# Unified Analysis of Generalized Selection Diversity with Normalized Threshold Test per Branch

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*Abstract* — **The ability to capture significant amount of transmitted signal energy present in the resolvable multipaths using only a modest number of rake fingers (correlators) is an important receiver design consideration for ultra-wideband (UWB) and wideband CDMA communication systems. This, however, is achieved at the expense of rake receiver performance. Motivated by this need, several suboptimal hybrid receiver structures have been proposed in the literature. The study on generalized selection diversity combining with normalized threshold test per branch ($\mathrm{T-GSC}(\mu, L)$) is also important from a theoretical standpoint because this model encapsulates both the traditional selection diversity and maximal-ratio combining (coherent detection) or post-detection equal-gain combining (noncoherent detection) schemes as limiting cases. However, mathematical analysis for $\mathrm{T-GSC}(\mu, L)$ has been limited to i.i.d Rayleigh fading channels. This paper derives simple to evaluate formulas for the moment generating function (mgf) of $\mathrm{T-GSC}(\mu, L)$ output SNR with $L$ resolvable multipaths. In addition to providing new results for many cases that heretofore had resisted solution in a simple form, our approach also also allows some of the previously obtained results to be simplified both analytically and computationally.**

## I. INTRODUCTION

Even though the MRC-rake receiver solution (i.e., combining all the $L$ resolvable multipaths using maximal-ratio combining technique) is optimum from a performance standpoint, it is not be desirable for practical implementation when a large number of multipaths are available for several obvious reasons. To reduce the receiver complexity and total power consumption of handheld portable units, several suboptimal hybrid rake receiver structures have been proposed in the literature including generalized selection diversity combining and partitioned diversity schemes [1]-[7].

Recognizing that the relative diversity improvement diminishes with increasing diversity order, it seems reasonable to combine only a few "strongest" multipaths to achieve the desired level of performance. This is the main motivation behind the development of suboptimal coherent generalized selection combining ($\mathrm{GSC}(N, L)$) technique [1] in which a subset of $N \le L$ paths with highest signal-to-noise ratios (SNRs) are optimally weighted and summed. For noncoherent and differentially coherent communications, the need for noncoherent $\mathrm{GSC}(N, L)$ is further emphasized owing to the noncoherent combining loss phenomenon inherent in the post-detection equal-gain combining (hereafter, referred to as PDEGC-rake) receiver [2]. In this case, there exists an optimum $N$ which minimizes the error probability performance for a given average SNR/bit and $L$.

While $\mathrm{GSC}(N, L)$ has a fixed processing complexity (since $N$ is decided a priori), it suffers from the fact that it potentially discards from combination of many paths which may significantly improve the receiver performance or alternatively, includes some "weak" paths at the expense of increased total processing power consumption. In [5], Sulyman and Kousa suggested a threshold-based generalized selection diversity receiver, $\mathrm{T-GSC}(\mu, L)$, which alleviates the above-mentioned problem by combining only the paths that has normalized SNR (ratio of the instantaneous SNR of each branch to that of the best branch) greater than $\mu$ ($0 \le \mu \le 1$). An analytical framework for analyzing such a receiver is provided in [6]. Nevertheless, the results are limited to Rayleigh fading with independent and identically distributed (i.i.d) diversity paths. In this paper, we develop a mathematical framework for analyzing both the coherent and noncoherent $\mathrm{T-GSC}(\mu, L)$ receiver performance over practical wireless channels. The key to our solution is the transformation of multivariate nested integrals that arise in the computation of mgf of SNR and $Pr\{N = k\}$ into a product form of univariate integrals.

