Error Rates for Nakagami-m Fading Multichannel Reception of Binary and M-ary Signals

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***Abstract*—This paper derives new closed-form formulas for error probabilities of single and multichannel communications in Rayleigh and Nakagami-m fading. Closed-form solutions to three generic trigonometric integrals are presented as part of the main result, providing a unified method for the derivation of exact closed-form average symbol-error probability expressions for binary and M-ary signals with L independent channel diversity reception. Both selection-diversity and maximal-ratio combining (MRC) techniques are considered. The results are generally applicable for arbitrary two-dimensional signal constellations that have polygonal decision regions operating in a slow Nakagami-m fading environments with positive integer fading severity index. MRC with generically correlated fading is also considered. The new expressions are applicable in many cases of practical interest. The closed-form expressions derived for a single channel reception case can be extended to provide an approximation for the error rates of binary and M-ary signals that employ an equal-gain combining diversity receiver.**

***Index Terms*—Digital communications, diversity methods, Nakagami fading, wireless communications.**

I. Introduction

An EXPLODING demand for wireless communications has rekindled interest in accurate methods for characterizing the performance of digital communications systems over fading channels. Nakagami-m distribution (also known as the m-distribution) is the widely accepted statistical model due to both its good fit with experimental results and its versatility. The m-distribution covers a wide range of fading scenarios by varying its fading index m, includes the Rayleigh and one-sided Gaussian distributions as special cases for the respective fading figures of $m = 1$ and $m = 0.5$, and can closely approximate[1] the Rician and Hoyt distributions. Nakagami-m distribution also models channel conditions more severe than Rayleigh fading (when $0.5 \leq m < 1$). Given this, the theoretical performance of communications systems in Nakagami-m fading is valuable. Particularly important are exact closed-form error probability expressions because such expressions facilitate the rapid calculation of error rates, provide useful insights into system behavior and can assist design parameter optimization.

Over the past four decades, many solutions ranging over bounds, approximations, integral expressions, and closed-form formulas have been presented for a variety of modulation formats, diversity reception techniques and fading distributions. Nevertheless, exact closed-form solutions are not available for the multichannel reception of arbitrary two-dimensional (2-D) M-ary signaling constellations with polygonal decision boundaries, M-ary differential phase-shift keying (MDPSK), and differential quadrature phase-shift keying (DQPSK) over Nakagami-m fading (positive integer fading severity index). In fact, previous related solutions have been generally limited to integral expressions. Where closed-form solutions were available, they were restricted to Rayleigh fading and/or confined to M-ary phase-shift keying (MPSK) and M-ary quadrature amplitude modulation (MQAM) signaling. To the best of our knowledge, this paper provides the most general solution to date.

In [1] and [2], Proakis derives the average symbol-error probability (ASER) of MPSK with maximal-ratio combining (MRC) in Rayleigh fading. His final expression [1, eq. (22)] involves the evaluation of an $(L-1)$th-order derivative, which can be cumbersome for large alphabet sizes $(M > 4)$ with a high-order of diversity. In [3], Chennakeshu and Anderson derive an alternative closed-form ASER for MRC and selection-diversity combining (SDC) in Rayleigh fading $(m = 1)$. The key to their solution is an expression for the distribution of the signal phase of MPSK signaling in Rayleigh fading, obtained after a lengthy derivation. An approximate solution for MPSK with equal-gain combining (EGC) is also obtained. Ekanayake [4] derived closed-form solutions for MPSK over Rayleigh fading channels. In [5], Miyagaki *et al.* derive a recursive formula for the ASER of MPSK in a Nakagami-m fading channel with positive integer m. An approximate solution for the MRC case is also provided. Aalo and Pattaramalai [6] extend [3] for MRC reception of MPSK in Nakagami-m fading for positive integer m. In [7], Lu *et al.* present a closed-form solution for M-ary QAM employing SDC and MRC in Rayleigh fading. The MRC analysis is extended to Nakagami-m fading in [8]. Tellambura *et al.* [9] and Alouini *et al.* [10] present finite-range integral expressions for MPSK and MDPSK in Rician and Nakagami-m fading, respectively. This approach is further refined by Dong *et al.* [11], [12] by exploiting Craig's method [13] to analyze the performance of 2-D M-ary signals with polygonal decision regions. Their final expressions are also not in closed form. More recently, Cho *et al.* [14] investigated the performance of MPSK in correlated Rayleigh fading using Chennakeshu's approach. In this work, we add to the above by deriving new exact closed-form solutions for

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[1]Nakagami-m distribution only closely approximates the Rice and Nakagami–Hoyt distributions at high and moderate error rates. The discrepancy can be large at the deep tails of the distributions.

the multichannel reception (MRC and SDC) of arbitrary 2-D M-ary signals with polygonal decision regions, MDPSK and $\pi/4$-DQPSK over Nakagami-m fading for positive integer m. Our approach also directly leads to closed-form solutions for MRC in generically correlated Nakagami-m fading (with positive integer fading severity index).

In Section II, we derive closed-form solutions to three generic trigonometric integrals that arise in the analysis of binary and M-ary signals over Rayleigh and Nakagami-m fading channels. Several new closed-form solutions for single channel reception of M-ary signals in Nakagami-m fading with positive integer m are derived in Section III. In Section IV, we present new results for M-ary signals employing MRC in both independent and correlated fading cases by exploiting the integral identities derived in Section II. Next, in Section V, closed-form solutions for M-ary signals employing SDC are derived. In Section VI, we derive approximate closed-form solutions for arbitrary 2-D M-ary signals with polygonal decision regions that employ an EGC receiver. Finally, the main points of this paper are summarized in Section VII.

II. Derivation of Closed-Form Solutions for a Class of Trignometric Integrals

In this section, we derive closed-form solutions for definite trigonometric integrals of the form $\int_{\theta_L}^{\theta_U}(\cos^2\theta/a+\cos^2\theta)^b\,d\theta$, $\int_{\theta_L}^{\theta_U}(\sin^2\theta/a+\sin^2\theta)^b\,d\theta$, and $\int_{\theta_L}^{\theta_U}[1+c\cos\theta/a+1+c\cos\theta]^b\,d\theta$ for positive integer $b\geq 1$. These integrals are not listed in well-known standard tables of integrals such as [15], yet the integrals are known to arise in the analysis of single and multichannel reception of both binary and M-ary signals in Rayleigh and Nakagami-m fading channels. Although it is possible to express the solution to these integrals by equivalence, with the results obtained from [3] or [6], here we derive them directly. Interestingly, the final results obtained via these two approaches are slightly different (see the Appendix). But more importantly, our derivation is short and simple, unlike the development in [3]. Several applications of these new expressions will be discussed in Sections III–VI.

To begin, let us define

$$I_C(\theta_L,\theta_U,a,b)=\int_{\theta_L}^{\theta_u}\left[\frac{\cos^2\theta}{a+\cos^2\theta}\right]^b d\theta,\qquad a>0$$
$$=(\theta_U-\theta_L)+\sum_{n=1}^{b}(-1)^n\binom{b}{n}\int_{\theta_L}^{\theta_U}\left[1+\frac{1}{a}\cos^2\theta\right]^{-n}d\theta. \tag{1}$$

Letting $c=(1/a)$ and subsequently using variable substitution $z=\sqrt{1+c}\cot\theta$, we immediately get

$$\int_{\theta_L}^{\theta_U}\frac{d\theta}{[1+c\cos^2\theta]^n}=\frac{1}{(1+c)^{n-1/2}}\int_{\sqrt{1+c}\cot\theta_U}^{\sqrt{1+c}\cot\theta_L}\frac{(1+c+z^2)^{n-1}}{(1+z^2)^n}dz$$
$$=\sum_{r=0}^{n-1}\frac{c^r}{(1+c)^{n-1/2}}\binom{n-1}{r}\int_{\sqrt{1+c}\cot\theta_U}^{\sqrt{1+c}\cot\theta_L}\frac{1}{(1+z^2)^{r+1}}dz. \tag{2}$$

