Closed Form and Infinite Series Solutions for the MGF of a Dual-Diversity Selection Combiner Output in Bivariate Nakagami Fading

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***Abstract*—Using a circular contour integral representation for the generalized Marcum-Q function, $Q_m(a,b)$, we derive a new closed-form formula for the moment generating function (MGF) of the output signal power of a dual-diversity selection combiner (SC) in bivariate (correlated) Nakagami-m fading with positive integer fading severity index. This result involves only elementary functions and holds for any value of the ratio a/b in $Q_m(a,b)$. As an aside, we show that previous integral representations for $Q_m(a,b)$ can be obtained from a contour integral and also derive a new, single finite-range integral representation for $Q_m(a,b)$. A new infinite series expression for the MGF with arbitrary m is also derived. These MGFs can be readily used to unify the evaluation of average error performance of the dual-branch SC for coherent, differentially coherent, and noncoherent communications systems.**

***Index Terms*—Digital communications, diversity reception, fading channels, wireless communications.**

I. Introduction

The error performance of dual-diversity selection combiners (SCs) over correlated Rayleigh and/or Nakagami-m fading channels has been analyzed by many authors (see [1]–[4] and references therein). In [3], the authors differentiate an integral representation for the cumulative density to get the probability density function (PDF). Their resulting expressions depend on the branch power ratios and the power correlation coefficient ρ [3, eq. (2)]. They then use the PDF to average the conditional error probability for different modulation formats. Alternatively, if one derives the SC output moment generating function (MGF) first, the performance of a broad class of modulation formats can be obtained at once [5]. Motivated by [5] and recognizing the fact that SC output MGF is the key to the performance analysis, we attempted to derive closed-form solutions for the average symbol-error probability (ASER) of binary noncoherent and differentially coherent modulation formats using an approach similar to that of [5], which utilizes certain integral representations of $Q_m(a,b)$. These closed-form expressions take different forms depending on $a>b$, $a=b$, or $a<b$ [6], [7] and they are omitted here for the sake of brevity. During our attempt, we have discovered a number of related results which are worthy of reporting. Our results provide some new insights and supplement [2] and [3]. Our major results and comments are as follows.

(1) We show that the trigonometric integral representations for $Q_m(a,b)$ in [6] and [7] can be obtained directly from a circular contour integral representation for $Q_m(a,b)$ [8] by an appropriate variable substitution. In addition, we also derive a new single integral representation for $Q_m(a,b)$ that is valid for $a>b$, $a=b$, or $a<b$.

(2) Utilizing the contour integral representation for $Q_m(a,b)$, we derive an exact closed-form expression for the SC output MGF in correlated Nakagami-m fading while the fading severity index m is a positive integer. Unlike [3], the resulting formula applies regardless of the values of the branch power ratios and ρ. For instance, the independent fading case can be treated directly by setting $\rho=0$. As such, it leads to a compact, unified analysis of a broad class of modulation formats for dual-diversity SC in Nakagami-m fading.

II. Integral Representations for $Q_m(a,b)$

Proakis [8, p. 885] provides the following contour integral representation for the generalized Marcum-Q function

$$Q_m(a,b)=\frac{1}{2\pi j}\oint_{\Gamma}\frac{e^{g(z)}}{z^m(1-z)}dz \tag{1}$$

where $g(z)=a^2((1/z)-1)/2+b^2(z-1)/2$ and Γ is a circular contour of radius r that encloses origin. The singularities of the integrand are at $z=0$ and $z=1$. Therefore, by Cauchy's theorem, we can choose any $0<r<1$. Now if we choose $r=1$, then we need to remove the singularity at $z=1$ on Γ by suitably deforming Γ [see Fig. 1(b)]. This representation holds regardless of $a>b$, $a=b$, or $a<b$, and for any positive integer m. In the following, we will show that both Helstrom's [6] and Simon's [7] integral representations readily follow from (1) for integer values of m.