## II. $\mathrm{T-GSC}(\mu, L)$ COMBINER OUTPUT STATISTICS

### *A. Independent and Identical Fading Statistics*

Suppose $\gamma_1, \gamma_2, \ldots, \gamma_L$ are i.i.d fading SNR random variables having pdf $f(x)$, cumulative distribution function (cdf) $F(x)$ and marginal mgf $\phi(s, x) = \int_x^\infty e^{-st} f(t) dt$. If the instantaneous SNRs are arranged in the increasing order of magnitude, we also have $\gamma_{1:L} \le \gamma_{2:L} \le \ldots \le \gamma_{L:L}$ as the order statistics. Then the instantaneous $\mathrm{T-GSC}(\mu, L)$ output SNR is given by

$$\gamma_{Tgsc} = \sum_{k=1}^{L} w_k \gamma_{\mathrm{gsc}} \tag{1}$$

where $\gamma_{\text{gsc}} = \sum_{k=L-N+1}^{L} \gamma_{k:L}$ is the $\text{GSC}(N, L)$ combiner output SNR, $w_k = \begin{cases} 1 & \text{if } \gamma_{k:L} \geq \mu\gamma_{L:L} \\ 0 & \text{otherwise} \end{cases}$ and $0 \leq \mu \leq 1$.

Clearly, the sum $N = \sum_{k=1}^{L} w_k$ is a random variable and $1 \leq N \leq L$. As such, this scheme may be viewed as a conventional $\text{GSC}(N, L)$ receiver whose number of paths being combined $N$ is random rather than fixed.

Now consider the following cases:

(a) If $\gamma_{1:L} \geq \mu\gamma_{L:L}$, we obtain $N = L$;

(b) If $\gamma_{2:L} \geq \mu\gamma_{L:L}$ but $\gamma_{1:L} < \mu\gamma_{L:L}$, we have $N = L - 1$;

(c) If $\gamma_{L-k+1:L} \geq \mu\gamma_{L:L}$ but $\gamma_{L-k:L} < \mu\gamma_{L:L}$, we have $N = k$;

and so on.

Obviously, $\text{T-GSC}(0, L)$ and $\text{T-GSC}(1, L)$ correspond to the classical MRC and SDC schemes respectively.

By now it should be apparent that once $Pr\{N = k\}$ is determined, the pdf, cdf and the mgf of $\text{T-GSC}(\mu, L)$ output SNR can be expressed as a weighted sum of the traditional $\text{GSC}(N, L)$ output SNR statistics. For an example, the mgf of $\text{T-GSC}(\mu, L)$ output SNR is given by

$$\phi_{\gamma_{Tgsc}}(s, \mu, L) = \sum_{k=1}^{L} Pr\{N = k\}\phi_{\gamma_{\text{gsc}}}(s, k, L) \tag{2}$$

where $\phi_{\gamma_{\text{gsc}}}(s, k, L)$ may be computed efficiently using [7]

$$\phi_{\gamma_{\text{gsc}}}(s, k, L) = k\binom{L}{k}\int_0^{\infty} e^{-sx} f(x)[F(x)]^{L-k}[\phi(s, x)]^{k-1} dx \tag{3}$$

in a variety of fading channel models (including Rayleigh, Rician, Nakagami-m fading models). Closed-form formulas for the pdf, cdf and the marginal mgf for the above channel models can be found in [7].

For $N = k$, we know that $\gamma_{L-k+1:L} \geq \mu\gamma_{L:L}$ and $\gamma_{L-k:L} < \mu\gamma_{L:L}$. From this definition, we have

$$Pr\{N = k\} = L!\int_0^{\infty} f(x_L)\int_{\mu x_L}^{x_L} f(x_{L-1})\int_{\mu x_L}^{x_{L-1}} f(x_{L-2})\ldots\int_{\mu x_L}^{x_{L-k+2}} f(x_{L-k+1})$$
$$\times\int_0^{\mu x_L} f(x_{L-k})\int_0^{x_{L-k}} f(x_{L-k-1})\ldots\int_0^{x_2} f(x_1)dx_1\ldots dx_L \tag{4}$$

