Exact Analysis of Equal-Gain Diversity Systems over Fading Channels

A. Annamalai[1], Jing Su[2] and C. Tellambura[3]

1. Bradley Department of Electrical and Computer Engineering, Virginia Polytechnic Institute and State University 7054 Haycock Road, Falls Church, VA 22043, USA, Tel: +1-703-535-3459, Fax: +1-703-518-8085
2. Mobile Communications Technology Center, Mitsubishi Wireless Communications, San Diego, CA 92121, USA Tel: +1-619-535-8841, Fax: +1-619-535-8801, E-mail: jing.su@cmtsd.mea.com
3. School of Computer Science and Software Engineering, Monash University, Clayton, Victoria 3168, Australia Tel: +1-613-9905-3296, Fax: +1-613-9905-3402, E-mail: chintha@dgs.monash.edu.au

ABSTRACT – Performance evaluation of equal-gain (EG) diversity systems is notoriously difficult — a classical problem dating back to Lord Rayleigh himself almost a century ago. A number of references to this problem and to its difficulty can be found in the research literature and textbooks. In this paper, we derive exact analytical expressions for the EG receiver performance in Rayleigh, Rician, Nakagami-m and Nakagami-q fading channels in terms of the Appell hypergeometric function. Our methodology and analytical framework can readily handle any order of diversity as well as arbitrary fading statistics (including mixed-fading). The method also applies to a broad class of coherent, differentially coherent and noncoherent modulation formats. For coherent BPSK and BFSK, the solution is particularly simple because a desirable exponential integral representation for the Gaussian probability integral is available.

I. Introduction

Despite its practical and theoretical importance, the literature on EG diversity systems is barren compared to that of other diversity combining methods. This lack may have stemmed from the difficulty of computing the probability density function (PDF) of the EG output signal-to-noise ratio (SNR) which depends on the square of a sum of L-fading amplitudes. In fact, a closed-form solution to the PDF of the sum has been elusive for almost a century, and indeed even for the case of Rayleigh fading (mathematically simplest distribution), no solution exists for $L > 2$.

In the past, some approximation techniques have been employed to characterize the EG receiver performance (see [1]-[8]). While the convergent infinite series technique suggested in [4] and [5] is accurate, the solution is still an approximation. Zhang [9] and Annamalai, Tellambura and Bhargava [6] were first to derive exact solutions for $L \geq 2$ in Rayleigh and Nakagami-m fading channels, respectively. In [9], only binary signalling schemes are considered and the author's closed-form solutions are restricted to second (for noncoherent detection) or third order diversity (for coherent detection) in Rayleigh fading. More recently, [6] developed a powerful frequency-domain technique and subsequently derived a single finite-range integral expression for calculating the error probabilities of binary PSK and multilevel QAM with L-branch EG diversity over the Nakagami-m channel. The analysis was extended to different modulation formats and a variety of fading channel models in [8] and [10]. Some closed-form solutions for binary signalling schemes in Nakagami-m fading were also derived in [10]. Building on our previous work, in this paper we derive exact analytical expressions for a broad class of modulation formats with EG diversity in all common fading channel models (Rayleigh, Rician, Nakagami-m and Nakagami-q). Different from [6] and [8], our final result is expressed in terms of the Appell hypergeometric function. The key to our solution is the reformulation of the error probability expression in the frequency domain [6, 10] and the solution to a Laplace transform integral involving the product of Kummer functions. As an aside, we show that Zhang's closed-form formulas in Rayleigh fading as special instances of our general expression.

The contributions of this paper include: (a) the development of a simple and direct approach for calculating the error rates for coherent binary signalling schemes by exploiting a desirable exponential form for the Gaussian probability integral. This alternate form is suitable to perform averaging over fading amplitudes because the L-fold convolution integral may now be partitioned into a product of one-dimension integrals (without any cross-product terms). Moreover, we point out (through change of index) that the computation of a convergent Appell hypergeometric series will require only the evaluation of a single infinite series of finite sums (i.e., the upper limit for the rest of the series will be finite); (b) derivation of some closed-form and exact infinite series solutions for a broad class of binary and M-ary modulations in Rayleigh, Rician, Nakagami-m and Nakagami-q fading. It is further highlighted that the advantage of our infinite series solutions (inherent in the CHF of Rician and Nakagami-q random variables, due to the Bessel function) is two-fold: (i) first, its representation is exact — this is in contrast to the infinite series solution devised by Beaulieu because his representation introduces some systematic errors for unbounded random variables (i.e., signal reconstruction error in addition to the series truncation error); (ii) secondly, it is considerably simpler and computationally very efficient in comparison with the approximate infinite series method suggested in [5].