(1) Consider the case $a<b$ [see Fig. 1(a)] where the circular contour Γ encloses origin with a radius less than unity. Therefore, z in (1) can be written as $z=re^{j\theta}$ with $r<1$ and $0\le\theta<2\pi$. Now select $r=a/b$, so we immediately get

$$Q_m(a,b)=\frac{e^{\frac{-a^2}{2}-\frac{b^2}{2}}}{2\pi}\left(\frac{b}{a}\right)^m\int_0^{2\pi}\frac{e^{ab\cos\theta-j(m-1)\theta}}{\left(\frac{b}{a}-e^{j\theta}\right)}d\theta \tag{2}$$

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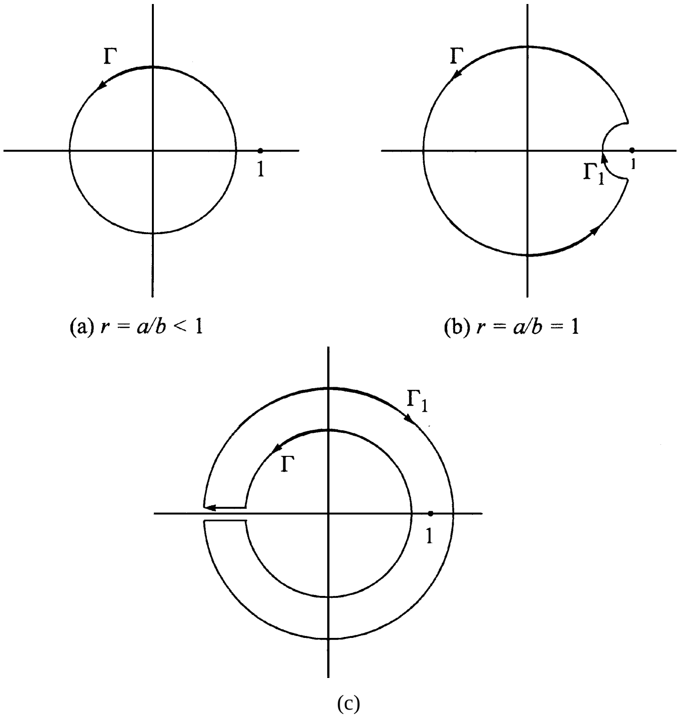

Fig. 1. Contours of a line integral. (a) $r < 1$. (b) $r = 1$. (c) $r > 1$.

where $m \in Z$ and $Z = \{1, 2, 3, \ldots\}$ is the set of positive integers. Taking the magnitude of the integrand, we obtain the new bound

$$Q_m(a,b) \le e^{\frac{-(a-b)^2}{2}} \left(\frac{b}{a}\right)^m \frac{a}{\sqrt{a^2+b^2}} \tag{3}$$

which holds for any integer $m \ge 1$, whereas the bound due to Simon [7, eq. (12)] holds only for $m = 1$.

(2) Consider the case $a = b$ [see Fig. 1(b)]. Now $g(z) = a^2(z + (1/z))/2 - a^2$ and Γ as shown in Fig. 1(b). Hence

$$Q_m(a,b) = \frac{e^{-a^2}}{2\pi} \int_{\epsilon}^{2\pi-\epsilon} \frac{e^{a^2\cos\theta - j(m-1)\theta}}{(1-e^{j\theta})} d\theta + \frac{1}{2\pi j} \oint_{\Gamma_1} \frac{e^{g(z)}}{z^m(1-z)} dz \tag{4}$$

where Γ_1 is the half-circle contour centered on $z = 1$ with radius ε. On Γ_1, $z = 1 - \varepsilon e^{j\alpha}$, and $-\pi/2 \le \alpha \le \pi/2$. Taking the real value of the first integral on the right-hand side and letting $\varepsilon \to 0$, we obtain

$$Q_m(a,a) = \frac{1}{2} + \frac{1}{2} \int_0^{2\pi} e^{-a^2(1-\cos\theta)} \frac{\sin\left[\left(m - \frac{1}{2}\right)\theta\right]}{\sin\left(\frac{\theta}{2}\right)} \frac{d\theta}{2\pi} \tag{5}$$

for $m \in Z$. This result [i.e., (5)] is, in fact, identical to [6, p. 528] derived by Helstrom.