From [15, eq. (2.148.4)], we know that

$$P(a,b,n)=\int_a^b\frac{dx}{(1+x^2)^n}$$
$$=\frac{(2n-3)!!}{2^{n-1}(n-1)!}(\tan^{-1}b-\tan^{-1}a)+\frac{(2n-3)!!}{(n-1)!}\sum_{k=1}^{n-1}\frac{(n-k-1)!}{2^k(2n-2k-1)!!}\times\left[\frac{b}{(1+b^2)^{n-k}}-\frac{a}{(1+a^2)^{n-k}}\right],\qquad n\geq 1. \tag{3}$$

where $(2n-3)!!=1.3\cdots(2n-3)$ and $(n-1)!=1.2.3\cdots(n-1)$. Now we have a complete closed-form solution for $I_C(\theta_L,\theta_U,a,b)$

$$I_C(\theta_L,\theta_U,a,b)=(\theta_U-\theta_L)+\sum_{n=1}^{b}(-1)^n\sum_{r=0}^{n-1}\frac{a^{n-r-1/2}}{(1+a)^{n-1/2}}\binom{b}{n}\binom{n-1}{r}\times P\left(\sqrt{\frac{1+a}{a}}\cot\theta_U,\sqrt{\frac{1+a}{a}}\cot\theta_L,1+r\right) \tag{4}$$

where $P(.,.,.)$ is defined in (3). Furthermore, recognizing that

$$I_S(\theta_L,\theta_U,a,b)=\int_{\theta_L}^{\theta_U}\left[\frac{\sin^2\theta}{a+\sin^2\theta}\right]^b d\theta=\int_{\theta_L+\pi/2}^{\theta_U+\pi/2}\left[\frac{\cos^2\theta}{a+\cos^2\theta}\right]^b d\theta \tag{5}$$

deriving a closed-form solution for $I_S(\theta_L,\theta_U,a,b)$ becomes straightforward

$$I_S(\theta_L,\theta_U,a,b)=(\theta_U-\theta_L)+\sum_{n=1}^{b}(-1)^n\sum_{r=0}^{n-1}\frac{a^{n-r-1/2}}{(1+a)^{n-1/2}}\binom{b}{n}\binom{n-1}{r}\times P\left(\sqrt{\frac{1+a}{a}}\tan\theta_L,\sqrt{\frac{1+a}{a}}\tan\theta_U,1+r\right). \tag{6}$$

In [5, eq. (24)], Miyagaki *et al.* presented a recursive solution to $\int_0^{\theta_U}(d\theta/(1+c\sin^2\theta)^n)$ for $c>0$ and integer $n\geq 1$. Using a variable substitution $z=\sqrt{1+c}\tan\theta$ and then following the development of (2), it is possible to derive a nonrecursive formula for computing the sine integral

$$\int_{\theta_L}^{\theta_U}\frac{d\theta}{(1+c\sin^2\theta)^n}=\sum_{r=0}^{n-1}\frac{c^r}{(1+c)^{n-1/2}}\binom{n-1}{r}\times P(\sqrt{1+c}\tan\theta_L,\sqrt{1+c}\tan\theta_U,1+r). \tag{7}$$

In fact, our closed-form solution (7) is more general yet simpler than [5, eq. (24)]. Note that by mimicking the development of (1) and then using (7) in (5), we can again obtain (6).

Now we shall consider evaluation of the trigonometric integral

$$I_V(\theta_L,\theta_U,c,a,b)=\int_{\theta_L}^{\theta_U}\left(\frac{1+c\cos\theta}{1+a+c\cos\theta}\right)^b d\theta, \qquad b\geq 1 \tag{8}$$

which is commonly encountered in the analysis of MDPSK and $\pi/4$-DQPSK over Rayleigh and Nakagami-m fading channels. Following the development of (1), the above expression can restated as

$$\begin{aligned} I_V(\theta_L,\,\theta_U,\,c,\,a,\,b) &= (\theta_U-\theta_L)+\sum_{n=1}^{b}(-1)^n\binom{b}{n}\left(\frac{a}{1+a+c\cos\theta}\right)^n d\theta \\ &= (\theta_U-\theta_L)+\sum_{n=1}^{b}\binom{b}{n}\left(\frac{-a}{1+a}\right)^n\int_{\theta_L}^{\theta_U}\frac{d\theta}{(1+\tilde{c}\cos\theta)^n} \end{aligned} \tag{9}$$

where $\tilde{c}=c/(1+a)$. Using trigonometric identity $\cos(2x)=2\cos^2(x)-1=1-2\sin^2(x)$, it can readily be shown that

$$\begin{aligned}\int_{\theta_L}^{\theta_U}\frac{d\theta}{(1+\tilde{c}\cos\theta)^n} &= \frac{2}{(1-\tilde{c})^n}\int_{\theta_L/2}^{\theta_U/2}\frac{d\theta}{\left[1+\frac{2\tilde{c}}{1-\tilde{c}}\cos^2\theta\right]^n} \\ &= \frac{2}{(1+\tilde{c})^n}\int_{\theta_L/2}^{\theta_U/2}\frac{d\theta}{\left[1-\frac{2\tilde{c}}{1+\tilde{c}}\sin^2\theta\right]^n}\end{aligned} \tag{10}$$

and thus a complete solution for $I_V(\theta_L,\,\theta_U,\,c,\,a,\,b)$ can be obtained using (10) and (2) or (7)

$$\begin{aligned} I_V(\theta_L,\,\theta_U,\,c,\,a,\,b) &= (\theta_U-\theta_L)+2\sum_{n=1}^{b}\binom{b}{n}\left(\frac{a}{c-a-1}\right)^n\sum_{r=0}^{n-1} \\ &\quad\times\binom{n-1}{r}\sqrt{\frac{1+a-c}{1+a+c}}\times\left(\frac{-2c}{1+a+c}\right)^r \\ &\quad\times P\left[\sqrt{\frac{1+a-c}{1+a+c}}\tan\frac{\theta_L}{2},\,\sqrt{\frac{1+a-c}{1+a+c}}\tan\frac{\theta_U}{2},\,1+r\right].\end{aligned} \tag{11}$$

Note that it is fairly easy to program $I_C(.,.,.,.)$, $I_S(.,.,.,.)$ and $I_V(.,.,.,.,.)$ in common mathematical software such as Matlab and Mathematica. Furthermore, the binomial coefficients can be easily computed in a recursive fashion. Thus, exact error rates can be computed in closed form for many different cases (see the following sections).

III. Single Channel Reception

For $L=1$, the moment generating function (MGF) of signal-to-noise ratio (SNR) in Nakagami-m fading is given by [10]

$$\begin{aligned}\phi_\gamma(s) &= \int_0^\infty \exp(-s\gamma)p_\gamma(\gamma)\,d\gamma \\ &= \left(\frac{m}{m+s\overline{\gamma}}\right)^m, \qquad m\geq\frac{1}{2}\end{aligned} \tag{12}$$

where $p_\gamma(.)$ is the probability density function (pdf) of instantaneous SNR and $\overline{\gamma}$ is the average SNR per symbol. Next, we derive closed-form solutions for several modulation formats in Nakagami-m fading with positive integer m using the MGF method [9], [10] and the trigonometric identities derived in the preceding section.

A. BPSK and Binary Frequency-Shift Keying (BFSK)

The conditional-error probability (CEP) for coherent binary phase-shift keying and frequency-shift keying can be expressed as [9, eq. (26)], [10, eq. (5)]

$$\begin{aligned}P_b(\gamma) &= \tfrac{1}{2}\,\mathrm{erfc}(\sqrt{a\gamma}) \\ &= \frac{1}{\pi}\int_0^{\pi/2}\exp\left(-\frac{a\gamma}{\cos^2\theta}\right)d\theta \\ &= \frac{1}{\pi}\int_0^{\pi/2}\exp\left(-\frac{a\gamma}{\sin^2\theta}\right)d\theta\end{aligned} \tag{13}$$

where $a=1$ for BPSK and $a=1/2$ for BFSK. The advantage of these exponential representations for analysis over fading channels is that the final average error rates can be expressed strictly in terms of the MGF. Therefore, the ABER can be readily shown as

$$\begin{aligned}\overline{P}_b &= \frac{1}{\pi}\int_0^{\pi/2}\phi_\gamma\left(\frac{a}{\cos^2\theta}\right)d\theta \\ &= \frac{1}{\pi}\int_0^{\pi/2}\left(\frac{\cos^2\theta}{\cos^2\theta+a\overline{\gamma}/m}\right)^m d\theta \\ &= \frac{1}{\pi}\,I_c(0,\,\pi/2,\,a\overline{\gamma}/m,\,m)\end{aligned} \tag{14}$$

or alternatively

$$\overline{P_b}=\frac{1}{\pi}\int_0^{\pi/2}\phi_\gamma\left(\frac{a}{\sin^2\theta}\right)d\theta=\frac{1}{\pi}\,I_S(0,\,\pi/2,\,a\overline{\gamma}/m,\,m) \tag{15}$$

where the closed-form solutions for $I_C(.,.,.,.)$ and $I_S(.,.,.,.)$ are given by (4) and (6) (or the equivalent expressions in the Appendix). A slightly more general solution for this specific case (i.e., $\theta_L=0$ and $\theta_U=\pi/2$) can be obtained by expressing $\mathrm{erfc}(.)$ in terms of the incomplete Gamma function and then using identity [15, eq. (6.455.1)], viz.,

$$\begin{aligned}I_C(0,\,\pi/2,\,c,\,m) &= \frac{\Gamma(m+1/2)}{2m\Gamma(m)(1+c)^m}\sqrt{\frac{\pi c}{1+c}}\,{}_2F_1\left(1,\,m+\frac{1}{2};m+1;\,\frac{1}{1+c}\right)\end{aligned} \tag{16}$$

which holds for any positive real $m\geq 0.5$.