II. Error Rates for Coherent Binary Signals

Using an alternative exponential integral representation for the Gaussian probability integral [10, Eq. (18)], it is easy to verify that

$$a\operatorname{erfc}(bx) = a - \frac{2a}{\pi}\int_0^\infty \frac{1}{t}e^{-t^2}\sin(2btx)dt \tag{1}$$

Thus, the averaging over the PDF of sum of fading amplitudes for the average symbol or bit error rates calculations for coherent BFSK ($a = 1/2, b = 1/\sqrt{2}$) and BPSK ($a = 1/2, b = 1$) can be accomplished without much difficulty (since the L-fold average can be partitioned into the product of L independent one-dimensional integrals):

$$P_b = \int_0^\infty \cdots \int_0^\infty \left[a - \frac{2a}{\pi} \int_0^\infty \frac{e^{-t^2}}{t} \mathrm{Imag}\left\{ \exp\left(j2bt \sqrt{\frac{E_b}{LN_0}} \sum_{l=1}^{L} \alpha_l \right) \right\} dt \right] \times f_{\alpha_1}(\alpha_1) \ldots f_{\alpha_L}(\alpha_L) d\alpha_1 \ldots d\alpha_L$$

$$= a - \frac{2a}{\pi} \int_0^\infty \frac{e^{-t^2}}{t} \mathrm{Imag}\left\{ \prod_{l=1}^{L} \int_0^\infty e^{j\frac{2bt}{\sqrt{L}}\sqrt{\frac{E_b}{N_0}}\alpha_l} f_{\alpha_l}(\alpha_l) d\alpha_l \right\} dt$$

$$= a - \frac{2a}{\pi} \int_0^\infty \frac{e^{-t^2}}{t} \mathrm{Imag}\left\{ \prod_{l=1}^{L} \phi_{\upsilon_l}\left(\frac{2bt}{\sqrt{L}} \right) \right\} dt$$

$$= a - \frac{a}{\pi} \int_0^\infty \frac{e^{-t}}{t} \mathrm{Imag}\left\{ \prod_{l=1}^{L} \phi_{\upsilon_l}\left(2b\sqrt{\frac{t}{L}} \right) \right\} dt \quad (2)$$

where L denotes the diversity order, E_b/N_0 is the SNR per bit, $f_{\alpha_l}(.)$ corresponds to the PDF of the fading amplitude in the l-th branch (α_l's are assumed to be either Rayleigh, Rician, Nakagami-m or Nakagami-q random variables) and $\phi_{\upsilon_l}(.)$ is the CHF of the fading envelope where $\upsilon_l = \alpha_l \sqrt{E_b/N_0}$.

Letting $t = \tan\theta$ in (2), we may also express the ABER in terms of a finite-range integral expression:

$$P_b = a - \frac{4a}{\pi} \int_0^{\frac{\pi}{2}} \frac{\exp(-\tan^2\theta)}{\sin(2\theta)} \mathrm{Imag}\left\{ \prod_{l=1}^{L} \phi_{\upsilon_l}\left(\frac{2b\tan\theta}{\sqrt{L}} \right) \right\} d\theta \quad (3)$$

The CHF of the fading envelopes for all common fading channel models are listed in Table 1.

A. Identical Fading

For independent and identically distributed (iid) diversity branches, (2) can be further simplified as

$$P_b = a - \frac{a}{\pi} \sum_{\substack{k=1 \\ k \text{ odd}}}^{L} (-1)^{\frac{k-1}{2}} \binom{L}{k} G(\mathrm{Re}[\phi_{\upsilon_1}(z)], \mathrm{Im}[\phi_{\upsilon_1}(z)], k, L) \quad (4)$$

where

$$G(A, B, x, y) = \int_0^\infty t^{-1} e^{-t} B^x A^{y-x} dt \quad (5)$$

and $z = 2b\sqrt{t/L}$, by exploiting $(A+B)^L = \sum_{k=0}^{L} \binom{L}{k} B^k A^{L-k}$

Note that (4) holds for all values of $L \geq 1$ (i.e., no restriction is imposed on the order of diversity). At this juncture, it is also worth noting that both the real and imaginary parts of the CHF of the signal amplitude can be expressed in terms of only the confluent hypergeometric series and/or the exponential functions (see Table 1). Hence, it is apparent that only the solution to the integral in the form of (6) (i.e., involving the product of Kummer functions) is further required to obtain an exact analytical solution for the EG receiver performance in a myriad of mobile radio environments:

$$I = \int_0^\infty t^{v-1} e^{-bt} \prod_{k=1}^{n} \Phi(a_k; b_k; c_k t) dt \quad (6)$$

