Computation of the Continuous-Time PAR of an OFDM Signal with BPSK Subcarriers

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***Abstract*—A procedure for computing the continuous-time peak-to-average power ratio (PAR) of an orthogonal frequency-division multiplexing (OFDM) signal, with binary phase-shift keying (BPSK) subcarriers, is developed here. It is shown that the instantaneous envelope power function (EPF) can be transformed into a linear sum of Chebyshev polynomials. Consequently, the roots of the derivative of EPF can be obtained by solving a polynomial. Using the procedure to evaluate the difference between the continuous-time and discrete-time PAR, it is shown that an oversampling factor of four is accurate.**

***Index Terms*—Chebyshev polynomials, OFDM, PAR.**

I. Introduction

Orthogonal frequency-division multiplexing (OFDM) has been proposed for both digital TV broadcasting and high speed wireless networks over multipath channels [1]. The principal drawback of OFDM is that the peak transmitted power may be up to N times the average, where N is the number of subcarriers. Linear amplifiers that can handle the peak power are less efficient. Hard limiting of the transmitted signal generates intermodulation products that can interfere with adjacent channels. Given these problems, following [2], which first described a block coding technique to reduce the signal peaks, many peak-to-average power ratio (PAR)-reduction techniques have been proposed in the open literature [3]–[5].

This letter does *not propose* any PAR-reduction technique, instead develops a procedure for computing the continuous-time PAR an OFDM signal, with binary phase-shift keying (BPSK) subcarriers. In many cases, the continuous-time PAR is approximated using the discrete-time PAR, which is obtained from the samples of the OFDM signal. The sampling rate is the Nyquist rate or a multiple of it (oversampling). An important question is how large the oversampling factor should be in order for the approximation to be accurate. Since our procedure yields the exact PAR value, this question can be confidently settled.

To compute the continuous-time PAR, the roots of the derivative of the envelope power function (EPF) are required. At first, finding the required roots appears very difficult. This is because the derivative in this case is a sum of sinusoidal functions. As such, it may require the use of a general root finding algorithm for nonlinear functions. Fortunately, there is a way around this difficulty. Using a inverse cosine-based transformation, the EPF can be converted to a sum of Chebyshev polynomials. Add to that, the required roots are now trapped within the interval 0 to 1. So the original root finding problem is reduced to a root finding problem for a polynomial. Reliable algorithms for finding all roots of a polynomial (recall that a polynomial of order n will have n roots) are well known. Even common mathematical software, such as MATLAB, contains functions that can handle polynomials of orders up to several thousands. Consequently, using this approach, the absolute peak of the envelope power function can be evaluated exactly.

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II. Mathematical Definitions

The complex envelope of the transmitted OFDM signal is represented by

$$s(t) = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} a_n e^{i2\pi nt} \tag{1}$$

where $i = \sqrt{-1}$ and $a_n \in \{1, -1\}$. We shall write an ordered N-tuple $a = (a_0, \ldots, a_{N-1})$. The instantaneous envelope *power* of the signal is the real-valued function $P_a(t) = |s(t)|^2$.

It is easy to show that [6], [3]

$$P_a(t) = \sum_{k=0}^{N-1} \beta_k \cos(2\pi kt) \tag{2}$$

where

$$\beta_k = \begin{cases} 1, & k = 0 \\ \dfrac{2}{N} \displaystyle\sum_{n=0}^{N-1-k} a_n a_{n+k}, & k = 1, 2, \ldots, N-1. \end{cases} \tag{3}$$

Equation (2) illustrates why our procedure is limited to BPSK subcarriers. With complex modulation techniques, such as QPSK, the EPF (2) would consist of both $\cos(2\pi kt)$ and $\sin(2\pi kt)$ terms. From here, we have found that the EPF cannot be transformed into a linear combination of Chebyshev polynomials.

The continuous-time PAR is defined as

$$||P_a(t)||_\infty = \max_t P_a(t). \tag{4}$$

Most of PAR-reduction techniques are concerned with reducing this quantity. However, since most systems employ discrete-time signals, the maximum amplitude of LN samples of $s(t)$ is used to approximate it, where L is the oversampling factor. The sampling can be implemented by an inverse discrete Fourier transform (IDFT).