B. MPSK and MDPSK

The CEP for MPSK is given by [13, eq. (5)]

$$P_S(\gamma)=\frac{1}{\pi}\int_0^{\pi-\pi/M}\exp\left(\frac{-\gamma\sin^2(\pi/M)}{\sin^2\theta}\right)d\theta. \tag{17}$$

Hence, the closed-form formula for the ASER in Nakagami-m fading with positive integer m can be obtained immediately by inspection

$$\overline{P_S} = \frac{1}{\pi} I_S(0, \pi - \pi/M, (\overline{\gamma}/m)\sin^2(\pi/M), m) \tag{18}$$

which is considerably simpler than [5, eq. (22)].

Pawula has recently shown that the CEP of MDPSK can be written as [16, eq. (3)]

$$P_S(\gamma) = \frac{1}{\pi}\int_0^{\pi-\pi/M,} \exp\left[\frac{-\gamma \sin^2(\pi/M)}{1+\cos(\pi/M)\cos\theta}\right] d\theta. \tag{19}$$

Therefore, the corresponding ASER is given by

$$\begin{aligned}\overline{P_S} &= \frac{1}{\pi}\int_0^{\pi-\pi/M} \phi_\gamma\left(\frac{\sin^2(\pi/M)}{1+\cos(\pi/M)\cos\theta}\right) d\theta \\ &= \frac{1}{\pi}\int_0^{\pi-\pi/M} \\ &\quad \times \left(\frac{1+\cos(\pi/M)\cos\theta}{1+\cos(\pi/M)\cos\theta + (\overline{\gamma}/m)\sin^2(\pi/M)}\right)^m d\theta \\ &= \frac{1}{\pi} I_V(0, \pi - \pi/M, \cos(\pi/M), (\overline{\gamma}/m)\sin^2(\pi/M), m)\end{aligned} \tag{20}$$

where $I_V(., ., ., ., .)$ may be computed using (11).

C. Noncoherent Detection of Equiprobable Correlated Binary Signals and $\pi/4$-DQPSK

The CEP for noncoherent detection of equal energy, equiprobable, binary nonorthogonal complex signals can be written as (with the aid of identity [16, eq. (10)])

$$\begin{aligned}P_S(\gamma) &= Q(a\sqrt{\gamma}, b\sqrt{\gamma}) - \frac{1}{2} I_0(ab\gamma)\exp\left[-\frac{\gamma}{2}(a^2+b^2)\right] \\ &= \frac{1}{2\pi}\int_0^{\pi} \exp\left(\frac{-\gamma(b^2-a^2)^2}{2(a^2+b^2) - 4ab\cos\theta}\right) d\theta\end{aligned} \tag{21}$$

where $a = \sqrt{(1-\sqrt{1-|\rho|^2}/2)}$, $b = \sqrt{(1+\sqrt{1-|\rho|^2}/2}$, and $0 \le |\rho| \le 1$ is the magnitude of the cross-correlation coefficient between the two signals. In a similar fashion to the development of (20), the desired ASER can be deduced by inspection

$$\overline{P}_S = \frac{1}{2\pi} I_V\left(0, \pi, \frac{-2ab}{(a^2+b^2)}, \frac{(b^2-a^2)^2(\overline{\gamma}/m)}{2(a^2+b^2)}, m\right). \tag{22}$$

In recent years, performance analysis of DQPSK has received considerable attention, owing to its adoption in the second generation of North American and Japanese digital cellular standards (see [17]–[19] and their references). The CEP for this modulation format can be expressed as

$$\begin{aligned}P_S(\gamma) &= \frac{1}{2\pi}\int_0^{\pi} \frac{\exp(-\gamma(2-\sqrt{2}\cos\theta))}{\sqrt{2}-\cos\theta} d\theta \\ &= \frac{1}{2\pi}\int_0^{\pi}\exp\left(\frac{-2\gamma}{2-\sqrt{2}\cos\theta}\right) d\theta\end{aligned} \tag{23}$$

and hence the ASER for $\pi/4$-DQPSK with Gray coding may be computed efficiently using

$$\overline{P}_S = \frac{1}{2\pi} I_V(0, \pi, -\sqrt{2}/2, \overline{\gamma}/m, m). \tag{24}$$

In fact, the ASER for this special case of interest can be obtained directly from (22) by substituting $a = \sqrt{2-\sqrt{2}}$ and $b = \sqrt{2+\sqrt{2}}$. Note also that (24) is considerably simpler than [18, eq. (52)] and does not require the evaluation of higher-order derivatives (which can be cumbersome when m is large).

D. MQAM and QPSK

By utilizing the exponential integral representations for $\mathrm{erfc}(.)$ and $\mathrm{erfc}^2(.)$, the exact CEP of M-ary square-QAM can be expressed as

$$\begin{aligned}P_S(\gamma) &= \frac{4q}{\pi}\int_0^{\pi/2}\exp\left(-\frac{p\gamma}{\sin^2\theta}\right) d\theta - \frac{4q^2}{\pi}\int_0^{\pi/4} \\ &\quad \times \exp\left(-\frac{p\gamma}{\sin^2\theta}\right) d\theta \\ &= \frac{4q}{\pi}\int_0^{\pi/2}\exp\left(-\frac{p\gamma}{\cos^2\theta}\right) d\theta - \frac{4q^2}{\pi}\int_{\pi/4}^{\pi/2} \\ &\quad \times \exp\left(-\frac{p\gamma}{\cos^2\theta}\right) d\theta\end{aligned} \tag{25}$$

where $p = 1.5/(M-1)$ and $q = 1 - 1/\sqrt{M}$. From these we immediately get closed-form solutions for the ASER in Nakagami-m fading (positive integer m)

$$\begin{aligned}\overline{P}_S &= \frac{4q}{\pi} I_S(0, \pi/2, p\overline{\gamma}/m, m) - \frac{4q^2}{\pi} \\ &\quad \times I_S(0, \pi/4, p\overline{\gamma}/m, m) \\ &= \frac{4q}{\pi} I_C(0, \pi/2, p\overline{\gamma}/m, m) - \frac{4q^2}{\pi} \\ &\quad \times I_C(\pi/4, \pi/2, p\overline{\gamma}/m, m)\end{aligned} \tag{26}$$

where $I_C(., ., ., .)$ and $I_S(., ., ., .)$ are calculated using (4) and (6), respectively. Although further simplifications of (4) and (6) are possible when $[\theta_L = 0, \theta_U = \pi/2]$ and $[\theta_L = 0, \theta_U = \pi/4]$, we refrain from doing so because once the generic closed-form expressions (4) and (6) are implemented, they can immediately be used to compute the error rates of a broad class of modulation formats (e.g., star-QAM, MPSK, arbitrary 2-D M-ary signal constellations)—with or without diversity reception, by simply varying their arguments. In addition, recognizing that the signal constellations for QPSK and 4-QAM are very similar, the performance of QPSK can be calculated directly from (26) by substituting $M = 4$.