Fortunately, the solution to this integral is known, and has been reported in [10]:

$$\int_0^\infty t^{v-1} e^{-bt} \prod_{k=1}^{n} \Phi(a_k; b_k; c_k t) dt = b^{-v} \Gamma(v) F_A\left[v; a_1, \ldots, a_n; b_1, \ldots, b_n; \frac{c_1}{b}, \ldots, \frac{c_n}{b} \right] \quad (7)$$

if $b_k > 0, v > 0, \sum c_k < b$. Notation $F_A(.;.;.;.)$ denotes the Appell hypergeometric function, which is defined as [12, Eq. (9.19)]

$$F_A(\alpha; \beta_1, \ldots, \beta_n; \gamma_1, \ldots, \gamma_n; z_1, \ldots, z_n) = \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \cdots \sum_{m_n=0}^{\infty} \frac{(\alpha)_{m_1+\ldots+m_n} (\beta_1)_{m_1} \cdots (\beta_n)_{m_n}}{(\gamma_1)_{m_1} \cdots (\gamma_n)_{m_n} m_1! \ldots m_n!} z_1^{m_1} z_2^{m_2} \ldots z_n^{m_n} \quad (8)$$

Table 1. PDF and CHF of fading signal amplitude for several fading channel models.

Channel Model	PDF $f_{\upsilon_l}(.)$ and CHF $\phi_{\upsilon_l}(.)$ of fading amplitude of the l-th branch $\upsilon_l = \alpha_l \sqrt{E_s/N_0}$
Rayleigh	$f_{\upsilon_l}(z) = \frac{2z}{\bar{\gamma}_l} \exp\left(\frac{-z^2}{\bar{\gamma}_l} \right),\ z \geq 0$ where $\bar{\gamma}_l = \frac{E_s}{N_0} E[\alpha_l^2]$ = average SNR per symbol $\phi_{\upsilon_l}(\omega) = \Phi\left(1; \frac{1}{2}; \frac{-\bar{\gamma}_l \omega^2}{4} \right) + j\omega \sqrt{\frac{\pi \bar{\gamma}_l}{4}} \exp\left(\frac{-\bar{\gamma}_l \omega^2}{4} \right)$
Rician	$f_{\upsilon_l}(z) = \frac{2(1+K_l)z}{\bar{\gamma}_l} \exp\left(-K_l - \frac{(1+K_l)z^2}{\bar{\gamma}_l} \right) \times I_0\left(2z \sqrt{\frac{K_l(K_l+1)}{\bar{\gamma}_l}} \right),\ z \geq 0$ where Rice factor $K_l \geq 0$ $\phi_{\upsilon_l}(\omega) = \exp(-K_l) \left[\sum_{i=0}^{\infty} \frac{K_l^i}{i!} \Phi\left(i+1; \frac{1}{2}; \frac{-\bar{\gamma}_l \omega^2}{4(1+K_l)} \right) + j\omega \sqrt{\frac{\bar{\gamma}_l}{1+K_l}} \sum_{i=0}^{\infty} \frac{\Gamma(i+3/2) K_l^i}{(i!)^2} \Phi\left(i + \frac{3}{2}; \frac{3}{2}; \frac{-\bar{\gamma}_l \omega^2}{4(1+K_l)} \right) \right]$
Nakagami-q ($-1 \leq b \leq 1$)	$f_{\upsilon_l}(z) = \frac{2z}{\bar{\gamma}_l \sqrt{1-b_l^2}} \exp\left(\frac{-z^2}{[1-b_l^2]\bar{\gamma}_l} \right) I_0\left(\frac{b_l z^2}{[1-b_l^2]\bar{\gamma}_l} \right),\ z \geq 0$ where $b_l = (1-q_l^2)/(1+q_l^2)$ and the fading parameter $0 \leq q_l \leq \infty$ $\phi_{\upsilon_l}(\omega) = \sqrt{1-b_l^2} \sum_{i=0}^{\infty} \frac{(b_l/2)^{2i} \Gamma(2i+1)}{(i!)^2} \times \Phi\left(2i+1; \frac{1}{2}; \frac{-\bar{\gamma}_l [1-b_l^2] \omega^2}{4} \right) + j\omega \sqrt{\bar{\gamma}_l}\, [1-b_l^2] \times \sum_{i=0}^{\infty} \frac{(b_l/2)^{2i} \Gamma(2i+3/2)}{(i!)^2} \Phi\left(2i + \frac{3}{2}; \frac{3}{2}; \frac{-\bar{\gamma}_l [1-b_l^2] \omega^2}{4} \right)$
Nakagami-m	$f_{\upsilon_l}(z) = \frac{2}{\Gamma(m_l)} \left(\frac{m_l}{\bar{\gamma}_l} \right)^{m_l} z^{2m_l - 1} \exp\left(\frac{-m_l z^2}{\bar{\gamma}_l} \right),\ z \geq 0$ where Nakagami-m fading parameter $m_l \geq 0.5$ $\phi_{\upsilon_l}(\omega) = \Phi\left(m_l, \frac{1}{2}; \frac{-\bar{\gamma}_l \omega^2}{4m_l} \right) + j\omega \frac{\Gamma(m_l + 1/2)}{\Gamma(m_l)} \sqrt{\frac{\bar{\gamma}_l}{m_l}} \Phi\left(m_l + \frac{1}{2}; \frac{3}{2}; \frac{-\bar{\gamma}_l \omega^2}{4m_l} \right)$

