On the Peak Factors of Sampled and Continuous Signals

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***Abstract*—Motivated by a recent letter on an infinite difference between the peak-to-average power ratios (PAPRs) of samples (a series) and a band-limited function constructed from that series, we investigate the amplitude and variations of a band-limited function and present some relevant bounds. Related aspects on sampling theorems and sampling series are also discussed.**

***Index Terms*—Band-limited function, peak-to-average power ratio, sampling series, sampling theorems.**

I. Introduction

In peak-to-average power ratios (PAPR) reduction techniques for OFDM, the PAPR of a continuous analog signal (denoted by PF_c) is approximately evaluated from that of the signal's samples (denoted by PF_s). In [1], it is shown that reducing PF_s does not necessarily result in a similar reduction of PF_c. Recently, Wulich [2] constructed a band-limited function from a series (of samples) and showed that PF_s is finite but PF_c is infinite. At first sight, the example appears to be wrong because the variations of a band-limited function must be bounded (i.e., such a function cannot take infinite values between two finite samples). This example therefore raises a number of fundamental questions. Firstly, "under what conditions, if any, can a band-limited function take infinite values between finite samples?" To answer this, we present several bounds for a function $x(t)$, its first derivative $x'(t)$ and its variation $|x(t) - x(t+\tau)|$ in terms of bandwidth and signal energy or power. These bounds follow readily from the use of the Cauchy–Schwarz inequality and are instructive in their own right. Secondly, "does an arbitrary sequence $\{x_n\}$ represent the samples of a band-limited function?" We show that Wulich's sequence does not satisfy the necessary conditions.

II. Wulich's Example

In OFDM, we are dealing with periodic signals which have infinite energy. Hence, the average power is used for normalization purposes. The example in [2] deals with a nonperiodic function. While the peak of such a function can still be found, it is not meaningful to define the PAPR because the average power in this case would be zero for a finite energy function. Thus, [2, Definition (2)] appears to be ill-advised but the example worked only because the function used is of infinite energy. On the other hand, the PAPR for an OFDM signal is upper bounded as $PF_c \le N$, where N is the number of subcarriers. This means Wulich's example is clearly not relevant for normal OFDM. Let us consider Wulich's example, [2, eq. (5)], which can be expressed as

$$x[n] = e^{j\pi n} + 2\delta[n] - 2u[n]e^{j\pi n} \tag{1}$$

where $\delta[n]$ is the discrete-time Dirac delta function and $u[n]$ is the discrete-time unit step function. Applying $\mathcal{F}\{e^{j\theta_o n}y[n]\} = Y_d(e^{j(\theta-\theta_o)})$ where $\mathcal{F}\{\}$ indicates Fourier transform and $Y_d(e^{j\theta}) = \mathcal{F}\{y[n]\}$, the Fourier transform of $x[n]$ is given by [3]

$$X_d(e^{j\theta}) = \frac{2}{1+e^{j\theta}}. \tag{2}$$

Consequently, the Fourier transform of $x(t)$, denoted by $X(f)$, is given by

$$X(f) = \frac{1}{f_s}\,\text{rect}\left(\frac{f}{f_s}\right)\frac{2}{(1+e^{j2\pi f/f_s})}. \tag{3}$$

Wulich's example can be explained qualitatively as follows. In time domain, $x(t)$, [2, eq. (1)], is an infinite sum of sinc functions. $\text{sinc}(t) \triangleq \sin(\pi t)/\pi t$ varies as $1/t$ and changes sign as t crosses integer values. We know that the infinite series

$$S = \sum_{n=1}^{n=\infty} \frac{1}{n} \tag{4}$$

diverges to infinity. It can be shown that Wulich's function behaves approximately as this series and hence has an infinite amplitude for any time instant that is not a sampling instant. Mathematically, for a time instant t with $|t| < K/f_s$ but $t \ne n/f_s$, (n an integer), the function value is given by

$$x(t) = \sum_{|n|=0}^{K-1} x[n]\text{sinc}(f_s t - n) + \sum_{|n|=K}^{\infty} x[n]\text{sinc}(f_s t - n).$$

If $x[n]$ for $|n| \ge K$ are alternating signs with equal amplitude and $x[n] = x[-n]$, then the second summation in the above equation will behave as in (4) while the first summation has a finite amplitude, hence resulting in an infinite amplitude for that time instant. Wulich's samples are exactly of the type described above. Hence, except at sampling instants (where the contributions from all other samples are all zeros), the function has infinite amplitude for all other time instants, excluding $t = \pm\infty$.