E. 2-D M-ary Signals

In [13], Craig derived the CEP for arbitrary 2-D M-ary signaling constellations with polygonal decision boundaries, using geometric relations, namely

$$P_S(\gamma) = \frac{1}{2\pi}\sum_{k=1}^{S} W_k \int_0^{\eta_k}\exp\left[\frac{-\gamma a_k \sin^2(\varphi_k)}{\sin^2(\theta+\varphi_k)}\right] d\theta \tag{27}$$

where S is the total number of signal points or decision subregions, W_k is the *a priori* probability of the symbol to which subregion k corresponds, a_k, η_k and φ_k are parameters relating to decision subregion k and they are independent of the instantaneous SNR γ. Since most constellations have symmetry, the number of subregions with distinct geometries is usually less than S. For our subsequent development, it will be more convenient to express (27) as

$$P_S(\gamma) = \frac{1}{2\pi}\sum_{k=1}^{S} W_k \int_{\varphi_k}^{\eta_k+\varphi_k} \exp\left[\frac{-\gamma a_k \sin^2(\varphi_k)}{\sin^2\theta}\right] d\theta \tag{28}$$

so that we can immediately get a closed-form solution for the ASER

$$\overline{P}_S = \frac{1}{2\pi}\sum_{k=1}^{S} W_k I_S[\varphi_k,\, \eta_k+\varphi_k,\, (\overline{\gamma}/m)a_k \sin^2\varphi_k,\, m] \tag{29}$$

where $I_S(.,.,.,.)$ is defined in (6). The previous result in the literature has been restricted to a finite-range integral expression for any $m \neq 1$(see [11, eq. (7)]).

As an illustrative example, let us consider the error rate calculation of 16-star QAM in Nakagami-m fading with positive integer m. The corresponding ASER can be readily evaluated using (29) with the following parameters [11]:

$$\begin{aligned}
&S = 4\\
&a_1 = a_3 = a_4 = \frac{2}{1+\beta^2}\left[(\beta-1)^2 + \left(\sqrt{2}-1\right)^2(\beta+1)^2\right]\\
&a_2 = \frac{8}{1+\beta^2} \qquad \eta_1 = \eta_3 = \tan^{-1}\left[\left(\sqrt{2}-1\right)\frac{\beta+1}{\beta-1}\right]\\
&\eta_2 = \pi - \eta_1 \qquad \eta_4 = \frac{7\pi}{8} - \eta_1 \qquad \varphi_1 = \varphi_3 = \frac{\pi}{2} - \eta_1\\
&\varphi_2 = \frac{\pi}{8} \qquad \varphi_4 = \frac{\pi}{8} + \eta_1 \qquad W_k = 1, \qquad \text{for } 1,\ldots,4
\end{aligned}$$

where β denotes the ring ratio and equally-likely transmissions of signals are assumed. For the particular case of $m = 1$, (29) reduces to [11, eq. (13)] as expected.

IV. Maximal-Ratio Diversity

In this section, we show that it also straightforward to derive closed-form ASER expressions for a broad class of modulation formats that employ MRC in Nakagami-m fading. Specifically, the following four cases can be treated easily: 1) independent and identically distributed (i.i.d.) diversity branches where mL is a positive integer; 2) dissimilar diversity branches with integer m_l, $l = 1, \ldots, L$; 3) identical $m_l/\overline{\gamma_l}$ across the diversity branches and $\sum_l m_l$ is a positive integer; and 4) generically correlated fading with integer m.

A. Independent and Identical Fading

When all the diversity branches are i.i.d., the MGF of the combiner output SNR is given by

$$\phi_{\gamma_s}(s) = \left(\frac{m}{m+s\overline{\gamma}}\right)^{mL}, \qquad m \geq \frac{1}{2} \tag{30}$$

where $\overline{\gamma}$ denotes the average received SNR per symbol per branch. Comparison between (30) and (12) reveals that both MGFs are almost identical except for the power constant. Given this, closed-form ASER formulas for the i.i.d. MRC case can be easily obtained from the results in Section III by replacing $m \to mL$ in the last argument of $I_C(.,.,.,.)$, $I_S(.,.,.,.)$ and $I_V(.,.,.,.,.)$. Consequently, we can immediately obtain the following closed-form solutions by inspection:

a) BPSK and BFSK: $\overline{P}_b = (1/\pi)I_C(0,\, \pi/2,\, a\overline{\gamma}/m,\, mL)$;
b) MPSK: $\overline{P}_S = (1/\pi)I_S(0,\, \pi-\pi/M,\, (\overline{\gamma}/m)\sin^2(\pi/M),\, mL)$;
c) MDPSK: $\overline{P}_S = (1/\pi)I_V(0,\, \pi-\pi/M,\, \cos(\pi/M),\, (\overline{\gamma}/m)\sin^2(\pi/M),\, mL)$;
d) DQPSK with Gray coding: $\overline{P}_S = (1/2\pi)I_V(0, \pi-\sqrt{2}/2,\, \overline{\gamma}/m,\, mL)$;
e) MQAM: $\overline{P}_S = (4q/\pi)I_S(0,\, \pi/2,\, p\overline{\gamma}/m,\, mL) - (4q^2/\pi)I_S(0,\, \pi/4,\, p\overline{\gamma}/m,\, mL)$;
f) 2-D M-ary signals: $\overline{P}_S = (1/2\pi)\sum_{k=1}^{S} W_k I_S[\varphi_k,\, \eta_k+\varphi_k,\, (\overline{\gamma}/m)a_k\sin^2\varphi_k,\, mL]$.

It should be noted that the restriction of a positive integer fading severity index is rather stringent and may be unnecessary. In fact, we can obtain closed-form formulas as long as mL is an integer. Thus, many more cases such as $\{m = 0.5,\, L = 2\}$ and $\{m = 1.2,\, L = 5\}$ could be handled using the above expressions.

B. Independent and Nonidentical Fading

If the diversity branches are distinct and all m_l's assume integer values, we can show that the MGF of $\gamma_s = \sum_{l=1}^{L}\gamma_l$ can be expressed (by partial fractions) as

$$\begin{aligned}
\phi_{\gamma_s}(s) &= \prod_{l=1}^{L}\left(\frac{m_l}{m_l+s\overline{\gamma}_l}\right)^{m_l}\\
&= \sum_{l=1}^{L}\sum_{k=1}^{m_l}\Lambda_l^{(m_l-k)}\left(\frac{m_l}{m_l+s\overline{\gamma}_l}\right)^{k}
\end{aligned} \tag{31}$$

where

$$\begin{aligned}
\Lambda_l^{(m_l-k)} = &\frac{(\overline{\gamma}_l/m_l)^k}{(m_l-k)!}\left[\prod_{i=1}^{L}\left(\frac{m_i}{\overline{\gamma}_i}\right)^{m_i}\right]\\
&\times \frac{d^{m_l-k}}{dt^{m_l-k}}\left[\prod_{i=1,\, i\neq l}^{L}\frac{1}{(t+m_i/\overline{\gamma}_i)^{m_i}}\right]\Bigg|_{t=-m_l/\overline{\gamma}_l}.
\end{aligned} \tag{32}$$

Once again, by noting the similarities between (31) and (12), we can directly obtain closed-form solutions for the ASER of different modulation formats in Nakagami-m fading (positive integer m) with dissimilar statistics. For instance, performance of the MDPSK and any 2-D signal constellation with polygonal decision regions may be computed using (33) and (34), respectively

$$\begin{aligned}
\overline{P}_S = &\frac{1}{\pi}\sum_{l=1}^{L}\sum_{k=1}^{m_l}\Lambda_l^{(m_l-k)}\\
&\times I_V\left[0,\, \pi-\frac{\pi}{M},\, \cos\left(\frac{\pi}{M}\right),\, \left(\frac{\overline{\gamma}_l}{m_l}\right)\sin^2(\pi/M),\, k\right]
\end{aligned} \tag{33}$$

$$\begin{aligned}
\overline{P}_s = &\frac{1}{2\pi}\sum_{l=1}^{L}\sum_{k=1}^{m_l}\Lambda_l^{(m_l-k)}\sum_{z=1}^{S}\\
&\times W_z I_S\left[\varphi_z,\, \eta_z+\varphi_z,\, (\overline{\gamma}_l/m_l)\, a_z\sin^2\varphi_z,\, k\right].
\end{aligned} \tag{34}$$

For all the other modulation schemes discussed in Section III, the corresponding ASER formulas in closed form can be obtained by inspection, similar to the two examples illustrated above.