which is a product of separable but nested integral, and $0 \leq x_1 < x_2 < \ldots < x_L < \infty$. Eq. (4) can be simplified as

$$Pr\{N = k\} = k\binom{L}{k}\int_0^{\infty} f(x)[F(\mu x)]^{L-k}[F(x) - F(\mu x)]^{k-1} dx \tag{5}$$

using the integral identity

$$\int_a^{x_M} f(x_{M-1})\ldots\int_a^{x_2} f(x_1)dx_1\ldots dx_{M-1} = \frac{[F(x_M) - F(a)]^{M-1}}{(M-1)!}, M \geq 2 \tag{6}$$

which transforms the multivariate integral into a product of univariate integrals. A proof for (6) is provided in [9] using the principles of mathematical induction.

In [9], we also provide an alternative derivation for (5), which exploits the knowledge of the joint pdf of $(\gamma_{L-k:L}, \gamma_{L-k+1:L}, \gamma_{L:L})$ instead of using (4).

For the special case of Rayleigh fading, (5) reduces into

$$Pr\{N = k\} = k\binom{L}{k}\sum_{i=0}^{L-k}(-1)^i\binom{L-k}{i}\sum_{j=0}^{k-1}(-1)^j\binom{k-1}{j}$$
$$\times 1/[1 + j + \mu(k - j - 1 + i)] \tag{7}$$

which is in perfect agreement with [6]. However, note that our derivation is much more general yet concise.

*B. Independent and Nonidentical Fading Statistics*

Suppose $\gamma_1, \gamma_2, \ldots, \gamma_L$ are independently distributed, $\gamma_k$ having cdf $F_k(x)$, pdf $f_k(x)$ and marginal mgf $\phi_k(s, x)$. In this case, $\phi_{\gamma_{\text{gsc}}}(s, k, L)$ can be evaluated as [8]

$$\phi_{\gamma_{\text{gsc}}}(s, k, L) = \sum_{\sigma \in S_L}\int \ldots \int_{0 \leq x_1 < x_2 < \ldots < x_L < \infty} e^{-s\sum_{i=L-k+1}^{L} x_i}\prod_{i=1}^{L} f_{\sigma(i)}(x_i)dx_L\ldots dx_1 \tag{8}$$

where $S_n$ is the set of all permutations of integers $\{1, 2, \ldots, n\}$ and $\sigma \in S_n$ denotes the specific function $\sigma = (\sigma(1), \sigma(2), \ldots, \sigma(n))$ which permutes the integers $\{1, 2, \ldots, n\}$. The cardinality of $S_n$ is equal to $n!$. The process of constructing all members of $S_n$ is recursive and most mathematical packages have explicit commands for this purpose. For instance, $S_n$ is constructed by the command *perms*([1,2,...,n]) in MATLAB.

Using (A.1) and (A.2), (8) can be simplified into

$$\phi_{\gamma_{\text{gsc}}}(s, k, L) = \sum_{\sigma \in T_{L,k}}\int_0^{\infty} e^{-sx} f_{\sigma(L-k+1)}(x)\left[\prod_{i=1}^{L-k} F_{\sigma(i)}(x)\right]$$
$$\times\left[\prod_{j=L-k+2}^{L}\phi_{\sigma(j)}(s, x)\right]dx, \quad 1 \leq k \leq L \tag{9}$$

or alternatively, as

$$\phi_{\gamma_{\text{gsc}}}(s, k, L) = \sum_{\sigma \in T_{L,k+1}}\int_0^{\infty} f_{\sigma(L-k)}(x)\left[\prod_{j=L-k+1}^{L}\phi_{\sigma(j)}(s, x)\right]$$
$$\times\left[\prod_{i=1}^{L-k-1} F_{\sigma(i)}(x)\right]dx, \quad 1 \leq k < L \tag{10}$$

where $\sum_{\sigma \in T_{L,k}} = \sum_{\substack{\sigma \in S_L, \sigma(1) < \sigma(2) < \ldots < \sigma(L-k) \\ \sigma(L-k+2) < \ldots < \sigma(L)}}$.