Although the computation of (8) appears difficult (i.e., need to evaluate the multiple infinite series), but it can be greatly simplified for many cases of practical interest. For instance, (8) reduces to a finite series if all $\beta_i, i \in \{1, \ldots, n\}$ are either zero or negative integer regardless of the values of z_i. This is because $(\beta_i)_{m_i} = 0$ for all $m_i > -\beta_i$. Furthermore, if the Appell hypergeometric function is absolutely convergent (i.e., if all $|z_i| < 1$, $i \in \{1, \ldots, n\}$), then it is possible to replace the multiple infinite series by a single infinite series of finite sums. This property is particularly useful in numerical computation, and is further elaborated in the Appendix in connection with the Cauchy product of infinite series. As it will become apparent in the later part of this section, all of our calculations for the Appell hypergeometric series fall in one of the above two categories. Also, for the particular case of $n = 2$, the Appell function in (7) can be replaced by the series

$$\begin{aligned} F_2(v;a_1, a_2;b_1, b_2;c_1, c_2) &= \sum_{n=0}^{\infty} \frac{(v)_n (a_1)_n}{(b_1)_n n!} c_1^n \, {}_2F_1(v+n, a_2;b_2;c_2) \\ &= \sum_{n=0}^{\infty} \frac{(v)_n (a_2)_n}{(b_2)_n n!} c_2^n \, {}_2F_1(v+n, a_1;b_1;c_1) \end{aligned} \tag{9}$$

given that the conditions $[|c_1| + |c_2| < 1, b_k > 0]$ are satisfied. Following our preceding discussion, it is evident that (9) will reduce to a finite series, if v or a_1 or a_2 (or all of them) is either zero or a negative integer. In addition to this, the Appell hypergeometric series of the second kind has the following property:

$$\begin{aligned} F_2(b;a_1, a_2;b, b;x, y) &= (1-x)^{-a_1}(1-y)^{-a_2} {}_2F_1\left[a_1, a_2;b;\frac{xy}{(1-x)(1-y)}\right] \end{aligned} \tag{10}$$

Next, we will summarize some exact infinite series expressions and closed-form formulas (where applicable) for the error rates of coherent binary PSK and FSK employing EG receiver over mobile radio channels.

A.1 Rayleigh

From Table 1, we know that

$$A = \mathrm{Re}\left\{\phi_{v_1}\left(2b\sqrt{\frac{t}{L}}\right)\right\} = \exp\left(\frac{-\bar{\gamma}_1 b^2 t}{L}\right)\Phi\left(\frac{-1}{2};\frac{1}{2};\frac{\bar{\gamma}_1 b^2 t}{L}\right)$$

$$B = \mathrm{Im}\left\{\phi_{v_1}\left(2b\sqrt{\frac{t}{L}}\right)\right\} = \sqrt{\frac{\pi\bar{\gamma}_1 b^2 t}{L}}\exp\left(\frac{-\bar{\gamma}_1 b^2 t}{L}\right)$$