Now we shall consider a special case where $m_l/\overline{\gamma}_l = \vartheta$, $l = 1, \ldots, L$, and $\sum_l m_l = D$, where D is a positive integer. In this case, the random variable γ_s (i.e., combiner output SNR) also has a Gamma pdf, viz.,

$$p_{\gamma_s}(\gamma) = \frac{\vartheta^D}{(D-1)!}\,\gamma^{D-1}\exp(-\vartheta\gamma). \tag{35}$$

The Laplace transform of (35) yields the MGF

$$\phi_{\gamma_s}(s) = \left(\frac{1}{1+s/\vartheta}\right)^D. \tag{36}$$

Therefore, closed-form solutions for the ASER can be obtained following our treatment in Section III. As an example, the performance of MPSK is given by

$$\overline{P}_s = \frac{1}{\pi}\,I_S\left[0,\,\pi-\pi/M,\,\frac{1}{\vartheta}\sin^2(\pi/M),\,D\right] \tag{37}$$

where we have replaced the final argument of $I_S(.,.,.,.)$ in (18) (single channel reception case) from $m \to D$ and let $m/\overline{\gamma} \to \vartheta$ in its third argument. In case of DQPSK or MDPSK, the final solution is attained from the closed-form ASER formula for single channel reception case by substituting $m/\overline{\gamma} \to \vartheta$ and $m \to D$ in $I_V(.,.,.,.,.)$'s fourth and fifth arguments, respectively.

C. Correlated Fading

When the diversity branches are correlated, the analysis proceeds in a similar manner to the independent fading scenario. In fact, the MGF approach for MRC only requires knowledge of the MGF of the combiner output SNR in order to calculate the ASER. For an arbitrarily correlated Nakagami-m fading environment (with the assumption that the fading severity index is common to all the diversity branches), the desired MGF may be written in the form [8], [20]

$$\phi_{\gamma_s}(s) = \det(I + sR\Lambda)^{-m} = \prod_{l=1}^{L}\frac{1}{(1+s\lambda_l)^m} \tag{38}$$

where I is the $L \times L$ identity matrix, Λ is a positive definite matrix of dimension L (determined by the branch covariance matrix), R is a diagonal matrix defined as $R = \mathrm{diag}(\overline{\gamma}_1/m, \ldots, \overline{\gamma}_L/m)$, and λ_l is the lth eigenvalue of matrix $R\Lambda$.

Using partial fractions, (38) can be expressed in a more desirable form (i.e., which facilitates the derivation of closed-form ASER formulas in bivariate Nakagami-m fading for positive integer fading severity index)

$$\phi_{\gamma_s}(s) = \sum_{l=1}^{L}\sum_{k=1}^{m}\mu_l^{(m-k)}\left(\frac{1}{1+s\lambda_l}\right)^k \tag{39}$$

where

$$\mu_l^{(m-k)} = \frac{\lambda_l^{k-m}}{(m-k)!}\,\frac{d^{m-k}}{dt^{m-k}}\left[\prod_{i=1,\,i\neq l}^{L}\frac{1}{(\lambda_i t+1)^m}\right]\Bigg|_{t=-1/\lambda_l}. \tag{40}$$

For the particular case of $m = 1$ (correlated Rayleigh fading), (39) reduces to

$$\phi_{\gamma_s}(s) = \sum_{l=1}^{L}\frac{\mu_l^{(0)}}{(1+s\lambda_l)} = \sum_{l=1}^{L}\frac{1}{1+s\lambda_l}\prod_{i=1,\,i\neq 1}^{L}\frac{\lambda_l}{\lambda_l-\lambda_i}. \tag{41}$$

Recognizing that the MGFs depicted in (39) and (31) are almost identical, we can therefore derive closed-form ASER expressions for a broad class of binary and M-ary modulation schemes in correlated Rayleigh and Nakagami-m fading cases (as we have done for the independent diversity branches with dissimilar fading statistics scenario). Thus, the following closed-form solutions are obtained by inspection from Section III:

a) BPSK and BFSK: $\overline{P}_b = (1/\pi)\sum_{l=1}^{L}\sum_{k=1}^{m}\mu_l^{(m-k)}I_C\,[0,\,(\pi/2),\,a\lambda_l,\,k]$;
b) MPSK: $\overline{P}_S = (1/\pi)\sum_{l=1}^{L}\sum_{k=1}^{m}\mu_l^{(m-k)}I_S\,[0,\,\pi-(\pi/M),\,\lambda_l\sin^2(\pi/M),\,k]$;
c) MDPSK: $\overline{P}_S = (1/\pi)\sum_{l=1}^{L}\sum_{k=1}^{m}\mu_l^{(m-k)}I_V\,[0,\,\pi-(\pi/M),\,\cos(\pi/M),\,\lambda_l\sin^2(\pi/M),\,k]$;
d) Nonorthogonal signaling: $\overline{P}_S = (1/2\pi)\sum_{l=1}^{L}\sum_{k=1}^{m}\mu_l^{(m-k)}I_V\,[[0,\,\pi,\,(-2ab/(a^2+b^2)),\,((b^2-a^2)^2\lambda_l/2(a^2+b^2)),\,k]$;
e) DQPSK with Gray coding: $\overline{P}_S = (1/2\pi)\sum_{l=1}^{L}\sum_{k=1}^{m}\mu_l^{(m-k)}I_V\,[0,\,\pi,\,-\sqrt{2}/2,\,\lambda_l,\,k]$;
f) MQAM: $\overline{P}_S = (4q/\pi)\sum_{l=1}^{L}\sum_{k=1}^{m}\mu_l^{(m-k)}(I_S[0,\,\pi/2,\,p\lambda_l,\,k] - qI_S\,[0,\,\pi/4,\,p\lambda_l,\,k])$;
g) 2-D M-ary signals: $\overline{P}_S = (1/2\pi)\sum_{l=1}^{L}\sum_{k=1}^{m}\mu_l^{(m-k)}\sum_{z=1}^{s}W_z I_S\,[\varphi_z,\,\eta_z+\varphi_z,\,\lambda_l a_z\sin^2\varphi_z,\,k]$.

To the best of the authors' knowledge, all of the above expressions are new. Previous results have been limited to MPSK in Rayleigh fading [14] or a finite-range integral expression (e.g., [8]).

V. SELECTION DIVERSITY

Recognizing that the MGF of combiner output SNR is the key to unified analysis of a wide range of digital modulations over fading channels, in this section, we first derive the desired MGF in Nakagami-m fading (positive integer fading severity index). Subsequently, closed-form ASER formulas for several modulation schemes employing SDC are derived with the aid of trigonometric integral identities presented in Section II.

By utilizing the Laplace transformation of a derivative property,[2] it is possible to describe the MGF in terms of the cumulative distribution function (cdf) instead of the usual definition involving the pdf

$$\phi_{\gamma_s}(s) = s\int_0^\infty \exp(-s\gamma)F_{\gamma_s}(\gamma)\,d\gamma \tag{42}$$

since $F_{\gamma_s}(0) = 0$ for all common fading channel models, and the cdf of SDC output SNR is simply the product of the cdf of

[2]Given an original function $f(t)$ and its Laplace transform $G(s)$, $(d/dt)f(t) \Leftrightarrow sG(s) - f(0)$.

SNR from individual diversity branches, viz.,

$$F_{\gamma_s}(\gamma) = \prod_{l=1}^{L} F_l(\gamma) \tag{43}$$

where $F_l(\gamma) = \gamma(m_l, m_l\gamma/\overline{\gamma}_l)/\Gamma(m_l)$ for Nakagami-m fading and notation $\gamma(a, x) = \int_0^x e^{-t}t^{a-1}\,dt$ denotes the incomplete Gamma[3] function.

A. Independent and Identical Fading

The SDC output SNR in Nakagami-m fading (integer m) with i.i.d. diversity branches can be expressed as

$$\begin{aligned} F_{\gamma_s}(\gamma) &= \left[1 - \exp\left(\frac{-m\gamma}{\overline{\gamma}}\right)\sum_{k=0}^{m-1}\left(\frac{m\gamma}{\overline{\gamma}}\right)^k \frac{1}{k!}\right]^L \\ &= \sum_{n=0}^{L}(-1)^n\binom{L}{n}\exp\left(\frac{-nm\gamma}{\overline{\gamma}}\right)\left[\sum_{k=0}^{m-1}\left(\frac{m\gamma}{\overline{\gamma}}\right)^k \frac{1}{k!}\right]^n \\ &= \sum_{n=0}^{L}(-1)^n\binom{L}{n}\exp\left(\frac{-nm\gamma}{\overline{\gamma}}\right)\sum_{k=0}^{n(m-1)}\beta_{kn}\left(\frac{m\gamma}{\overline{\gamma}}\right)^k \end{aligned} \tag{44}$$

after expanding the first expression in (44) binomially and then using multinomial theorem. Coefficients β_{kn} in (44) may be computed using [21]

$$\beta_{kn} = \sum_{i=k-m+1}^{k} \frac{\beta_{i(n-1)}}{(k-i)!} I_{[0,\,(n-1)(m-1)]}(n) \tag{45}$$

where $\beta_{00} = \beta_{0n} = 1$, $\beta_{k1} = 1/k!$, $\beta_{1n} = n$ and

$$I_{[a,\,b]}(n) = \begin{cases} 1, & a \le n \le b \\ 0, & \text{otherwise.} \end{cases}$$