The construction of all permutations in the group $T_{L,k}$ can be automated using only four command lines in MATLAB. It should also be emphasized that $\phi_{\gamma_{\text{gsc}}}(s, k, L)$ can be evaluated efficiently using (9) for $1 \leq k \leq \lfloor L/2 \rfloor$ and (10) while $\lceil L/2 \rceil \leq k < L$. This is because the complexity of (9) is mainly dictated by the time required to compute $k\binom{L}{k}$

one-dimensional integrals while (10) involves the complexity of evaluation of $(L-k)\binom{L}{k}$ one-dimensional integrals.

For the special case of i.i.d fading, (9) reduces to (3), as expected. It is also not very difficult to show that

$$Pr\{N=k\} = \sum_{\sigma \in S_L} \int_0^{\infty} f_{\sigma(L)}(x_L) \int_{\mu x_L}^{x_L} f_{\sigma(L-1)}(x_{L-1}) \int_{\mu x_L}^{x_{L-1}} f_{\sigma(L-2)}(x_{L-2})$$

$$\times \ldots \int_{\mu x_L}^{x_{L-k+2}} f_{\sigma(L-k+1)}(x_{L-k+1}) \int_0^{\mu x_L} f_{\sigma(L-k)}(x_{L-k})$$

$$\times \int_0^{x_{L-k}} f_{\sigma(L-k-1)}(x_{L-k-1}) \ldots \int_0^{x_2} f_{\sigma(1)}(x_1) dx_1 \ldots dx_{L-1} dx_L \quad (11)$$

which may be rewritten concisely as

$$Pr\{N=k\} = \sum_{\substack{\sigma \in S_L,\, \sigma(1) < \ldots < \sigma(L-k) \\ \sigma(L-k+1) < \ldots < \sigma(L-1)}} \int_0^{\infty} f_{\sigma(L)}(x_L) \left[ \prod_{k=1}^{L-k} F_{\sigma(k)}(\mu x_L) \right]$$

$$\times \prod_{i=L-k+1}^{L-1} [F_{\sigma(i)}(x_L) - F_{\sigma(i)}(\mu x_L)] dx_L \quad (12)$$

with the aid of (A.1). Substituting (12) and (9) (or (10)) into (2), we obtain an expression for the mgf of $\mathrm{T-GSC}(\mu, L)$ output SNR with i.n.d diversity paths. This mgf can used to unify the performance evaluation of a broad range of digital modulation/detection schemes in practical wireless channels. Details can be found in [9].

## III. COMPUTATIONAL RESULTS AND REMARKS

From (2), we immediately obtain a generic expression for the average symbol error rate (ASER) with $\mathrm{T-GSC}(\mu, L)$:

$$\bar{P}_{S-Tgsc}(\mu, L) = \sum_{k=1}^{L} Pr\{N=k\} \bar{P}_{S-\mathrm{gsc}}(k, L) \quad (13)$$

Similarly, the outage probability can be calculated as

$$P_{out-Tgsc}(\gamma^*, \mu, L) = \sum_{k=1}^{L} Pr\{N=k\} P_{out-\mathrm{gsc}}(\gamma^*, k, L) \quad (14)$$

For the purpose of illustrating our generic results for ASER and outage probability numerically, we shall focus primarily on the i.i.d fading case. Fig. 1 shows the ASER variation for QPSK with coherent $\mathrm{T-GSC}(\mu, 5)$ in Rician channels when the average SNR/symbol/path is fixed at 10 dB. The curves for $\mu = 1$ and $\mu = 0$ coincide with the plots obtained for $\mathrm{GSC}(1, 5)$ and $\mathrm{GSC}(5, 5)$ respectively. These curves also provide upper and lower bounds for other threshold values. Not only does the ASER decreases in the presence of a stronger specular component (i.e., a higher $K$ value), the rate at which the ASER decays also increases with smaller $\mu$ values. Thus, lowering $\mu$ significantly improves the receiver performance in less severely faded environments. For example, as the threshold changes from $\mu = 0.7$ to $\mu = 0.15$, the ASER decreases from $3 \times 10^{-5}$ to $5 \times 10^{-7}$ (approximately two order of magnitude) at $K = 2$ and by almost three order of magnitude at $K = 5$.