Thus, we have

$$G(A, B, k, L) = \left[\frac{\pi\bar{\gamma}_1 b^2}{L(1+\bar{\gamma}_1 b^2)}\right]^{k/2}\Gamma\left(\frac{k}{2}\right)$$

$$\times F_A\left[\frac{k}{2};\underbrace{\frac{-1}{2},\ldots,\frac{-1}{2}}\,;\underbrace{\frac{1}{2},\ldots,\frac{1}{2}}_{(L-k)\ \text{times}}\,;\underbrace{\frac{\bar{\gamma}_1 b^2/L}{1+\bar{\gamma}_1 b^2},\ldots,\frac{\bar{\gamma}_1 b^2/L}{1+\bar{\gamma}_1 b^2}}\right] \tag{11}$$

with the aid of (7). Now, substituting (11) into (4), we have a complete solution for CPSK and CFSK in Rayleigh fading. For the special cases of $L = 1$, $L = 2$ and $L = 3$, we can show that this new expression agrees with those furnished in [9] and [10]:

(a) $L = 1$ (single channel reception)

$$P_b = a - \frac{a}{\pi}G(A, B, 1, 1) = a\left[1 - \sqrt{\frac{\bar{\gamma}_1 b^2}{1+\bar{\gamma}_1 b^2}}\right] \tag{12}$$

(b) $L = 2$

$$\begin{aligned} P_b &= a - \frac{2a}{\pi}G(A, B, 1, 2) \\ &= a - ab\sqrt{\frac{2\bar{\gamma}_1}{1+\bar{\gamma}_1 b^2}}\,{}_2F_1\left(\frac{1}{2}, \frac{-1}{2};\frac{1}{2};\frac{\bar{\gamma}_1 b^2}{2(1+\bar{\gamma}_1 b^2)}\right) \\ &= a\left[1 - \frac{b\sqrt{\bar{\gamma}_1(2+\bar{\gamma}_1 b^2)}}{1+\bar{\gamma}_1 b^2}\right] \end{aligned} \tag{13}$$

(c) $L = 3$

$$\begin{aligned} P_b &= a - \frac{a}{\pi}[3G(A, B, 1, 3) - G(A, B, 3, 3)] \\ &= a - ab\sqrt{\frac{3\bar{\gamma}_1}{1+\bar{\gamma}_1 b^2}}F_A\left[\frac{1}{2};\frac{-1}{2}, \frac{-1}{2};\frac{1}{2}, \frac{1}{2};\frac{\bar{\gamma}_1 b^2}{3(1+\bar{\gamma}_1 b^2)}, \frac{\bar{\gamma}_1 b^2}{3(1+\bar{\gamma}_1 b^2)}\right] \\ &\quad + \frac{a\pi}{2}\left[\frac{\bar{\gamma}_1 b^2}{3(1+\bar{\gamma}_1 b^2)}\right]^{3/2} \end{aligned} \tag{14}$$

Now applying (10) into (14), we obtain the solution derived by Zhang, as expected.

A.2 Nakagami-m

For Nakagami-m fading, we have

$$A = \exp\left(\frac{-\bar{\gamma}_1 b^2 t}{m_1 L}\right)\Phi\left(\frac{1}{2} - m_1, \frac{1}{2};\frac{\bar{\gamma}_1 b^2 t}{m_1 L}\right)$$

$$B = \frac{2b\Gamma(m_1 + 1/2)}{\Gamma(m_1)}\sqrt{\frac{\bar{\gamma}_1 t}{m_1 L}}\exp\left(\frac{-\bar{\gamma}_1 b^2 t}{m_1 L}\right)\Phi\left(1 - m_1;\frac{3}{2};\frac{\bar{\gamma}_1 b^2 t}{m_1 L}\right)$$

Hence, it can be readily shown that

$$G(A, B, k, L) = \left[\frac{2\Gamma(m_1 + 1/2)}{\Gamma(m_1)}\sqrt{\frac{\bar{\gamma}_1 b^2}{L(m_1 + \bar{\gamma}_1 b^2)}}\right]^k \Gamma\left(\frac{k}{2}\right) \times F_A\left[\frac{k}{2};\right.$$

$$\underbrace{1-m_1, \ldots, 1-m_1}_{k}\,,\ \underbrace{\frac{1}{2}-m_1, \ldots, \frac{1}{2}-m_1}_{L-k}\,;\ \underbrace{\frac{3}{2}, \ldots, \frac{3}{2}}_{k}\,,$$

$$\left.\underbrace{\frac{1}{2}, \ldots, \frac{1}{2}}_{L-k}\,;\underbrace{\frac{\bar{\gamma}_1 b^2}{L(m_1 + \bar{\gamma}_1 b^2)}, \ldots, \frac{\bar{\gamma}_1 b^2}{L(m_1 + \bar{\gamma}_1 b^2)}}_{L}\right] \tag{15}$$

after simplifications using integral identity (7).