Thus, the MGF of γ_s is given by (obtained by substituting the final expression in (44) into (42) and after simplifications)

$$\phi_{\gamma_s}(s) = \sum_{n=0}^{L}(-1)^n\binom{L}{n}\sum_{k=0}^{n(m-1)}\beta_{kn}\frac{k!m^k s\overline{\gamma}}{(\overline{\gamma}s + nm)^{k+1}} \tag{46}$$

which can be restated as

$$\begin{aligned} \phi_{\gamma_s}(s) &= \sum_{n=0}^{L}(-1)^n\binom{L}{n}\sum_{k=0}^{n(m-1)}\beta_{kn}\frac{k!}{n^k} \\ &\quad \times \left[\frac{1}{[1+s(\overline{\gamma}/mn)]^k} - \frac{1}{[1+s(\overline{\gamma}/nm)]^{k+1}}\right]. \end{aligned} \tag{47}$$

For the particular case of $m = 1$, (46) reduces to the familiar expression [9, eq. (17)].

Now it should be apparent that we can also immediately derive closed-form ASER expressions for SDC systems in Nakagami-m fading, following our treatment for the MRC case. Thus, the solutions for a variety of digital modulations with SDC receivers can be obtained by inspection. For instance, error rate expressions for MDPSK and MPSK are given by (48) and (49), respectively

$$\begin{aligned} \overline{P}_S &= \frac{1}{\pi}\sum_{n=0}^{L}(-1)^n\binom{L}{n}\sum_{k=0}^{n(m-1)}\beta_{kn}\frac{k!}{n^k} \\ &\quad \times \left\{ I_V\left[0,\, \pi - \frac{\pi}{M},\, \cos\left(\frac{\pi}{M}\right),\, \frac{\overline{\gamma}}{nm}\sin^2(\pi/M),\, k\right]\right. \\ &\qquad - I_V\left[0,\, \pi - \frac{\pi}{M},\, \cos\left(\frac{\pi}{M}\right),\right. \\ &\qquad\quad \left.\left.\frac{\overline{\gamma}}{nm}\sin^2(\pi/M),\, k+1\right]\right\} \end{aligned} \tag{48}$$

$$\begin{aligned} \overline{P}_S &= \frac{1}{\pi}\sum_{n=0}^{L}(-1)^n\binom{L}{n}\sum_{k=0}^{n(m-1)}\beta_{kn}\frac{k!}{n^k} \\ &\quad \times \left\{ I_S\left[0,\, \pi - \frac{\pi}{M},\, \frac{\overline{\gamma}}{nm}\sin^2(\pi/M),\, k\right]\right. \\ &\qquad \left. - I_S\left[0,\, \pi - \frac{\pi}{M},\, \frac{\overline{\gamma}}{nm}\sin^2(\pi/M),\, k+1\right]\right\}. \end{aligned} \tag{49}$$

Next, we demonstrate that it is possible to derive an even more concise solution for the ASER if we derive the MGF of SDC output SNR from its pdf. For the i.i.d. case, the pdf is given by

$$\begin{aligned} p_{\gamma_s}(\gamma) &= \frac{L}{\Gamma(m)}\left(\frac{m}{\overline{\gamma}}\right)^m \gamma^{m-1}\exp\left(\frac{-m\gamma}{\overline{\gamma}}\right) \\ &\quad \times \left[1 - \exp\left(\frac{-m\gamma}{\overline{\gamma}}\right)\sum_{k=0}^{m-1}\left(\frac{m\gamma}{\overline{\gamma}}\right)^k\frac{1}{k!}\right]^{L-1} \\ &= \frac{L}{\Gamma(m)}\sum_{n=0}^{L-1}(-1)^n\binom{L-1}{n}\exp\left[\frac{-m\gamma(n+1)}{\overline{\gamma}}\right] \\ &\quad \times \sum_{k=0}^{n(m-1)}\beta_{kn}\left(\frac{m}{\overline{\gamma}}\right)^{m+k}\gamma^{k+m-1}. \end{aligned} \tag{50}$$

Therefore, the desired MGF can be readily shown as

$$\begin{aligned} \phi_{\gamma_s}(s) &= \frac{L}{\Gamma(m)}\sum_{n=0}^{L-1}(-1)^n\binom{L-1}{n}\sum_{k=0}^{n(m-1)}\beta_{kn}\frac{\Gamma(k+m)}{(n+1)^{m+k}} \\ &\quad \times \left[\frac{1}{1+s\overline{\gamma}/[m(n+1)]}\right]^{m+k} \end{aligned} \tag{51}$$

with the aid of identity [15, eq. (3.351.3)]. Now using (51), we get another closed-form solution for the ASER of MDPSK with SDC

$$\begin{aligned} \overline{P}_S &= \frac{L}{\pi}\sum_{n=0}^{L-1}(-1)^n\binom{L-1}{n}\sum_{k=0}^{n(m-1)}\beta_{kn}\frac{\Gamma(k+m)}{\Gamma(m)(n+1)^{m+k}} \\ &\quad \times I_V\left[0,\, \pi - \frac{\pi}{M},\, \cos\left(\frac{\pi}{M}\right),\, \frac{\overline{\gamma}}{m(n+1)}\right. \\ &\qquad \left. \times \sin^2(\pi/M),\, m+k\right]. \end{aligned} \tag{52}$$

[3]It is well known that $\gamma(n, x) = (n-1)![1 - e^{-x}\sum_{k=0}^{n-1}(x^k/k!)]$ for postive integer n.

Since (52) is considerably simpler than (48), it is recommended that (51) be used to derive closed-form error rate expressions for other modulation formats. For instance, we can immediately write the closed-form ASER formula for MQAM with SDC by inspection [using (51) and (26)]

$$\overline{P}_S = \frac{4qL}{\pi} \sum_{n=0}^{L-1} (-1)^n \binom{L-1}{n} \sum_{k=0}^{n(m-1)} \beta_{kn} \frac{\Gamma(k+m)}{\Gamma(m)(n+1)^{m+k}} \times \left\{ I_S \left[0, \frac{\pi}{2}, \frac{p\overline{\gamma}}{m(n+1)}, m+k\right] - qI_S \left[0, \frac{\pi}{4}, \frac{p\overline{\gamma}}{m(n+1)}, m+k\right] \right\}. \quad (53)$$

Using the above technique, one can easily derive closed-form solutions for all other modulation schemes discussed in Section III (omitted here for the sake of brevity). Moreover, it should be noted that the advantage of the new results derived in this section are twofold. First of all, our new expressions hold for any arbitrary L and this general case can be treated in a single formula. More importantly, our solutions are in closed form. Both these points contrast with the recent results put forward by Dong *et al.* [12] for the special case of 2-D signal constellations with polygonal decision boundaries. For the second and the third-order diversity, [12, eqs. (27) and (28)] require an evaluation of a single finite-range integral. Whereas for the general case of arbitrary L, [12, eq. (25)] involves a doubly integral expression. It is therefore evident that our approach yields both exact closed-form and computationally efficient results for the Nakagami-m fading with integer fading severity index. Note also that [12, eqs. (27) and (28)] can be readily transformed into an integral form identical to (5) (using variable substitution $\Theta = \theta + \psi$) and we can immediately express these integrals in closed form

$$P_{L=2} = \frac{1}{\pi} I_S \left[\psi, \eta+\psi, \frac{\Lambda^2 a}{m} \sin^2 \psi, m\right] - \sum_{k=0}^{m-1} \frac{1}{2^{m+k}\pi} \binom{m+k-1}{k} I_S \times \left[\psi, \eta+\psi, \frac{\Lambda^2 a}{2m} \sin^2 \psi, m+k\right] \quad (54)$$

$$P_{L=3} = \frac{3}{2\pi} I_S \left[\psi, \eta+\psi, \frac{\Lambda^2 a}{m} \sin^2 \psi, m\right] - \sum_{k=0}^{m-1} \frac{3}{2^{m+k}\pi} \binom{m+k-1}{k} I_S \times \left[\psi, \eta+\psi, \frac{\Lambda^2 a}{2m} \sin^2 \psi, m+k\right] + \sum_{k=0}^{m-1} \sum_{j=0}^{m-1} \frac{(m+k+j-1)!}{(m-1)!k!j!} \frac{3^{1-m-j-k}}{2\pi} I_S \times \left[\psi, \eta+\psi, \frac{\Lambda^2 a}{3m} \sin^2 \psi, m+j+k\right]. \quad (55)$$