(3) If $a > b$, $r = a/b$ is greater than unity. So we need to consider the closed contour shown in Fig. 1(c). The inner circle Γ has a radius less than unity, while the outer circle Γ_1 has a radius of a/b. Inside the closed contour, the only singularity of the integrand occurs at $z = 1$. Hence, using Cauchy's theorem, we find

$$\lim_{z=1} \frac{e^{g(z)}}{z^m} = \frac{1}{2\pi j} \oint_{\Gamma} \frac{e^{g(z)}}{z^m(1-z)} dz + \frac{1}{2\pi j} \oint_{\Gamma_1} \frac{e^{g(z)}}{z^m(1-z)} dz. \tag{6}$$

The first integral is $Q_m(a,b)$ and $z = (a/b)e^{j\theta}$ on Γ_1. Therefore, we get

$$Q_m(a,b) - 1 = \frac{e^{\frac{-a^2}{2} - \frac{b^2}{2}}}{2\pi} \left(\frac{b}{a}\right)^m \int_0^{2\pi} \frac{e^{ab\cos\theta - j(m-1)\theta}}{\left(\frac{b}{a} - e^{j\theta}\right)} d\theta \tag{7}$$

for $m \in Z$.

Note that (2) and (7) are identical to Helstrom's results [6], except the integrands are in a complex format, and hence, are slightly more compact. Since the integrals are real valued, taking the real parts of the integrands in (2) and (7) gives the exact same integral representations of Helstrom.

Similarly, Simon's results [7] are very closely related. For instance, consider the $a < b$ case. As in the derivation of (2), we can select $z = (a/b)je^{j\theta}$. Note that the magnitude of z is still less than unity, i.e., $|z| < 1$. Hence, using this new substitution in (1), we find

$$Q_m(a,b) = \frac{e^{\frac{-a^2}{2} - \frac{b^2}{2}}}{2\pi} \left(\frac{b}{a}\right)^m \int_0^{2\pi} \frac{e^{-ab\sin\theta - j(m-1)\left[\theta + \frac{\pi}{2}\right]}}{\left(\frac{b}{a} - je^{j\theta}\right)} d\theta \tag{8}$$

for $m \in Z$. Again, this representation is very compact, yet it is identical to [7, eqs. (7) and (10)].

A. Yet Another Simple Integral Representation for $Q_m(a,b)$

While the contour in Fig. 1(b) is used for the $a = b$ case, it can also be used when $a \ne b$. Therefore, if we use $z = e^{j\theta}$ in (1) along with the contour in Fig. 1(b), we may obtain a new integral representation for the generalized Marcum-Q function. Now $g(z) = (1/2)(a^2 + b^2)[\cos\theta - 1] + (j/2)(b^2 - a^2)\sin\theta$. Hence

$$Q_m(a,b) = \frac{1}{2\pi} \int_{\epsilon}^{2\pi-\epsilon} \frac{e^{\frac{1}{2}[a^2+b^2](\cos\theta - 1 - j\sin\theta) - j(m-1)\theta}}{(1-e^{j\theta})} d\theta + \frac{1}{2\pi j} \oint_{\Gamma_1} \frac{e^{g(z)}}{z^m(1-z)} dz \tag{9}$$

Taking the real value of the first integral on the right-hand side and letting $\varepsilon \to 0$, we obtain

$$\begin{aligned} Q_m(a,b) &= \frac{1}{2} + \frac{1}{2} \int_0^{2\pi} \frac{e^{\frac{1}{2}(\cos\theta - 1)[a^2+b^2]}\Psi(\theta)}{1-\cos\theta} \frac{d\theta}{2\pi} \\ &= \frac{1}{2} + \frac{1}{4\pi} \int_0^{2\pi} e^{-\sin^2\left(\frac{\theta}{2}\right)[a^2+b^2]} \frac{\sin\left[\frac{(2m-1)\theta + (a^2-b^2)\sin\theta}{2}\right]}{\sin\left(\frac{\theta}{2}\right)} d\theta \end{aligned} \tag{10}$$

where $\Psi(\theta) = \cos[(m-1)\theta + t(\theta)] - \cos[m\theta + t(\theta)]$, $t(\theta) = (1/2)(a^2 - b^2)\sin\theta$, and $m \in Z$. To the best of our knowledge, the above expression is *new*, and it holds for $a < b$, $a = b$, or $a > b$. Also notice that if $a = b$, (10) reduces to (5), as anticipated.