Consider the IDFT of length LN of a expressed as

$$A_k = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} a_n e^{i2\pi nk/LN}, \qquad k = 0, 1, \ldots, LN-1. \tag{5}$$

The discrete-time PAR is defined as

$$\|P_a(t)\|_{\text{dis}} = \max_{0 \le k < LN} |A_k|^2. \tag{6}$$

The $L > 1$ corresponds to oversampling. Of course, if $L \gg 1$, the discrete-time PAR should approach the continuous-time PAR. It is, therefore, clear that

$$\|P_a(t)\|_{\text{dis}} \le \|P_a(t)\|_{\infty}. \tag{7}$$

III. Computation of Continuous-Time PAR

To compute Eq. (4) exactly, the roots of $dP_a(t)/dt = 0$ are needed. Let us define

$$Q_a(t) = P_a\left[\frac{\cos^{-1} t}{2\pi}\right] = \sum_{k=0}^{N-1} \beta_k T_k(t) \tag{8}$$

where $T_k(t) = \cos(k \cos^{-1} t)$ is the kth-order Chebyshev polynomial [7, p. 1054]. Note that $T_0(t) = 1$, $T_1(t) = t$, $T_2(t) = 2t^2 - 1$ and so on [explicit expressions for the coefficients of $T_n(t)$ for any n are available]. Since $Q_a(t)$ is a polynomial of degree $(N-1)$, its first derivative is a polynomial of degree $(N-2)$. Recall that a polynomial of degree n will have n roots. These roots can be real or complex and many algorithms exist with which one can find all the roots of a polynomial [8, p. 369]. For example, a companion matrix can be constructed whose eigenvalues are the desired roots. This is the method used in our numerical examples.

Since

$$\frac{dP_a(t)}{dt} = \frac{dQ_a(\cos(2\pi t))}{dt} = -2\pi \sin(2\pi t)\, \frac{dQ_a[\cos(2\pi t)]}{d\cos(2\pi t)} \tag{9}$$

the derivative vanishes both at $t = 0,\ 1/2$ and at the transforms of the real roots of $dQ_a(t)/dt$ that lie between -1 and $+1$. Let those roots be $\xi_1, \xi_2, \ldots, \xi_M$ where $M \le N-2$. Define the set

$$\Lambda = \left\{0, \frac{1}{2}, \frac{\cos^{-1}\xi_1}{2\pi}, \ldots, \frac{\cos^{-1}\xi_M}{2\pi}\right\}. \tag{10}$$

It is clear that all the required roots of the derivative of $P_a(t)$ are in this set. Note that $P_a(t)$ is a periodic function and only the roots between 0 to 1 need to be considered. Therefore, the continuous-time PAR is obtained by

$$\|P_a(t)\|_{\infty} = \max_{t \in \Lambda} P_a(t). \tag{11}$$

The following is an outline of the complete procedure.

Step 1) Calculate the correlation coefficients β_k for a given data vector a.

Step 2) Obtain the coefficients of $Q_a(t)$ in (8).

Step 3) Solve for the roots of the derivative of $Q_a(t)$.

Step 4) Evaluate $P_a(t)$ on Λ and pick the maximum.

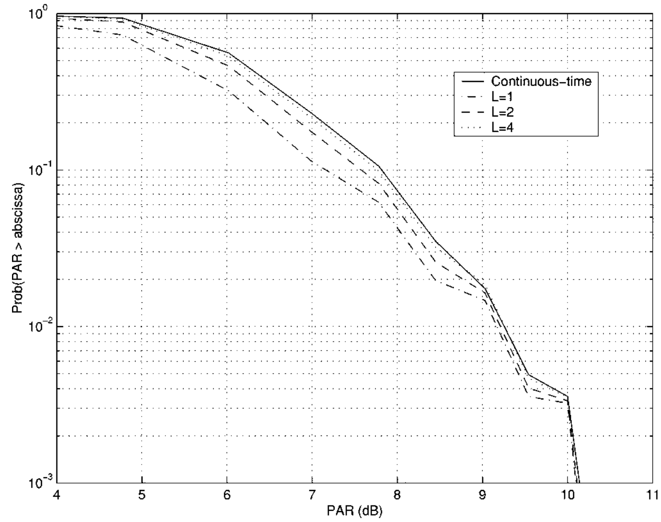

Fig. 1. PAR distribution for 32-carrier BPSK-OFDM.

IV. Discussion and Conclusions

In Fig. 1, results using the above method are shown for a 32-BPSK-subcarrier OFDM system. One million 32-bit codewords are generated randomly. The continuous-time PAR, computed using the above procedure, is compared with the discrete-time PAR (6). The difference can be as high as 1 dB, where $L = 1$. When $L = 4$, the difference is negligible. These results are applicable for an unperturbed system in which the signal amplitude is Rayleigh distributed (for large N, $s(t)$ is an approximately complex Gaussian random process).