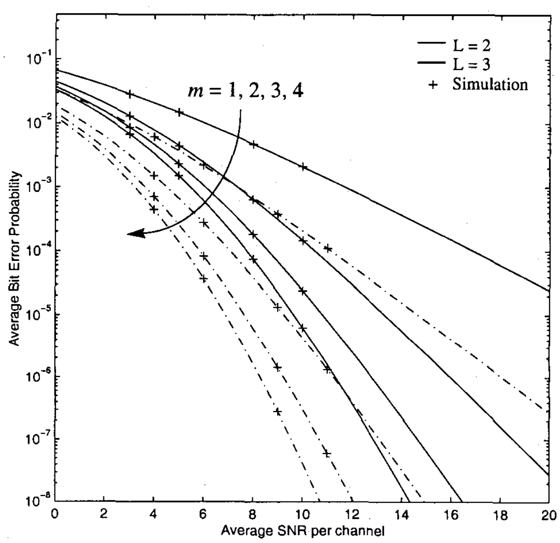

Fig. 1. Error rates for coherent BPSK employing second and third order EGC diversity over Nakagami-m channels.

A.3 Rician

The analysis in Rician fading is more involved compared to the Rayleigh and Nakagami-m cases, owing to unavailability of a closed-form solution for the CHF of the fading amplitude. However, the real and imaginary parts of the CHF can be expressed in terms of a single infinite series formula:

$$A = \exp\left(\frac{-\bar{\gamma}_1 b^2 t}{L(1+K_1)} - K_1\right)\sum_{i=0}^{\infty}\frac{K_1^i}{i!}\Phi\left(-i - \frac{1}{2};\frac{1}{2};\frac{\bar{\gamma}_1 b^2 t}{L(1+K_1)}\right)$$

$$\begin{aligned} B = &\sqrt{\frac{4\bar{\gamma}_1 b^2 t}{L(1+K_1)}}\exp\left(\frac{-\bar{\gamma}_1 b^2 t}{L(1+K_1)} - K_1\right) \\ &\times \sum_{i=0}^{\infty}\frac{\Gamma(i+3/2)K_1^i}{(i!)^2}\,\Phi\left(-i;\frac{3}{2};\frac{\bar{\gamma}_1 b^2 t}{L(1+K_1)}\right) \end{aligned}$$

Next, we will show the application of (4) and (7) to derive an exact infinite series formula for the ABER of coherent binary signalling schemes employing dual diversity EG receiver:

$$G(A, B, 1, 2) = \exp(-2K_1)\sqrt{\frac{2\pi\bar{\gamma}_1 b^2}{1+K_1+\bar{\gamma}_1 b^2}}\sum_{n=0}^{\infty}\sum_{l=0}^{n}\frac{K_1^n}{(l!)^2}$$
$$\times\frac{\Gamma(l+3/2)}{(n-l)!}F_A\left[\frac{1}{2};-n+l-\frac{1}{2},-l;\frac{1}{2},\frac{3}{2};\frac{\bar{\gamma}_1 b^2}{2(1+K_1+\bar{\gamma}_1 b^2)},\frac{\bar{\gamma}_1 b^2}{2(1+K_1+\bar{\gamma}_1 b^2)}\right] \quad (16)$$

By substituting (16) into (4), we obtain the desired result. It is apparent that the above technique can be still applied for the analysis of EG systems with a higher order of diversity. But, further simplifications of the resultant expressions involving multiple infinite series to a single infinite series of finite sums is recommended (details may be found in the Appendix).

A.4 Nakagami-q

The analysis for Nakagami-q fading can proceed in a similar manner to the Rician fading scenario since the CHF of the fading amplitude can be represented by an infinite series expression. An illustrative example for this fading environment is provided in [13].

B. Nonidentical Fading

It is also possible to extend the analysis for the iid case to examine a more realistic nonidentical fading channel model. Our observation here is that if we can rewrite the imaginary part of product of CHFs in (2) as a partial sum, then we can derive the desired results using (7), as we have done in the iid case. A sketch of our derivation is outlined below:

$$\prod_{i=1}^{n}(A_i+B_i) = \left(\prod_{i=1}^{n}A_i\right)\prod_{i=1}^{n}\left(1+\frac{B_i}{A_i}\right)$$
$$= \prod_{i=1}^{n}A_i + \sum_{i=1}^{n}B_i\prod_{\substack{k=1\\k\neq i}}^{n}A_k + \sum_{i=1}^{n}\sum_{\substack{j=1\\j>i}}^{n}B_iB_j\prod_{\substack{k=1\\k\neq i,j}}^{n}A_k + \ldots$$
$$+\sum_{i=1}^{n}\sum_{\substack{j=1\\j>i}}^{n}\cdots\sum_{\substack{w=1\\w>\ldots>j>i}}^{n}B_iB_j\ldots B_w\prod_{\substack{k=1\\k\neq i,j,\ldots,w}}^{n}A_k+\ldots+\prod_{i=1}^{n}B_i \quad (17)$$