III. Derivation of the MGF of SNR at the Output of Dual-Branch SC Combiner

The PDF of the signal envelope at the output of SC combiner is [4]

$$f(r) = \Psi_2(r)\left[1 - Q_m\left(r\sqrt{\frac{2\rho}{(1-\rho)\Omega_2}}, r\sqrt{\frac{2}{(1-\rho)\Omega_1}}\right)\right] + \Psi_1(r)\left[1 - Q_m\left(r\sqrt{\frac{2\rho}{(1-\rho)\Omega_1}}, r\sqrt{\frac{2}{(1-\rho)\Omega_2}}\right)\right] \quad (11)$$

where $\Psi_j(r) = (2r^{2m-1})/(\Gamma(m)\Omega_j^m)\exp(-r^2/\Omega_j)$, $j \in \{1,2\}$, is introduced for ease of notation. Now, the desired MGF may be evaluated as

$$\phi(s) = \int_0^\infty \exp(-sr^2) f(r) dr. \quad (12)$$

A. Integer Fading Severity Index

Clearly, the solution to the definite integral (13) is required to solve our problem on hand

$$I_m(p,a,b) = \int_0^\infty x^{2m-1}\exp(-px^2)\left[1 - Q_m(ax,bx)\right]dx = \frac{\Gamma(m)}{2p^m} - \int_0^\infty x^{2m-1}\exp(-px^2)Q_m(ax,bx)dx \quad (13)$$

where $\{p\} > 0$. Now substituting (1) into (13) and performing the integration with respect to x first (changing the order of integration is valid because both the integrals are convergent), we find

$$I_m(p,a,b) = \frac{\Gamma(m)}{2}\left[\frac{1}{p^m} - \left(\frac{-2}{b^2}\right)^m \times \frac{1}{2\pi j}\oint_\Gamma \frac{dz}{(1-z)[z^2 - \alpha_0 z + \alpha_1]^m}\right] \quad (14)$$

where $\alpha_0 = (2p + a^2 + b^2)/b^2$ and $\alpha_1 = (a/b)^2$. The denominator of the integrand in (14) has two positive roots. It can be easily shown that one root is inside the unit circle and the other is outside. Therefore, applying the residue theorem and invoking Liebnitz's differentiation rule [9, (0.42)] and after simplifications, we obtain a closed-form solution for $I_m(p,a,b)$

$$I_m(p,a,b) = \frac{\Gamma(m)}{2}\left\{\frac{1}{p^m} - \frac{(2b)^{2m}}{[Y(Y-X)]^m} \times \sum_{k=0}^{m-1}\frac{\Gamma(m+k)}{k!\Gamma(m)}\left[\frac{1}{2}\left(1 - \frac{X}{Y}\right)\right]^k\right\} \quad (15)$$

where $X = 2p + a^2 - b^2$ and

$$Y = \sqrt{[2p + (a+b)^2][2p + (a-b)^2]}.$$

Obviously, (15) holds only for integer m. Besides, we would like to point out that (15) is equivalent to [2, eq. (6)], but simpler than the latter. Therefore, the MGF of SNR at the SC combiner output in bivariate Nakagami fading can be conveniently evaluated using

$$\phi(s) = \sum_{i=1, j\neq i}^{2}\frac{2}{\Gamma(m)\Omega_i^m} I_m\left(s + \frac{1}{\Omega_i}, \sqrt{\frac{2\rho}{(1-\rho)\Omega_i}}, \sqrt{\frac{2}{(1-\rho)\Omega_j}}\right) = \sum_{i=1, j\neq i}^{2}\left\{\frac{1}{(1+s\Omega_i)^m} - \frac{(2A_{ij})^{2m}}{[\Omega_i B_{ij}(B_{ij}-1)]^m} \times \sum_{k=0}^{m-1}\frac{\Gamma(m+k)}{k!\Gamma(m)}\left[\frac{1}{2}\left(1 - \frac{1}{B_{ij}}\right)\right]^k\right\} \quad (16)$$

where $A_{ij} = (\sqrt{\Omega_j(1-\rho)/2})/(\Omega_j/(\Omega_i - 1) + s\Omega_j\,(1-\rho))$ and $B_{ij} = (\sqrt{[s\Omega_j\Omega_j(1-\rho) + \Omega_i + \Omega_j]^2 - 4\rho\Omega_i\Omega_j})$ $/(s\Omega_j\Omega_j(1-\rho) + \Omega_j - \Omega_i)$.