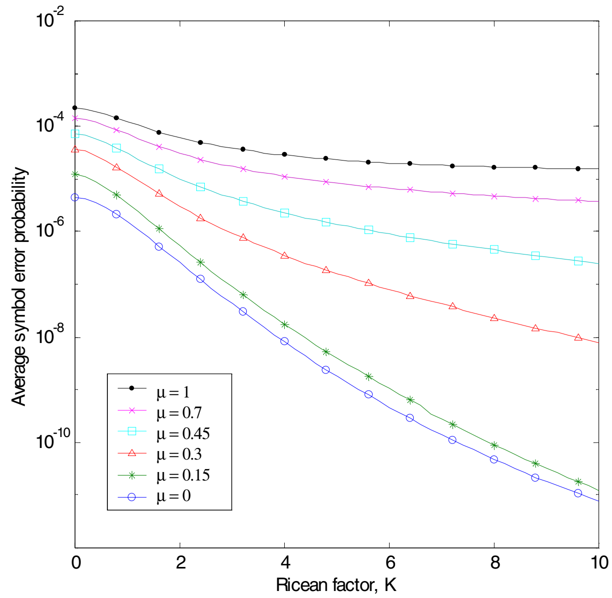


Fig. 1. ASER of QPSK plotted as as function of Rice factor for a coherent $\mathrm{T-GSC}(\mu, 5)$ receiver with i.i.d diversity paths when the average SNR/symbol/path is 10 dB.

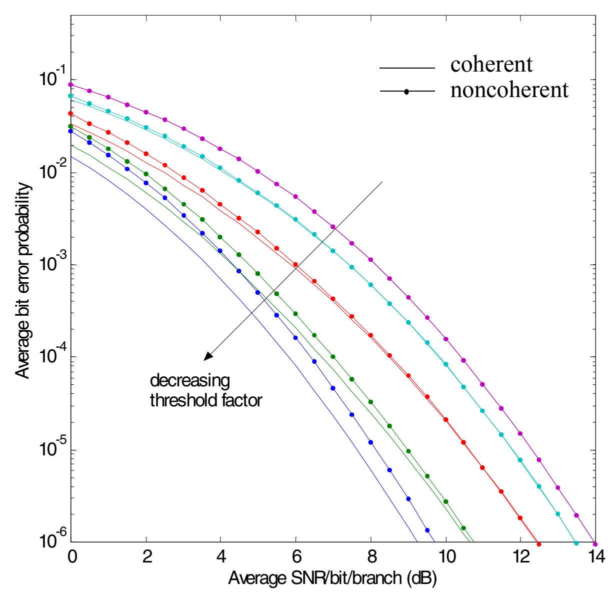


Fig. 2. ABER performance of DQPSK in a Nakagami-m channel (fading index $m = 1.75$) in conjunction with both coherent and noncoherent $\mathrm{T-GSC}(\mu, 5)$ (for $\mu = 0, 0.2, 0.4, 0.7, 1$) receiver implementations.

Fig. 2 illustrates the ABER performance of DQPSK in conjunction with coherent and noncoherent $\mathrm{T-GSC}(\mu, 5)$ receivers in a Nakagami-m channel (fading index $m = 1.75$). The difference between the coherent and noncoherent $\mathrm{T-GSC}(\mu, 5)$ performance curves diminishes as the average SNR/bit/branch increases particularly for $\mu \gg 0$.

