Further Results on the Beaulieu Series

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***Abstract*—A frequent problem in digital communications is the computation of the probability density function (pdf) and cumulative distribution function (cdf), given the characteristic function (chf) of a random variable (RV). This problem arises in signal detection, equalizer performance, equal-gain diversity combining, intersymbol interference, and elsewhere. Often, it is impossible to analytically invert the chf to get the pdf and cdf in closed form. Beaulieu has derived an infinite series for the cdf of a sum of RVs that has been widely used. In this letter, we rederive his series using the Gil–Pelaez inversion formula and the Poisson sum formula. This derivation has several advantages including both the bridging of the well-known sampling theorem with Beaulieu's series and yielding a simple expression for calculating the truncation error term. It is also shown that the pdf and cdf can be computed directly using a discrete Fourier transform.**

***Index Terms*—Characteristic functions, diversity, Fourier analysis, intersymbol interference and cochannel interference, outage.**

I. Introduction

A FREQUENT problem encountered in the performance evaluation of digital communications systems is the computation of the probability density function (pdf) $f(x)$ and cumulative distribution function (cdf) $F(x)$ of a random variable (RV) X given the characteristic function (chf) $\phi(\omega)$. This problem occurs in many applications including signal detection, linear equalizers, maximal-ratio and equal-gain diversity combining, intersymbol interference, outage probability, cochannel interference, coded modulation, and phase-jitter. Often, both the chf and the moment generating function (mgf) are readily available in closed form, but it is difficult or impossible to carry out the inverse Fourier transform (FT) or Laplace transform analytically to get the pdf and cdf in closed form. Instead, moment methods, numerical quadrature formulas, Chebyshev approximations, and other techniques have been developed [1]–[3].

One method that has proven particularly useful in communications applications is the infinite series for the cdf, derived by Beaulieu [4]. The cdf series was obtained by combining two well-known and valuable techniques in communication theory; wherein a Chernoff bound approach was applied to an approximate Fourier series expansion of a periodic square waveform. The Beaulieu series is important because it has been employed to solve many diverse problems [5]–[16]. While all of these papers deal with *the application of the Beaulieu series*, in this letter, we focus on some issues, not sufficiently elaborated in the original paper [4], surrounding the series itself. Importantly, we rederive the Beaulieu series using the Gil-Pelaez inversion formula and the Poisson sum formula. The alternative derivation exploits the fact that the pdf and chf together form an FT pair, and it provides insights into the uses and limitations[1] of the Beaulieu series. Specifically, our alternative derivation provides the following.

- For unbounded RVs, the total error has two components: the series truncation error and the aliasing error (or the domain truncation error). Explicit expressions for both the terms are derived.
- The series truncation error bound can now be expressed using the chf. Whereas the truncation error bound given in [4, eq. (16)] is less useful because it is expressed in terms of the pdf samples, which are unknown.
- The convergence conditions of the series are made more explicit.
- The relationship to the sampling theorem is made more explicit. For instance, the truncation error bound for the Beaulieu series has the same format as the aliasing error bound for the reconstruction of a nonband-limited signal that uses the sampling theorem.

Aside from [4], there have been several key studies dealing with the numerical computation of pdf and cdf given a chf [18]–[23]. Reference [18] is concerned with numerical inversion of Laplace transforms using the finite Fourier cosine transform and provides an approximate series for function $f(t)$ given its Laplace transform $F(s)$. This does not use either the Gil-Pelaez theorem or the Poisson sum formula. Reference [19] contributes a sine series for the cdf. In [20], the author proposes a windowing function to apply to the chf and derives the error terms. Reference [21] treats the fast FT inversion of z transforms. In [22], a general method for approximating a cdf from its chf is given. This paper also derives an error bound. Reference [23] considers the computation of the cdf of an integer-valued random variable.

II. Alternative Derviation

To begin, let X be an RV with pdf $f(x)$ and cdf $F(x)$ and complementary cdf $G(x)$ defined as

$$F(x) = \int_{-\infty}^{x} f(\xi)\,d\xi \quad \text{and} \quad G(x) = \int_{x}^{\infty} f(\xi)\,d\xi. \qquad (1)$$

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[1]The main limitation of the Beaulieu series is that it can lose accuracy when computing the tails of a distribution (e.g., error rates less than 10^{-8}). Helstrom thus suggests the use of saddle-point integration [17].