B. Independent but Nonidentical Fading

Following the development of (50), we can show that the pdf of γ_s at the output of a dual-branch SDC with dissimilar fading statistics is given by

$$p_{\gamma_s}(\gamma) = \sum_{l=1,\, z\neq l}^{2} \frac{1}{(m_l-1)!} \left(\frac{m_l}{\overline{\gamma}_l}\right)^{m_l} \gamma^{m_l-1} \exp\left(\frac{-m_l\gamma}{\overline{\gamma}_l}\right) \times \left[1 - \exp\left(\frac{-m_z\gamma}{\overline{\gamma}_l}\right) \sum_{k=0}^{m_z-1} \left(\frac{m_z\gamma}{\overline{\gamma}_z}\right)^k \frac{1}{k!}\right]. \quad (56)$$

Then, finding Laplace transform of (56), and after simplifications, the MGF can be expressed as

$$\phi_{\gamma_s}(\gamma) = \sum_{l=1,\, z\neq l}^{2} \left\{ \left(\frac{1}{1+s\overline{\gamma}_l/m_l}\right)^{m_l} - \sum_{k=0}^{m_z-1} \binom{k+m_l-1}{k} \times \frac{(\overline{\gamma}_l m_z)^k (\overline{\gamma}_z m_l)^{m_l}}{(\overline{\gamma}_l m_z + \overline{\gamma}_z m_l)^{k+m_l}} \times \left(\frac{1}{1+\frac{s\overline{\gamma}_l\overline{\gamma}_z}{\overline{\gamma}_l m_z+\overline{\gamma}_z m_l}}\right)^{k+m_l} \right\}. \quad (57)$$

Once again, the closed-form error rates for a broad class of digital modulations employing dual-branch SDC can be attained by inspection. For instance, the ASER of 2-D M-ary signal constellations and binary nonorthogonal signaling in Nakagami-m fading with integer m may be calculated using (58) and (59), respectively, shown at the bottom of the next page. It is also possible to derive closed-form error rate expressions for any $L > 2$ of practical interest. One way to achieve this is to substitute (43) into (42), and then evaluate the resultant integral (after the multiplication operation) using identity [15, eq. (3.351.3)]. Once the MGF is obtained, we can make use of the identities in Section II to get the closed-form solution for the ASER, as in our preceding examples. Alternatively, the other approach is to determine the pdf first using

$$p_{\gamma_s}(\gamma) = \sum_{l=1}^{L} \frac{1}{(m_l-1)!} \left(\frac{m_l}{\overline{\gamma}_l}\right)^{m_l} \gamma^{m_l-1} \exp\left(\frac{-m_l\gamma}{\overline{\gamma}_l}\right) \times \prod_{z=1,\, z\neq 1}^{L} \left[1-\exp\left(\frac{-m_z\gamma}{\overline{\gamma}_z}\right) \sum_{k=0}^{m_z-1} \left(\frac{m_z\gamma}{\overline{\gamma}_z}\right)^k \frac{1}{k!}\right] \quad (60)$$

and then compute its Laplace transform to get the MGF (after expanding the product term). The latter method is identical to the development of (57).

VI. Equal-Gain Diversity

Unlike MRC, EGC receiver analysis can be rather involved. The principle difficulty is finding the pdf of the combiner output SNR, which depends on the square of a sum of L random fading amplitudes. No closed-form solution for this sum is known for the Nakagami-m fading case. Even for the Rayleigh fading case, no solution is found for $L > 2$. In [22, eq. (82)], Nakagami-m approximated the pdf of a sum of L i.i.d. Nakagami-m variates by another Nakagami-m distribution, namely

$$p_\xi(x) \cong \frac{2}{\Gamma(m)}\left(\frac{\hat{m}}{\hat{\Omega}}\right)^{\hat{m}} x^{2\hat{m}-1}\exp\left(-\frac{\hat{m}}{\hat{\Omega}}x^2\right) \tag{61}$$

where $\xi = \sum_{l=1}^{L}\alpha_l$, $\alpha_l (l = 1, \ldots, L)$, are Nakagami-$m$ distributed fading amplitudes, $\Omega_l = E[\alpha_l^2]$, $\Omega_1 = \cdots = \Omega_L = \Omega$ for i.i.d. case

$$\hat{m} \approx mL$$

and

$$\hat{\Omega} = L\Omega + L(L-1)\Omega\frac{\Gamma^2(m+1/2)}{m\Gamma^2(m)} \approx L^2\Omega\left(1-\frac{1}{5m}\right).$$

Using this approximate pdf, we can show that the MGF of SNR at the combiner output is given by

$$\begin{aligned}\phi_{\gamma_s}(s) &\cong \left[\frac{mL}{mL+s\overline{\gamma}[1+(L-1)\Gamma^2(m+1/2)/m\Gamma^2(m)]}\right]^{mL}\\ &\simeq \left[\frac{m}{m+s\overline{\gamma}(1-1/5m)}\right]^{mL}\end{aligned} \tag{62}$$

where $\gamma_s = (E_s/LN_0)\xi^2$ is the instantaneous SNR at the EGC output, and $\overline{\gamma}$ is the average received SNR per symbol of a single diversity branch.

Noting the similarities between (62) and (12) or (30), we can immediately derive the approximate ASER formulas in closed form for different modulation formats employing EGC. In this case, the final argument of $I_C(.,.,.,.)$, $I_S(.,.,.,.)$, and $I_V(.,.,.,.,.)$ is equal to mL. Also, the third argument of $I_C(.,.,.,.,.)$ and $I_S(.,.,.,.,.)$ or the fourth argument of $I_V(.,.,.,.,.)$ should be multiplied by the factor $[1+(L-1)\Gamma^2(m+1/2)/m\Gamma^2(m)]/L \approx$ 1–1/5 m. For instance, the performance of arbitrary 2-D M-ary signal constellations with EGC in Nakagami-m fading (positive integer m) can be evaluated conveniently using (63)

$$\begin{aligned}\overline{P}_S &\cong \frac{1}{2\pi}\sum_{k=1}^{S}W_k I_S\\ &\quad\times\left[\varphi_k,\, \eta_k+\varphi_k,\, \left\{1+(L-1)\frac{\Gamma^2(m+1/2)}{m\Gamma^2(m)}\right\}\right.\\ &\quad\quad\left.\times\frac{\overline{\gamma}}{mL}\,a_k\sin^2\varphi_k,\, mL\right]\\ &= \frac{1}{2\pi}\sum_{k=1}^{S}W_k I_S\left[\varphi_k,\, \eta_k+\varphi_k,\, \left(1-\frac{1}{5m}\right)(\overline{\gamma}/m)\right.\\ &\quad\quad\left.\times a_k\sin^2\varphi_k,\, mL\right].\end{aligned} \tag{63}$$

We would like highlight that the accuracy of the approximation model improves as the fading index increases, and our numerical experiments reveal this model yields excellent approximation for $m \geq 2$, and thus very attractive from computational efficiency point of view. It is also straightforward to extend [22, eq. (82)] to take into account the effect of dissimilar signal strengths but assuming that the fading index m is identical across the diversity branches. In this case, we obtain

$$\hat{\Omega} = \sum_{l=1}^{L}\left\{\Omega_l + \frac{\Gamma^2(m+1/2)}{m\Gamma^2(m)}\sum_{k=1,\,k\neq l}^{L}\sqrt{\Omega_l\Omega_k}\right\}$$

since the mean $E[\alpha_1] = (\Gamma(m+1/2)/\Gamma(m))\sqrt{(\Omega_l/m)}$. Therefore, the MGF of SNR of the combiner output is given by

$$\phi_{\gamma_s}(s) \cong \left[\frac{mL}{mL+\dfrac{s}{L}\displaystyle\sum_{l=1}^{L}\left\{\overline{\gamma}_l+\frac{\Gamma^2(m+1/2)}{m\Gamma^2(m)}\sum_{k=1,\,k\neq l}^{L}\sqrt{\overline{\gamma}_l\overline{\gamma}_k}\right\}}\right]^{mL}. \tag{64}$$