Suppose $A_i = \text{Re}\{\phi_{\upsilon_i}(.)\}$ and $B_i = \sqrt{-1}\text{Im}\{\phi_{\upsilon_i}(.)\}$, then the imaginary part of the product (17) is simply a collection of the terms which consists of odd products of B_i (i.e., the second term, the fourth term, and so on). A few examples illustrating the application of (17) in the study of the impact of dissimilar statistics as well as mixed fading cases can be found in [13] (details are omitted here for the sake of brevity).

III. Error Rates for Noncoherent Binary Signals and M-ary Modulations

In this section, we are particularly interested in reporting how the analysis in the preceding section can be utilized and/or extended to consider several other modulation formats such as noncoherent detection of binary FSK, DPSK, QPSK and MQAM. These extensions are possible because the Fourier Transform (FT) of their conditional error probability (CEP) can be expressed in closed-form (also, in terms of confluent hypergeometric series)[1]. For the sake of illustration, let us consider the case of iid fading and assume that notations C and D correspond to the real and imaginary parts of the FT of the CEP, respectively. Then, it can be readily shown that the average error rate is given by

$$P_S = \frac{1}{\pi}\int_0^{\infty}\text{Re}\left\{(C+jD)\prod_{l=1}^{L}\phi_{\upsilon_l}\left(\frac{t}{\sqrt{L}}\right)\right\}dt$$
$$= \frac{1}{\pi}\int_0^{\infty}\text{Re}\{(C+jD)(A+jB)^L\}dt$$
$$= \frac{1}{\pi}\sum_{\substack{k=1\\k\text{ odd}}}^{L}(-1)^{\frac{k+1}{2}}\binom{L}{k}H(A,B,D,k,L)$$
$$+\frac{1}{\pi}\sum_{\substack{k=0\\k\text{ even}}}^{L}(-1)^{\frac{k}{2}}\binom{L}{k}H(A,B,C,k,L) \quad (18)$$

where $A = \text{Re}[\phi_{\upsilon_1}(t/\sqrt{L})]$, $B = \text{Im}[\phi_{\upsilon_1}(t/\sqrt{L})]$ and

$$H(\alpha,\beta,\delta,x,y) = \int_0^{\infty}\alpha^{y-x}\beta^x\delta dt \quad (19)$$

It suffices to state here that the integral (19) can be evaluated in closed-form for Rayleigh and Nakagami-m fading channel models with the aid of identity (7). Details may be found in [13]. Also, comparison between (4) and (18) reveals the similarities between both these expressions: they both rely on (7) for their subsequent simplifications to get the desired closed-form or infinite series solutions. However, it should be emphasized that we have employed two distinct frequency domain techniques to arrive at these expressions in the first place. The CHF method using Parseval's theorem is more general and applicable for a broader class of modulation formats. But the competing technique tends to yield a very concise formula if the CEP takes the form of a complementary error function.

Further extensions of (18) to take into account of the effect of nonidentical and/or mixed fading can also be treated quite easily (similar to the approach discussed in Section IIB). To summarize, in this section we have highlighted the availability of a simple yet general procedure for deriving closed-form and/or infinite series expressions for the ASER of several binary and M-ary modulation formats in a variety of fading environments. A comprehensive treatment of this subject will be dealt in our forthcoming work [13]. It is further noted that the mathematical approach as well as the results presented herein may also be directly used in the analysis of several other modulation formats such as MPSK, MDPSK, star-QAM, DQPSK etc. In these cases, the final results will be expressed in terms of a single finite range integral and the integrand will take a form similar to that of the results obtained for noncoherent BFSK or DPSK signalling scheme. Finally, we would like to point out that it is

1. The readers are referred to [6] and [10] for a more thorough discussions on the development of a frequency-domain technique which is utilized in this section for calculating the average symbol error rates (ASER) of a wide range of modulation formats over generalized fading channels. Also, FTs of the CEP for all common modulation formats are listed in [10].

also straight-forward to derive simple closed-form approximation formulas for a broad class of coherent, differentially coherent and noncoherent modulation formats in Nakagami-m channel with iid diversity branches and $\sum_k m_k$ is assumed to be a positive integer. To achieve this, we utilize the approximate PDF of the EG combiner output [3] and the solutions to three generic trigonometric integrals derived in [11].