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III. Bounds for a Band-Limited Function

The bandwidth is a measure of how fast a function varies; consequently, the variations of a band-limited function between its two adjacent samples will be finite and bounded, due to the finite bandwidth. This is provided that the function is either of a nonperiodic, finite energy type or of a periodic, finite power type. So, in general, the function cannot take infinite values if the two samples are finite. In the following, we explore the bounds for a band-limited function.

A. Periodic Function

Let us consider a band-limited periodic function $x(t)$, whose power spectral density is zero for $|f| > Lf_o$. It can be expressed by a Fourier series as

$$x(t) = \sum_{n=-L}^{L} c_n e^{j2n\pi f_o t} \tag{5}$$

where $\{c_n\}$ are Fourier series coefficients for $x(t)$. Applying the Cauchy–Schwarz inequality, $|\sum a_n b_n|^2 \leq \sum |a_n|^2 \sum |b_n|^2$, to (5), we obtain

$$|x(t)| \leq \sqrt{(2L+1)\sum_{n=-L}^{L}|c_n|^2}. \tag{6}$$

From Parseval's theorem we know $\sum_{n=-L}^{L}|c_n|^2 = P$, the total average power of $x(t)$. Hence, we obtain

$$|x(t)| \leq \sqrt{(2L+1)P}. \tag{7}$$

Similarly, we can proceed for the derivative of $x(t)$ and obtain the following:

$$|x'(t)| \leq 2\pi f_o \sqrt{\frac{PL(L+1)(2L+1)}{3}}. \tag{8}$$

Alternatively, we can find $|x(t+\tau) - x(t)|$ as follows:

$$x(t+\tau) - x(t) = \int_{-Lf_o}^{Lf_o} X(f)(e^{j2\pi f\tau} - 1)e^{j2\pi ft}\, df$$

$$|x(t+\tau) - x(t)| \leq \sqrt{\sum_{k=-L}^{L}|c_k|^2 4 \sum_{n=-L}^{L}\sin^2(n\pi f_o \tau)}.$$

Using $\sin^2(\phi) \leq \phi^2$ and Parseval's theorem, we get

$$|x(t+\tau) - x(t)| \leq 2\pi f_o \tau \sqrt{P\,\frac{L(L+1)(2L+1)}{3}}. \tag{9}$$

Hence, for an OFDM signal $x(t)$ with N subcarriers, similar to (7), we see that

$$|x(t)| \leq \sqrt{NP} \tag{10}$$

and consequently, $PF_c \leq N$ if signal constellations that have unity amplitude, such as quadrature phase-shift keying (QPSK), are used. We are also aware that

$$|x'(t)| \leq 2\pi f_o \sqrt{P(N-1)N(2N-1)/6} \tag{11}$$

$$|x(t+\tau) - x(t)| \leq 2\pi f_o \tau \sqrt{P(N-1)N(2N-1)/6} \tag{12}$$

where f_o is the subcarrier spacing of an OFDM signal. It is noted that the signal amplitude $|x(t)|$ does not depend on the absolute bandwidth, but only on the number of subcarriers N, regardless of f_o. But the variations of the signal $|x'(t)|$ and $|x(t+\tau) - x(t)|$ depend on both N and f_o.