$$\begin{aligned}\overline{P}_S = \frac{1}{2\pi}\sum_{l=1,\,z\neq 1}^{2}&\left\{\sum_{\nu=1}^{S}W_\nu I_S\left[\varphi_\nu,\, \eta_\nu+\varphi_\nu,\, \frac{\overline{\gamma}_l}{m_l}a_\nu\sin^2\varphi_\nu,\, m_l\right] - \sum_{k=0}^{m_z-1}\binom{k+m_l-1}{k}\frac{(\overline{\gamma}_l m_z)^k(\overline{\gamma}_z m_l)^{m_l}}{(\overline{\gamma}_l m_z+\overline{\gamma}_z m_l)^{k+m_l}}\right.\\ &\left.\times\sum_{\nu=1}^{S}W_\nu I_S\left[\varphi_\nu,\, \eta_\nu+\varphi_\nu,\, \left(\frac{\overline{\gamma}_l\overline{\gamma}_z}{\overline{\gamma}_l m_z+\overline{\gamma}_z m_l}\right)\times a_\nu\sin^2\varphi_\nu,\, k+m_l\right]\right\}\end{aligned} \tag{58}$$

$$\begin{aligned}\overline{P}_S = \frac{1}{2\pi}\sum_{l=1,\,z\neq 1}^{2}&\left\{I_V\left[0,\, \pi,\, \frac{-2ab}{(a^2+b^2)},\, \frac{\overline{\gamma}_l(b^2-a^2)^2}{2m_l(a^2+b^2)},\, m_l\right] - \sum_{k=0}^{m_z-1}\binom{k+m_l-1}{k}\frac{(\overline{\gamma}_l m_z)^k(\overline{\gamma}_z m_l)^{m_l}}{(\overline{\gamma}_l m_z+\overline{\gamma}_z m_l)^{k+m_l}}\right.\\ &\left.\times I_V\left[0,\, \pi,\, \frac{-2ab}{(a^2+b^2)},\, \frac{(b^2-a^2)^2}{2(a^2+b^2)}\left(\frac{\overline{\gamma}_l\overline{\gamma}_z}{\overline{\gamma}_l m_z+\overline{\gamma}_z m_l}\right),\, k+m_l\right]\right\}.\end{aligned} \tag{59}$$

As a check, we find that (64) reduces to (62) when all the diversity branches are i.i.d. As before, we can immediately get approximate closed-form solutions for a broad class of binary and M-ary modulation formats using (64). For instance, the performance of arbitrary 2-D M-ary modulations employing EGC with nonidentical fading in Nakagami-m fading can be evaluated using

$$\overline{P}_S \cong \frac{1}{2\pi}\sum_{k=1}^{S} W_k I_s \times \left[\varphi_k,\, \eta_k+\varphi_k,\, a_k \sin^2\varphi_k \frac{1}{mL^2}\sum_{l=1}^{L} \times \left\{\overline{\gamma}_l + \frac{\Gamma^2(m+1/2)}{m\Gamma^2(m)}\sum_{n=1,\, n\neq l}^{L}\sqrt{\overline{\gamma}_l\overline{\gamma}_n}\right\}, mL\right]. \tag{65}$$

VII. Conclusions

While the problem of M-ary signaling/diversity performance in Nakagami-m fading has been considered by many researchers, the closed-form solutions for many important cases have not been reported. In this paper, we have outlined a simple, unified approach for deriving closed-form error rates for single and multichannel reception of binary and M-ary signals in Nakagami-m fading with positive integer fading severity index. The solutions are exact for MRC and SDC receivers. Approximate solutions (but in closed form) for EGC receivers with i.i.d. and dissimilar branches have also been derived. Many new closed-form expressions have been derived in this paper including the cases such as 2-D M-ary signaling, DQPSK, MDPSK and so on. It is noted that the closed-form solutions presented here are not an exhaustive list, and many more cases such as analysis of narrowband digital frequency modulation with integrate and dump and sample and hold bit detection strategies, binary continuous-phase FSK with coherent and noncoherent detection schemes, maximum-likelihood differential detection of DPSK with block-by-block detection, exact pairwise-error probability for trellis and block coded modulations, all in Rayleigh and Nakagami-m fading with positive integer m, could be treated in a straightforward manner by utilizing the results derived in Section II. Similarly, closed-form solutions for a wide range of modulation formats in conjunction with imperfect MRC (i.e., Gaussian distributed weighting errors) or a hybrid SC/MRC receiver over Rayleigh fading can be derived easily. These and more related results (e.g., closed-form error rates for dual branch SDC in correlated Rayleigh fading) can be found in [23].

Appendix

In this appendix, we obtain closed-form solutions for $I_s(\theta_L, \theta_U, a, b)$ (where b is a positive integer) from the equivalence between [6, eq. (15)] or [3, eq. (21)] with (5). In fact, this solution is in a slightly different form compared to (6). From Section IV-A, we know that the ASER of MPSK for MRC with i.i.d. diversity branches in Rayleigh fading can be expressed as

$$\begin{aligned}\overline{P}_S &= \frac{1}{\pi}\int_0^{\pi-\pi/M}\left(\frac{\sin^2\theta}{\sin^2\theta+\overline{\gamma}\sin^2(\pi/M)}\right)^L d\theta\\ &= \frac{1}{\pi} I_S\left[0,\, \pi-\frac{\pi}{M},\, \overline{\gamma}\sin^2(\pi/M),\, L\right].\end{aligned} \tag{A.1}$$

Now, letting $a=\overline{\gamma}\sin^2(\pi/M)$ and $\theta_U = \pi-\pi/M$ in (A.1) as well as in [6, eq. (15)], we can get a closed-form solution for $I_S(0, \theta_U, a, L)$. Since $I_S(\theta_L, \theta_U, a, b) = I_S(0, \theta_U, a, b) - I_S(0, \theta_L, a, b)$, we have a complete solution for the integral (5)

$$\begin{aligned}&I_S(\theta_L, \theta_U, a, b)\\ &= \theta_U-\theta_L-\sqrt{\frac{a}{a+1}}\left[\frac{\pi}{2}+\alpha(a,\theta_U)\right]\sum_{k=0}^{b-1}\frac{1}{[4(a+1)]^k}\\ &\quad\times\left\{\binom{2k}{k}+\frac{1}{\pi}\sum_{l=0}^{k-1}\binom{2k}{l}\frac{\sin[2(k-l)\alpha(a,\theta_U)]}{k-l}\right\}\\ &\quad+\sqrt{\frac{a}{a+1}}\left[\frac{\pi}{2}+\alpha(a,\theta_L)\right]\times\sum_{k=0}^{b-1}\frac{1}{[4(a+1)]^k}\\ &\quad\times\left\{\binom{2k}{k}+\frac{1}{\pi}\sum_{l=0}^{k-1}\binom{2k}{l}\frac{\sin[2(k-l)\alpha(a,\theta_L)]}{k-l}\right\}\end{aligned} \tag{A.2}$$

where $\alpha(c, \theta) = \tan^{-1}[\sqrt{(c/1+c)}\cot(\pi-\theta)]$.

Using a similar approach, we can also derive yet another closed-form expression for $I_S(\theta_L, \theta_U, a, b)$ using [3, eq. (21)] instead of [6, eq. (15)]

$$\begin{aligned}&I_S(\theta_L, \theta_U, a, b)\\ &= \theta_U-\theta_L-\sqrt{\frac{a}{a+1}}[\alpha(a,\theta_U)-\alpha(a,\theta_L)]\\ &\quad\times\sum_{k=0}^{b-1}\binom{2k}{k}\frac{1}{[4(a+1)]^k}-\sqrt{\frac{a}{a+1}}\sin[\alpha(a,\theta_U)]\\ &\quad\times\sum_{k=1}^{b-1}\sum_{i=1}^{k}\frac{T_{ik}}{(a+1)^k}\cos^{2(k-i)+1}[\alpha(a,\theta_U)]\\ &\quad+\sqrt{\frac{a}{a+1}}\sin[\alpha(a,\theta_L)]\sum_{k=1}^{b-1}\sum_{i=1}^{k}\frac{T_{ik}}{(a+1)^k}\\ &\quad\times\cos^{2(k-i)+1}[\alpha(a,\theta_L)]\end{aligned} \tag{A.3}$$

where notation T_{ik} is defined as

$$T_{ik} = \frac{\binom{2k}{k}}{\binom{2(k-i)}{k-i}4^i[2(k-i)+1]}.$$

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