This trend can be attributed to the weight contribution for each of the possible diversity orders as depicted in Fig. 3. Also recall that for a fixed average SNR/bit/branch $\bar{\gamma}$, the difference in the ABER performance for DQPSK between the coherent and noncoherent classical $\mathrm{GSC}(N, 5)$ receivers is smaller at higher $\bar{\gamma}$ and for smaller $N$ values [7]. From Fig. 2, we can conclude that there is not much benefit that can be realized from using a coherent $\mathrm{T-GSC}(\mu, L)$ receiver over a noncoherent $\mathrm{T-GSC}(\mu, L)$ receiver if we are operating at high branch SNRs and high $\mu$ levels.

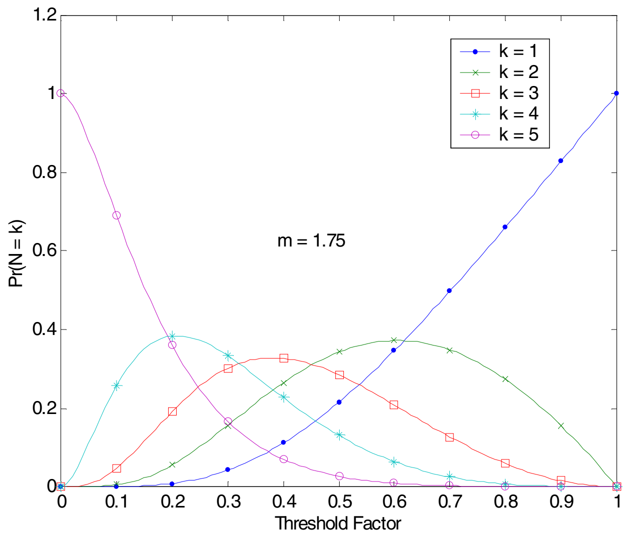


Fig. 3. Variation of the weighting probability $Pr\{N = k\}$ with the threshold factor $\mu$ in a Nakagami-m channel ($m = 1.75$).

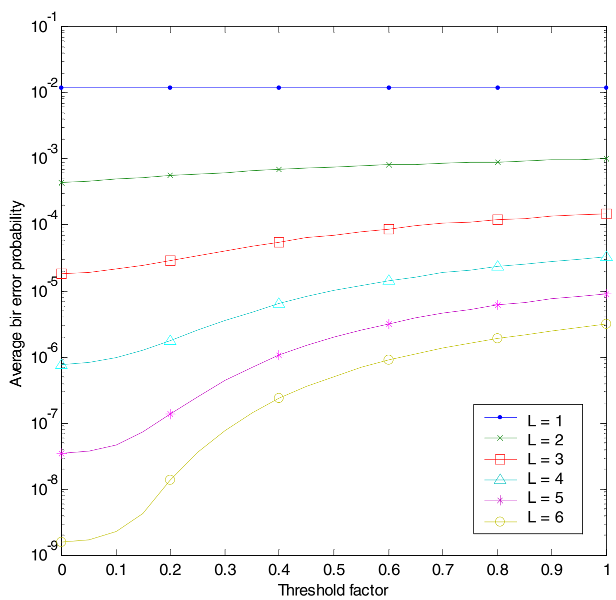


Fig. 4. Investigation on the effect of diversity order $L$ on the ABER performance of BPSK that employs $\mathrm{T-GSC}(\mu, L)$ in a Ricean channel ($K = 2$) and average SNR/bit/branch of 10 dB.

Fig. 4 illustrates the effect of increasing $L$ on the ABER performance of BPSK with $\mathrm{T-GSC}(\mu, L)$. For a higher diversity order $L$, the bit error probability decreases due to larger diversity gain. Since the improvement in ABER performance is much more pronounced at lower $\mu$, we may conclude that increasing $L$ is much more effective for a MRC receiver than for a SDC receiver to improve the overall receiver performance. It is also observed that the relative diversity improvement diminishes with increasing diversity order. This observation is more pronounced at a higher $\mu$ level as expected.