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Appendix

In this appendix, we show that it is possible to express the solution to a product of two or more convergent infinite series as a single infinite series of finite sums, by making a change of index. This facilitates rapid computation of our final error rate expressions for the EG receivers (usually involves the computation of the Appell hypergeometric series and/or as a product of multiple infinite series) in a myriad of fading environments. Aside from the above, the simplified expression allows the errors due to the series truncation to be quantified more systematically.

Let us first assume that a double infinite series is absolutely convergent. By introducing the change of index $n = m_1 + m_2$, it is quite easy to show that

$$\sum_{m_1=0}^{\infty}\sum_{m_2=0}^{\infty} A_{m_1,m_2} = \sum_{n=0}^{\infty}\sum_{m_2=0}^{n} A_{n-m_2,m_2} \quad (20)$$

since for $m_1 \geq 0$, the index m_2 must satisfy the condition $n - m_2 \geq 0$, or equivalently $m_2 \leq n$. Similarly, we can deduce that

$$\sum_{m_1=0}^{\infty}\sum_{m_2=0}^{\infty}\sum_{m_3=0}^{\infty} A_{m_1,m_2,m_3} = \sum_{w=0}^{\infty}\sum_{n=0}^{w}\sum_{m_3=0}^{n} A_{w-n,n-m_3,m_3} \quad (21)$$

by making the change of indexes $n = m_2 + m_3$ and $w = n + m_1$, given that the series on the left-side is absolutely convergent. Following this logic, we get

$$F_2(\alpha;\beta_1,\beta_2;\gamma_1,\gamma_2;z_1,z_2) = \sum_{n=0}^{\infty}\sum_{m_2=0}^{n} \frac{(\alpha)_n(\beta_1)_{n-m_2}(\beta_2)_{m_2}}{(\gamma_1)_{n-m_2}(\gamma_2)_{m_2}(n-m_2)!m_2!} z_1^{n-m_2} z_2^{m_2} \quad (22)$$

if $|z_1| < 1$ and $|z_2| < 1$.

The corresponding single infinite series expression for the hypergeometric series with multiple variables can also be derived in a similar fashion:

$$F_A(\alpha;\beta_1,\ldots,\beta_n;\gamma_1,\ldots,\gamma_n;z_1,\ldots,z_n) = \sum_{v=0}^{\infty}\sum_{u=0}^{v}\cdots\sum_{n=0}^{w}\sum_{m_n=0}^{n} \frac{(\alpha)_v(\beta_1)_{v-u}\cdots(\beta_{n-1})_{n-m_n}(\beta_n)_{m_n}}{(\gamma_1)_{v-u}\cdots(\gamma_{n-1})_{n-m_n}(\gamma_n)_{m_n}} \times \frac{1}{(v-u)!\ldots(n-m_n)!m_n!} z_1^{v-u}\ldots z_{n-2}^{w-n} z_{n-1}^{n-m_n} z_n^{m_n} \quad (23)$$

for all $|z_i| < 1$, $i \in \{1, \ldots, n\}$.

Finally, we would like to point out that (20) is related to the Cauchy product of two series. For example, if $\sum a_k$ and $\sum b_k$ are both absolutely convergent series, then so is their Cauchy product:

$$\sum_{m=0}^{\infty} a_m \sum_{k=0}^{\infty} b_k = \sum_{n=0}^{\infty} c_n \quad (24)$$

where

$$c_n = \sum_{k=0}^{n} a_k b_{n-k} = \sum_{k=0}^{n} a_{n-k} b_k$$

It is apparent that we can also replace the product of multiple convergent infinite series by a single infinite series of finite sums, as before. This technique, is therefore recommended to improve the computational efficiency of the average error probability expressions of EG systems, particularly for Rician and Nakagami-q channels.

Another useful property is the extension of Cauchy's product to the power formula:

$$\left(\sum_{n=0}^{\infty} a_n x^n\right)^p = \sum_{n=0}^{\infty} c_n x^n, \; p = 2, 3, 4, \ldots \quad (25)$$

where $c_0 = a_0^p$ and $c_n = \frac{1}{a_0 n}\sum_{k=1}^{n}(kp - n + k)a_k c_{n-k}, n \geq 1$.