B. Non-Periodic Function

For a nonperiodic function $x(t)$ with energy E and Fourier transform $X(f)$ where $X(f) = 0$ for $|f| > f_c$, using Cauchy–Schwarz inequality and Parseval's theorem would lead to the following bounds [4]:

$$|x(t)| \leq \sqrt{2f_c E}. \tag{13}$$

$$|x'(t)| \leq \sqrt{8\pi^2 f_c^3 E/3}. \tag{14}$$

$$|x(t+\tau) - x(t)| \leq \sqrt{E \cdot \left\{4f_c - \frac{2}{\pi\tau}\sin(2\pi f_c \tau)\right\}}. \tag{15}$$

Thus, both the function and its variations are bounded in terms of its energy and bandwidth. For $|f_c\tau| < 1$, we have the following expression:

$$x(\tau/2) - x(-\tau/2) = 2j \int_{-f_c}^{f_c} X(f)\mathrm{sgn}(f)\frac{\tau}{|\tau|}|\sin(\pi f\tau)|\, df. \tag{16}$$

Using the mean value theorem, we can obtain

$$\begin{aligned} x(\tau/2) - x(-\tau/2) &\geq m2j \int_{-f_c}^{f_c} |\sin(\pi f\tau)|\, df \\ &\geq \frac{4jm}{\pi\tau}[1 - \cos(\pi f_c \tau)] \end{aligned} \tag{17}$$

where $\min\{X(f)\mathrm{sgn}(f)(\tau/|\tau|)\} \leq m \leq \max\{X(f) \cdot\mathrm{sgn}(f)(\tau/|\tau|)\}$. The energy of Wulich's function is given by

$$E = \int_{-\infty}^{\infty}|x(t)|^2\, dt = \int_{-\infty}^{\infty}|X(f)|^2\, df = \infty. \tag{18}$$

Hence, (13) indicates that Wulich's function $x(t)$ can have infinite amplitudes, while (14) and (15) suggest that the variations of $x(t)$ can be infinite. Alternatively, (17) suggests that the variations of Wulich's function can be infinite since $X(f = \pm f_s/2)$ has an infinite amplitude.

IV. Sampling Series

In [2], Wulich uses a function defined in (1) as a sampling series and constructs a band-limited function. In what follows, we present related aspects on sampling. From [5] we know that the following sampling expansion

$$x(t+\tau) = \sum_{n=-\infty}^{\infty} x(t+nT)\,\frac{\sin(2\pi f_c[\tau - nT])}{2\pi f_c[\tau - nT]} \tag{19}$$

where $T = 1/(2f_c)$, does not hold, in general, for finite power signals. In fact, we can observe from (5) that it requires $2L+1$ coefficients of orthogonal kernels in order to completely define $x(t)$. From this fact, we can deduce that within one period (which is enough to represent the periodic signal) it requires $2L+1$ orthogonal samples (we consider only equally spaced samples) to completely define the signal without any additional knowledge of the signal or restrictions on the sampling. However, if we sample over more than one period, it is not necessary to do so at a frequency of $(2L+1)f_o$. It is sufficient to sample at a frequency that gives $2L+1$ samples in only one of the considered periods. If we sample over an infinite period of time, i.e., $-\infty < t < \infty$, then it is sufficient to use a sampling frequency f_s just greater than $2Lf_o$.

Consider a signal with $X(f) = 0$ for $|f| > f_c$. In this case, the support of $X(f)$ or bandwidth can be given by a closed interval, $[-f_c, f_c]$. For such a band-limited signal of finite energy, the sampling theorems of Shannon [6], Whittakers and Kotel'nikov [7] state that the required sampling frequency is $f_s \geq 2f_c$. Later, Campbell [8] extended the sampling theorem by considering a rather general scope, namely, distribution with bounded support beyond the originally considered case, which was of a band-limited finite energy signal. This scope encompassed band-limited periodic signals. It was emphasized that it is a requirement that the support of $X(f \pm nf_s)$ for the integer $n \neq 0$ be disjoint from the support of the Fourier transform of interpolating function (in our case, an ideal low pass filter). From [8], we can see that by means of disjoint supports, it avoids the case with a sampling frequency f_s and $X(|f| > f_s/2) = 0$ where $X(f = \pm f_s/2) \neq 0$ when extending the bounded support finite energy function to the bounded support distribution.