Comparison between Fig. 3 and Fig. 5 reveals that as the channel condition improves (i.e., a larger $m$ value), the mean of the density function for $Pr\{N = k\}$ shifts to the right. For a specified threshold $0 < \mu < 1$, the probability of choosing higher number of paths (larger $k$) is lowered when the channel experience more severe fading. This is expected because as the fading index decreases, the number of paths having signal levels greater than the threshold declines due to more frequent deep fades and wider dynamic range of received signal amplitude variations. In other words, for a specified $0 < \mu < 1$, the average number of paths combined $\bar{N}$ is higher in better channel conditions, viz.,

$$\bar{N} = \sum_{k=1}^{L} kPr\{N = k\} \tag{15}$$

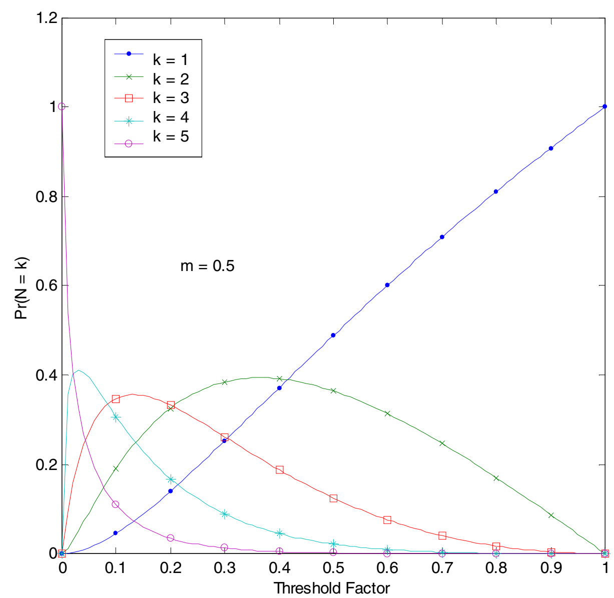


Fig. 5. Variation of the weighting probability $Pr\{N = k\}$ as a function of threshold factor $\mu$ in a Nakagami-m environment with fading index $m = 0.5$.

Fig. 6 depicts the normalized output SNR variations in Nakagami-m channels for different combinations of $\mu$ and fading index $m$. Even though $\bar{\gamma}_{Tgsc}/\bar{\gamma}$ decreases with $m$ for $\mu = 0.9$, it increases for $\mu = 0.15$. This behaviour can be explained as follows: If $\mu$ is small, the likelihood of higher order diversities (i.e., number of diversity paths that exceed threshold) is high; and hence weighting factors $Pr\{N = k\}$ for large $k$ values are also high. Moreover, the likelihood of combining majority of the available diversity paths at lower threshold levels increases with increasing $m$. Thus in a channel that experience severe fading (i.e., small $m$), $\bar{\gamma}_{Tgsc}/\bar{\gamma}$ increases for lower $\mu$ but start decreasing when $\mu$ gets very large. Finally, Fig. 7 shows the outage probability performance of $\mathrm{T-GSC}(\mu, 5)$ receiver operating in a Ricean fading environment.