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The characteristic function of X is given by

$$\phi(\omega) = \int_{-\infty}^{\infty} f(x) e^{j\omega x}\, dx \tag{2}$$

where $j^2 = -1$.

The Gil-Pelaez theorem [24] states

$$F(x) = \frac{1}{2} - \int_{-\infty}^{\infty} \frac{e^{-j\omega x}\phi(\omega)}{j2\pi\omega}\, d\omega \tag{3}$$

i.e., the cdf is given as an integral of the chf. In fact, if one approximates this integral by a trapezoidal sum, one gets the Beaulieu series. However, Beaulieu's original approach both avoids a problem with a discontinuity at the origin ($\omega = 0$) and provides rigorous bounds on all the systematic sources of error. Our approach both clarifies the method and improves the error bounds. The following theorem leads to our new derivation of the series.

Theorem 1: For a pdf $f(x)$, which is bandlimited or almost bandlimited (i.e., there are positive constants A and α for which $|\phi(\omega)| \leq A(1+|\omega|)^{-1-\alpha}$ for all real ω)

$$f(x) = \frac{4}{T} \sum_{n=1,\, n \text{ odd}}^{\infty} \Re\{e^{-jn\omega_0 x}\phi(n\omega_0)\} + E(f;x) \tag{4a}$$

$$F(x) = \frac{1}{2} - \sum_{n=1,\, n \text{ odd}}^{\infty} \frac{2\Im\{e^{-jn\omega_0 x}\phi(n\omega_0)\}}{n\pi} + E(F;x) \tag{4b}$$

where $\Re(z)$ and $\Im(z)$ denote the real and imaginary parts of z, T is a parameter governing the sampling rate in the frequency domain, $\omega_0 = 2\pi/T$, and the error terms are

$$E(f;x) = \sum_{|n|>0} (-1)^{n+1} f(x + nT/2) \tag{5a}$$

$$E(F;x) = \sum_{n=1}^{\infty} (-1)^n [G(x + nT/2) - F(x - nT/2)]. \tag{5b}$$

Note that the two series in (4) are periodic with period T and that (4a) can exactly compute $f(x)$ for either $|x| \leq T/4$ or $|x| \leq T/2$ for bounded two-sided or bounded one-sided positive RVs, respectively. The terms $E(f;x)$ and $E(F;x)$ are the domain truncation errors for unbounded RVs.

Proof: If $u(t)$ and $U(\omega)$ are an FT pair, the Poisson sum formula [25, p. 395] yields

$$\sum_{n=-\infty}^{\infty} u(t+nh) = \frac{1}{h} \sum_{n=-\infty}^{\infty} U(2\pi n/h) e^{j2\pi nt/h} \tag{6}$$

provided these series are convergent. The right-hand side (RHS) is convergent if $u(t)$ is bandlimited or almost bandlimited. If $u(t)$ is a pdf, then $u(t) \to 0$ as $|t| \to \infty$ and this is a necessary condition for the convergence of the left-hand side (LHS). Note that if $u(t) = 0$ for $|t| > h/2$, the right-hand side can exactly compute $u(t)$.

We will begin by proving (4a) and (5a). Suppose

$$u(t) = f(t) e^{-j\pi t/h} \tag{7}$$

then by the frequency shifting property of FTs

$$U(\omega) = \phi\left(-\omega - \frac{\pi}{h}\right). \tag{8}$$

Substituting (7) and (8) in (6), multiplying both sides by $e^{j\pi t/h}$, we get

$$\sum_{n=-\infty}^{\infty} f(t+nh) e^{-jn\pi} = \frac{1}{h} \sum_{n=-\infty}^{\infty} \phi\left(-\frac{2\pi n}{h} - \frac{\pi}{h}\right) e^{j\pi t(2n+1)/h}. \tag{9}$$

Setting $h = T/2$, using the relationship $\phi^*(\omega) = \phi(-\omega)$, while noting that the $n = 0$ term in the LHS gives $f(t)$, we can thus obtain (4a) and (5a).

Now we will consider the derivation of the cdf series (4b). Since $U(0)$ in the Poisson sum formula (6) is the area under $u(t)$, we have

$$\int_{-\infty}^{\infty} u(\zeta)\, d\zeta = h \sum_{n=-\infty}^{\infty} u(t+nh) - \sum_{|n|>0} U(2\pi n/h) e^{j2\pi nt/h}. \tag{10}$$

The first sum on the RHS is a trapezoidal rule for the integral. Note that if $U(\omega) = 0$ for $|\omega| > 2\pi/h$, this rule is exact. We can now set $t = 0$ or $t = h/2$ to obtain useful formulas for the integral.