B. Non-Integer Fading Severity Index

If the fading severity index is not an integer value, then the methods discussed thus far are no longer applicable because the trigonometric integrals for $Q_m(a,b)$ given in [6], [7], and (10) are restricted to positive integer m alone, and we cannot simplify the contour integral (1) through the use of residue theorem. In this case, we can utilize [2, eq. (6)] to get

$$\phi(s) = \frac{2^{2m}\Gamma(2m)}{\Gamma(m)\Gamma(m+1)}\sum_{i=1, j\neq i}^{2}\frac{(A_{ij})^{2m}}{[\Omega_i B_{ij}(1+B_{ij})]^m} \times {}_2F_1\left[1-m, m; 1+m; \frac{1}{2}\left(1 - \frac{1}{B_{ij}}\right)\right] \quad (17)$$

where A_{ij} and B_{ij} are as defined in (16), the Gauss hypergeometric series ${}_2F_1(a,b;c;z) = \sum_{n=0}^\infty((a)_n(b)_n)$ $/((c)_n)(z^n)/(n!)$ is convergent for $|z| < 1$, and notation $(.)_n$ denotes the Pochhammers symbol. To the best of our knowledge, the expression (17) is new and holds for arbitrary $m \geq 0.5$ values. This MGF can be used to unify the performance evaluation of various modulation formats in Nakagami-m fading with arbitrary parameters.

IV. ASER of Binary and M-ary Modulation Formats With Dual-Branch SC Diversity

This section presents the use of the new MGFs for a unified analysis of a broad class of modulation formats employing dual-branch SC in Nakagami fading with arbitrary parameters. Using Craig's results [10], it is not difficult to show that the conditional error probability for the binary and M-ary signaling constellations (coherent, differentially coherent, and noncoherent modulation formats) can be expressed as an exponential function of the SNR or an integral thereof [5]. Thus, the average error rate can be expressed in terms of only the MGF. In the following, two examples illustrate this process.

If the conditional error probability is in the exponential form, $P_S(\varepsilon|\gamma) = a\exp(-b\gamma)$, then we have a closed-form expression for the ASER. For instance, the average bit-error rate performance for binary differential phase-shift keying (DPSK) and noncoherent frequency-shift keying (FSK) with dual-branch SC is given by

$$P_S = a\phi(b) \tag{18}$$

where $\{a = 1/2, b = 1\}$ for binary DPSK and $\{a = 1/2, b = 1/2\}$ for binary orthogonal FSK. As well, when $m = 1$ (Rayleigh fading), we get

$$\phi(s) = \sum_{i=1, j\neq i}^{2} \frac{(2A_{ij})^2}{[\Omega_i B_{ij}(1 + B_{ij})]} \tag{19}$$

where A_{ij} and B_{ij} are as defined in (16). [5, eqs. (31) and (32)] follow at once from (16). Also notice that (16) (unlike [5, eq. (32)]) is independent of the ratio between the arguments of the generalized Marcum-Q function even when $\rho \neq 0$. Similarly, for integer m and $\rho = 0$, [5, eq. (57)] follows at once from (16).

If the conditional error probability is of the form $P_S(\varepsilon|\gamma) = a\operatorname{erfc}(\sqrt{b\gamma})$ (e.g., coherent binary PSK or FSK), then the ASER can be expressed as

$$P_s = \frac{2a}{\pi} \int_0^{\frac{\pi}{2}} \phi(b \csc^2 \theta) d\theta. \tag{20}$$

It is clear that the evaluation of (20) only involves a single integral with finite integration limits since we have a closed-form solution for the MGF. Unlike the development of [5, eq. (59)], no further manipulations are necessary. Following this technique, many other modulation schemes can be analyzed at once.

V. Conclusion

This letter makes several contributions: (a) a closed-form expression is derived for the MGF for integer m; (b) a hypergeometric series-based expression is derived for the MGF for noninteger m; and (c) a new, single-integral representation is derived for $Q_m(a, b)$ (with finite integration limits) that is valid for $a > b$, $a = b$, or $a < b$. To enable the error analysis, we derive the required MGF for both cases of identical and dissimilar mean received signal strengths. The evaluation for the independent fading case can be directly obtained by setting the power correlation coefficient to zero in our expressions. These closed-form formulas can be directly used to determine the error performance of a broad class of modulation formats with dual-diversity SC over independent and correlated Nakagami-m and Rayleigh fading channels.

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