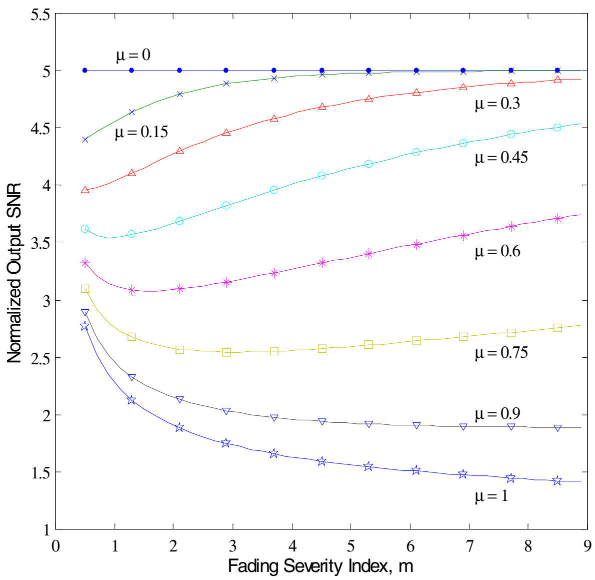


Fig. 6. Normalized output SNR $\bar{\gamma}_{Tgsc}/\bar{\gamma}$ plotted as a function fading severity index for different threshold levels and $L = 5$.

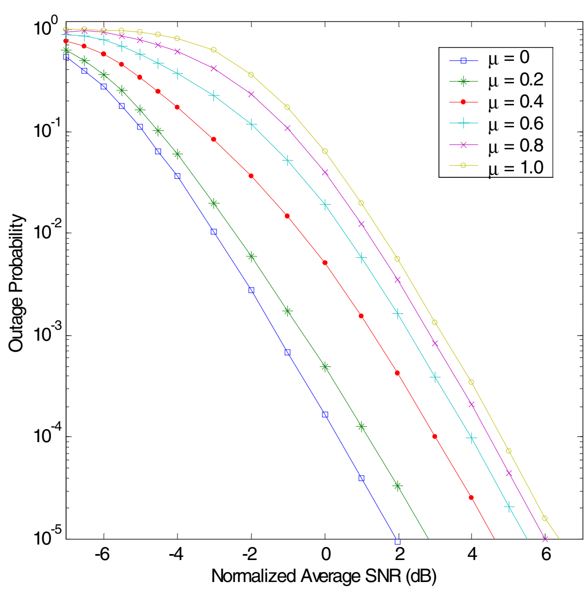


Fig. 7. Outage probability $P_{out-Tgsc}(\gamma^*, \mu, 5)$ plotted as a function of normalized mean SNR $\bar{\gamma}/\gamma^*$ in a Ricean channel with $K = 3$.

## APPENDIX

In this appendix, we provide two useful integral identities which can facilitate the transformation of multivariate nested integrals that arise in the computation of $\phi_{\gamma_{gsc}}(s, k, L)$ and $Pr\{N = k\}$ into a product of univariate integrals:

$$I_{M-1}(x_M, a) = \sum_{\sigma \in S_{M-1}} \int_a^{x_M} g_{\sigma(M-1)}(x_{M-1}) \ldots \int_a^{x_2} g_{\sigma(1)}(x_1) dx_1 \ldots dx_{M-1} = \prod_{k=1}^{M-1} G_{\sigma(k)}(x_M, a) \tag{A.1}$$

$$J_{M-1}(x_1) = \sum_{\sigma \in S_{M-1}} \int_{x_1}^{\infty} g_{\sigma(2)}(x_2) \ldots \int_{x_{M-1}}^{\infty} g_{\sigma(M)}(x_M) dx_M \ldots dx_2 = \prod_{k=1}^{M-1} H_{\sigma(M-k+1)}(x_1) \tag{A.2}$$

for any $M \geq 2$, provided $G_{\sigma(k)}(y, a) = \int_a^y g_{\sigma(k)}(x) dx < \infty$ and $H_{\sigma(k)}(y) = \int_y^{\infty} g_{\sigma(k)}(x) dx < \infty$ are absolutely integrable for $k = 1, 2, \ldots, M-1$.

A rigorous proof for (A.1) and (A.2) are given in [8]. Using these identities, we obtain a compact representation and computationally efficient formula for the mgf of $\mathrm{T-GSC}(\mu, L)$ output SNR over generalized fading channels [see (9), (10) and (12)].

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