Suppose

$$u(t) = \frac{e^{-jtx}\phi(t)}{j2\pi t} \tag{11}$$

then using the identity in (3), we have

$$U(\omega) = \int_{-\infty}^{\infty} u(t) e^{-j\omega t}\, dt = 1/2 - F(x + \omega). \tag{12}$$

Substituting (11) and (12) in (10), setting $t = h/2$ in (10), and letting $h = 4\pi/T$, we obtain (4b). The error term $E(F;x)$ in (5b) is obtained by combining $U(nT/2)$ and $U(-nT/2)$ in (10). ■

A. Relationship to the Sampling Theorem

The sampling theorem shows that a function can be reconstructed using its own samples [25, p. 378]

$$f(x) = \sum_{n=-\infty}^{\infty} f\left(\frac{n\pi}{\Omega}\right) \frac{\sin(\Omega x - n\pi)}{(\Omega x - n\pi)}.$$

This reconstruction is exact if $|\phi(\omega)| = 0$ for $|\omega| > \Omega$ where $\phi(\omega)$ is the FT of $f(x)$. Otherwise, undersampling and the concomitant spectral aliasing in the frequency domain occur. The aliasing error depends on the behavior of $\phi(\omega)$ for $|\omega| > \Omega$. By contrast, in (4a) $f(x)$ is reconstructed using its frequency-domain samples and the reconstruction is exact if $f(x) = 0$ for $|x| > T$ for some constant T. The aliasing error now depends on the behavior of $f(x)$ for $|x| > T$.

B. Convergence

The convergence of the two infinite series (4a) and (4b) is ensured if $f(x)$ is bandlimited (i.e., $\phi(\omega) = 0$ for $|\omega| \geq \omega_c$, in which case the series terminate for some finite index n) or almost bandlimited. Since

$$\left| \sum_n^{\infty} \Re\{e^{-jn\omega_0 x}\phi(n\omega_0)\} \right| \leq \sum_n^{\infty} \frac{A}{(1+n\omega_0)^{1+\alpha}} \qquad (13a)$$

$$\left| \sum_n^{\infty} \frac{\Im\{e^{-jn\omega_0 x}\phi(n\omega_0)\}}{n} \right| \leq \sum_n^{\infty} \frac{A}{n(1+n\omega_0)^{1+\alpha}} \qquad (13b)$$

convergence of the two series is assured.

C. Domain Truncation

Consider an RV with the range $a < X < b$, denoted by $[a, b]$, where a and b are finite. Therefore, $f(x) = 0$ if $x \notin [a, b]$. From (5a) and (5b), it follows at once:

$$E(f; x) = E(F; x) = 0, \qquad \text{for } x \in [a, b] \qquad (14)$$

if $T > 2(b-a)$. That is, the two infinite series expansions are *exact*. On the other hand, if a or b is infinite, one has to truncate the pdf so that most of its probability mass is concentrated on a bounded interval. One selects this interval such that $\Pr(X \notin [a, b]) \leq \epsilon$, and ϵ can be made arbitrarily small.

D. Series Truncation

Suppose the two series are truncated at $n = N$. The truncation error is given by

$$E_t(f) = \frac{4}{T} \sum_{n=N+2,\, n \text{ odd}}^{\infty} \Re\{e^{-jn\omega_0 x}\phi(n\omega_0)\}$$

$$|E_t(f)| \leq \frac{4}{T} \sum_{n=N+2,\, n \text{ odd}}^{\infty} |\phi(n\omega_0)|$$

$$\leq \frac{1}{\pi} \int_{N\omega_o}^{\infty} |\phi(\omega)|\, d\omega. \qquad (15)$$

It is assumed that $|\phi(\omega)|$ decreases monotonically for $|\omega| \geq N\omega_0$. It can similarly be shown that

$$|E_t(F)| \leq \frac{1}{\pi} \int_{N\omega_o}^{\infty} \frac{|\phi(\omega)|}{\omega}\, d\omega. \qquad (16)$$

This series truncation error bound is expressed in terms of a known chf. But the truncation error bound in [4] requires knowledge of the pdf samples which are unknown and hence further bounding techniques must be employed to use that bound. Interestingly, the truncation error bound (15) is twice the aliasing error bound [26] when a nonband-limited function $f(t)$ is reconstructed from its samples taken at $T/2N$ apart assuming $f(t)$ is bandlimited to $2\pi N/T$. Note also that (15) and (16) imply that if $|\phi(\omega)| = 0$ for $\omega \geq N\omega_0$, then the truncation error is zero.