Let us consider an interpolating function $p(t) = \sin(\pi f_s t)/\pi f_s t$ whose Fourier transform is given by

$$P(f) = \begin{cases} 1/f_s, & |f| < f_s/2 \\ 1/(2f_s), & |f| = f_s/2 \\ 0, & |f| > f_s/2. \end{cases} \tag{20}$$

For a band-limited finite energy signal $x(t)$ where $X(|f| > f_s/2) = 0$ but $X(f = \pm f_s/2) \neq 0$, the reconstruction of the signal from its sampling series will result in a frequency spectrum that is exactly the same as the original one if the original frequency spectrum has the same value at $|f| = f_s/2$ [i.e., $X(-f_s/2) = X(f_s/2)$]; $X(f)$ need not be symmetric. Even if this condition is not satisfied, the only difference between the original frequency spectrum and the reconstructed frequency spectrum will be at $|f| = f_s/2$. For a finite energy signal, the ratio of the energy content at $|f| = f_s/2$ to the total energy of the signal is zero. Hence, the reconstructed signal is essentially the same as the original one, justifying the use of the sampling theorems of Shannon, Whittakers and Kotel'nikov [6], [7].

For a band-limited periodic signal with a maximum frequency content of Lf_o, if the frequency spectra are the same at $|f| = f_s/2$, the reconstructed signal is exactly the same as the original one. An example of this type is $x(t) = \cos(2\pi Lf_o t)$. If the frequency spectra are not the same at $|f| = f_s/2$, the reconstructed spectrum will not be exactly the same as the original one. The reconstruction error would depend on the ratio of the error power at $|f| = f_s/2$ to the total power of the original signal. For an extreme case where $x(t) = \sin(2\pi Lf_o t)$, whose spectrum at $|f| = f_s/2$ are exactly opposite and the above power ratio is unity, there would be a total loss of the original signal. This fact can also be recognized from the signal's sampling series, which is a set of zeros. Hence, in general, for any band-limited signal (regardless of finite energy signal or periodic signal) whose frequency spectrum $X(f) = 0$ for $|f| > f_c$, the sufficient sampling frequency is $f_s > 2f_c$. This reflects the requirement of disjoint supports, as stated in [8]. This disjoint support requirement can also be observed in another treatment on sampling series for band-limited generalized functions (see [9, Lemma 1]). The above statement, however, does not mean that any function with bandwidth $[-f_c, f_c]$ requires $f_s > 2f_c$ but simply means that there are some functions with bandwidth $[-f_c, f_c]$ that require $f_s > 2f_c$, while the other functions require $f_s \geq 2f_c$.

Another interesting point, made by Jerri in [10], is that a sampling series $\{x(nT)\}$ given by an arbitrary sequence $\{x_n\}$, lacking the assertion of $\sum_{n=-\infty}^{\infty} |x_n| < \infty$, does not necessarily imply that the series represents a band-limited function. The sequence used in [2] is not a periodic signal and does not satisfy this assertion. Moreover, from its Fourier transform we can observe that it does not have a disjoint support as required in [8], [9]. Hence, this sequence and its corresponding function may not fall inside the scope of previous sampling theorems.

A closer look at the time domain function in Section II reveals that except at the sampling points where the function has values given by [2, eq.. (5)], the function has an infinite amplitude for all other time instants (excluding $t = \pm\infty$). This unusual nature of the function result in the unusual outcome of an infinite difference between the peak factor of the series PF_s and the peak factor of the function PF_c as reported in [2]. We may also conclude that arbitrarily chosen series may not necessarily represent a band-limited signal that is continuous or has finite jumps.

V. Conclusion

For a band-limited periodic signal, the amplitude is bounded by the total power and the number of constituent harmonic tones. The signal variation is bounded by the total power and the bandwidth. For a band-limited nonperiodic signal, the amplitude and variation are bounded by the total energy and the bandwidth. Using an arbitrarily chosen data sequence for the sampling series may not necessarily result in a band-limited signal of interest for communications systems.

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