauss multiplication formula [13] allows us to write

$$\Gamma(p - rs) = (2\pi)^{\frac{1-r}{2}} r^{p-rs-\frac{1}{2}} \prod_{k=1}^{r} \Gamma\left(\frac{p+k-1}{r} - s\right). \tag{33}$$

From (33) and (32), we obtain

$$\begin{aligned} I = &\frac{(2\pi)^{\frac{1-r}{2}} r^{p-\frac{1}{2}} z^{-p}}{2\pi j} \\ &\times \int_{c-j\infty}^{c+j\infty} \left(\frac{z^r}{\alpha r^r}\right)^s \Gamma(s) \prod_{k=1}^{r} \Gamma\left(\frac{p+k-1}{r} - s\right) ds. \end{aligned} \tag{34}$$

Now comparing (34) with the definition of Meijer's G function [10, (9.301)] we obtain (4) as desired.

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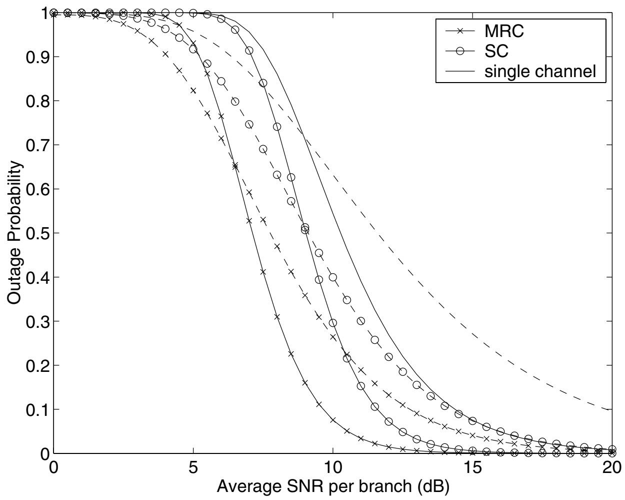

Fig. 1. Outage probability versus average SNR per branch for single channel reception, SC reception ($L = 2$) and MRC reception ($L = 2$) on Weibull fading channels when $m = 2$ (dashed lines) and $m = 4$ (solid lines) at outage threshold $\xi = 10\ dB$.

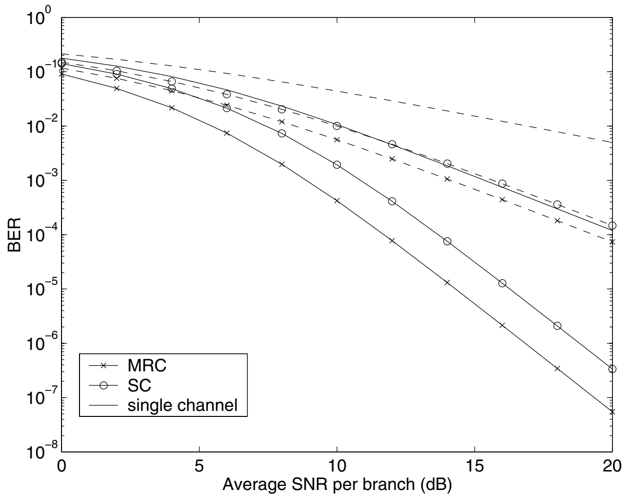

Fig. 2. Average BER versus average SNR per branch for CBFSK with single channel reception, SC reception ($L = 2$) and MRC reception ($L = 2$) on Weibull fading channels when $m = 2$ (dashed lines) and $m = 4$ (solid lines).

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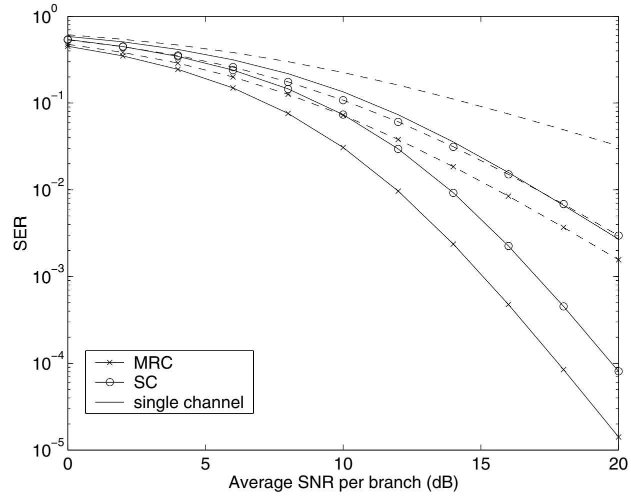

Fig. 3. Average SER versus average SNR per branch for 8-PSK with single channel reception, SC reception ($L = 2$) and MRC reception ($L = 2$) on Weibull fading channels when $m = 2$ (dashed lines) and $m = 4$ (solid lines).

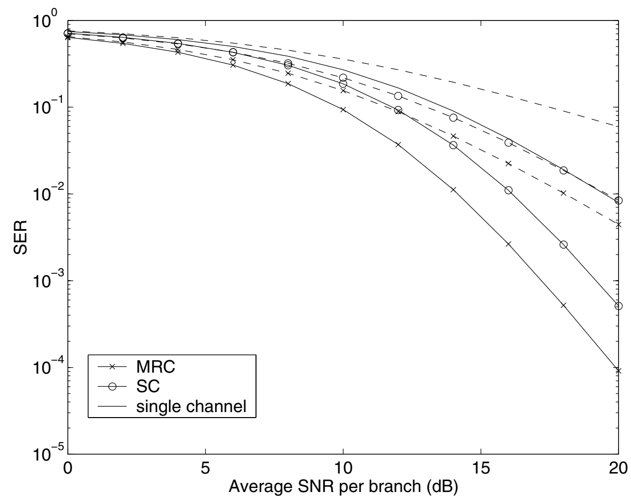

Fig. 4. Average SER versus average SNR per branch for 16-QAM with single channel reception, SC reception ($L = 2$) and MRC reception ($L = 2$) on Weibull fading channels when $m = 2$ (dashed lines) and $m = 4$ (solid lines).