E. Generalized Beaulieu Series

When deriving (4b), we set $t = h/2$ in the LHS of (10). However, if instead we set $t = \sigma$ (a variable parameter) and use (11) and (12) in (10), then we get a slightly more general infinite series for the cdf

$$F(x) = \frac{1}{2} - 2 \sum_{n=-\infty}^{\infty} \frac{e^{-j(\sigma + 4n\pi/T)x}\phi\left(\sigma + \dfrac{4n\pi}{T}\right)}{j(\sigma T + 4n\pi)} + E_\sigma(F; x) \qquad (17)$$

where

$$E_\sigma(F; x) = \sum_{|n|>0} [1/2 - F(x + nT/2)]\, e^{jn\sigma T/2}. \qquad (18)$$

This series depends on the sampling rate parameter T and an additional parameter σ. Note that if $\sigma = 2\pi/T$, then this reduces to (4b), the Beaulieu series. If $\sigma = 0$, (17) yields a slightly different series for the cdf. If σ is between 0 and $2\pi/T$, the series sum is complex.

III. Discrete FT (DFT) Implementation

Ignoring $E(f; x)$ (5a) for the moment, while setting $T = 2P$ and limiting the series (4a) to $N/2$ terms, we obtain

$$\hat{f}(x) = \frac{1}{P} \sum_{n=-N+1,\, n \text{ odd}}^{N-1} e^{-jn\pi x/P}\phi\left(\frac{n\pi}{P}\right). \qquad (19)$$

There is a clear resemblance between this sum and the DFT. This sum can also be recognized as a periodic Fourier series expansion for a function in the region $0 \leq x < P$. Now let $k = (n + N - 1)/2$ and $d_k \triangleq \phi[(2k + 1 - N)\pi/P]$. Thus

$$\hat{f}(x) = \frac{1}{P} e^{j\pi x(N-1)/P} \sum_{k=0}^{N-1} e^{-j2\pi xk/P}\, d_k. \qquad (20)$$

Note that the phase factor arises because we use (7). Since $f(x) \geq 0$ and is real-valued, we can write

$$\hat{f}\left(\frac{lP}{N}\right) = \frac{1}{P} |D_l|, \qquad l = 0, \ldots, N-1 \qquad (21)$$

where D_l are the N-point DFT of d_k.

Next, we ignore $E(F; x)$ (5b) for a moment, while setting $T = 2P$ and limiting the series in (4b) to $N/2$ terms. It is clear that this is a Fourier series expansion for a function in the region $0 \leq x < P$. Now, let $k = (n + N - 1)/2$ and $\overline{d}_k \triangleq \phi[(2k + 1 - N)\pi/P]/[j\pi(2k + 1 - N)]$. Thus

$$\hat{F}\left(\frac{lP}{N}\right) = \frac{1}{2} - e^{j\pi l(N-1)/N}\overline{D}_l, \qquad l = 0, \ldots, N-1 \qquad (22)$$

where $\overline{D}_l$ are the N-point DFT of $\overline{d}_k$.

The above development indicates that Beaulieu's series expressions for the cdf and pdf can be computed using a DFT. That is, if the cdf is to be computed at N equispaced points, the Beaulieu series (4b) does not need to be computed N times, rather a single DFT provides all the cdf samples. This is a significant reduction in computational complexity.

IV. Conclusions

An alternative derivation has been developed, using both the Gil-Pelaez inversion formula and the Poisson sum formula, for the Beaulieu series for the pdf and cdf. This derivation has several advantages including both the bridging of the well-known sampling theorem with Beaulieu's series and the yielding of a simple expression for the truncation error term. For example, if the pdf $f(x)$ is computed using the series, the truncation error depends on the area under the magnitude of the chf in "out-band;" hence, the truncation error is similar to the aliasing error in the sampling series. Another interesting conclusion is that when the two series are truncated, they resemble an N-point DFT expression. Therefore, both the pdf and cdf can be computed directly using DFTs. Beaulieu's series expressions (4a) and (4b) are very useful because the chf can be found easily in many problems where the pdf is much more difficult to obtain. Nearly ten years after their publication, the contribution of this letter is hence to enhance the understanding of the Beaulieu series to help their use.

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