The partial transmit sequence (PTS) approach provides improved PAR statistics for the OFDM signal. The input data block is divided into subblocks which are multiplied by rotational factors in order to reduce the PAR. Here, we study the same system above with four subblocks, while limiting the rotational factors r_k to -1 and $+1$. Our sole aim is to find the continuous-time PAR of this method. Consequently, the input data vector is modified as

$$\hat{a} = [a_0, a_1, \ldots, a_7, r_1 a_8, \ldots, r_1 a_{15}, \ldots, r_3 a_{24}, \ldots, r_3 a_{31}].$$

The optimal vector $[r_1, r_2, r_3]$ is chosen out of eight values for the purpose of minimizing the discrete-time PAR (6) (i.e., from one of the eight IDFT's per symbol), the distribution of which is shown in the PTS discrete-PAR curve in Fig. 2 for one million randomly generated 32-bit codewords. We can also use the procedure to compute $\|P_{\hat{a}}(t)\|_{\infty}$ of $\hat{a}$, labeled in the same figure as PTS continuous-PAR. While the discrete-time PAR is several decibels less than that of the uncoded case, the continuous-time PAR of PTS is reduced *much less*. This shows that both caution and care must be exercised when the discrete-time PAR is used as a measure of PAR-reduction. Again, when $L = 4$, the difference between the continuous-time and discrete-time PAR is negligible.

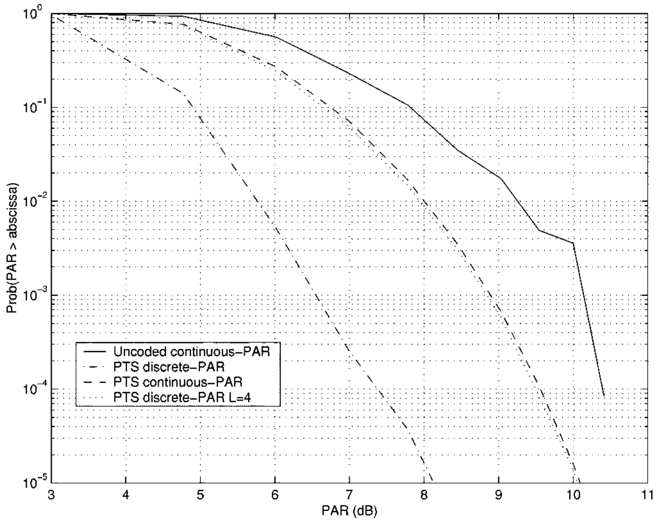

Fig. 2. PAR distribution for PTS with 32-carrier BPSK-OFDM.

In conclusion, the PAR-reduction problem for OFDM has received a great deal of attention recently. Despite this, exact computational methods for the continuous-time PAR have not been reported. In this letter, a procedure for computing the continuous-time PAR of an OFDM signal, with BPSK subcarriers, has been developed. The procedure may be useful for theoretical studies of PAR distributions. Using it to evaluate the difference between the continuous-time and discrete-time PAR, it has been shown that an oversampling factor of four is accurate.

REFERENCES

[1] J. A. C. Bingham, "Multicarrier modulation for data transmission: An idea whose time has come," *IEEE Commun. Mag.*, pp. 5–14, 1990.

[2] A. E. Jones, T. A. Wilkinson, and S. K. Barton, "Block coding scheme for reduction of peak to mean envelope power ratio of multicarrier transmission schemes," *Electron. Lett.*, vol. 30, pp. 2098–2099, 1994.

[3] P. Van Eetvelt, G. Wade, and M. Tomlinson, "Peak to average power reduction for OFDM schemes by selective scrambling," *Electron. Lett.*, vol. 32, pp. 1963–1964, Oct. 1996.

[4] L. J. Cimini Jr. and N. R. Sollenberger, "Peak-to-average power ratio reduction of an OFDM signal using partial transmit sequences," in *Proc. IEEE. ICC'99*, Vancouver, Canada, 1999.

[5] J. A. Davis and J. Jedwab, "Peak-to-mean power control in OFDM, Golay complementary sequences, and Reed–Muller codes," *IEEE Trans. Inform. Theory*, vol. 45, pp. 2397–2417, Nov. 1999.

[6] C. Tellambura, "Upper bound on the peak factor of N-multiple carriers," *Electron. Lett.*, vol. 33, pp. 1608–1609, Sept. 1997.

[7] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products*, 5th ed. New York: Academic, 1994.

[8] W. H. Press, S. A. Teukolsky, W. T. Vetterling, and B. P. Flannery, *Numerical Recipes in C—The Art of Scientific Computing*, 2nd ed. New York: Cambridge Univ. Press, 1992.