Transactions Letters

Efficient Computation of $\mathrm{erfc}(x)$ for Large Arguments

C. Tellambura, *Member, IEEE*, and A. Annamalai, *Member, IEEE*

***Abstract*—A new, infinite series representation for the error function is developed. It is especially suitable for computing $\mathrm{erfc}(x)$ for large x. For instance, for any $x \geq 4$, the error function can be evaluated with a relative error less than 10^{-10} by using only eight terms. Similarly, the error function can be evaluated with a relative error less than 8×10^{-7} for any $x \geq 2$ using just six terms. An analytical bound is derived to show that the total error due to series truncation and undersampling rapidly decreases as x increases. Comparisons with two other series are provided.**

***Index Terms*—Error function, Fourier analysis, numerical methods, sampling theorem.**

I. Introduction

In many communications problems, noise is often characterized using a Gaussian distribution. Therefore, the error function

$$\mathrm{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2}\, dt \tag{1}$$

expresses the error probability. The tail probability of a unit-variance Gaussian random variable $Q(x) = 1/2\mathrm{erfc}(x/\sqrt{2})$ is also called the error function. In this letter, we shall use the first definition. Without loss of generality, we only consider the case $x > 0$. There is no known closed-form expression for $\mathrm{erfc}(x)$. Several numerical or approximation methods have been given in the literature [1]–[4]. Some of these may not be accurate enough for some applications. For instance, the approximation [1] for $Q(x)$ gives only about three significant figure accuracy for $x \geq 0$. The series by Beaulieu [2] is much more accurate, with a relative error less than 6.3×10^{-7} for all $0 \leq x \leq 6$ using only 17 terms. However, we found that this series exhibits poor convergence for large x. Therefore, it is not efficient for computing the error function for large arguments. In fact, this series is best suited for computing the error function near the origin (as $x \to 0$).

Paper approved by N. C. Beaulieu, the Editor for Wireless Communication Theory of the IEEE Communications Society. Manuscript received February 2, 1999; revised July 29, 1999.

C. Tellambura is with the Communications and DSP Research Group, School of Computer Science and Software Engineering, Monash University, Clayton, Vic. 3168, Australia (e-mail: chintha@dgs.monash.edu.au).

A. Annamalai was with the Department of Electrical and Computer Engineering, University of Victoria, Victoria BC, Canada. He is now with the Bradley Department of Electrical and Computer Engineering, Virginia Polytechnic Institute and State University, Blacksburg, VA 22314 USA (e-mail: annamalai@vt.edu).

Publisher Item Identifier S 0090-6778(00)03640-0.

This paper derives an infinite series representation for the error function. This series, based on the sampling theorem, requires only eight terms to achieve a relative error less than 1×10^{-10} for all $x \geq 4$. Similarly, the error function can be evaluated with a relative error less than 8×10^{-7} for any $x \geq 2$ using just six terms.

II. Derivation of the Series

A. Sampling Theorem for Non-Bandlimited Functions

According to the Shannon sampling theorem, an arbitrary band-limited function can be reconstructed exactly from its samples taken at a sufficiently high rate. If one, however, attempts to reconstruct a *non-bandlimited function* using its samples, then

$$f(t) = \sum_{n=-\infty}^{\infty} f(nh)\, \mathrm{sinc}\left(\frac{t-nh}{h}\right) + \epsilon(t) \tag{2}$$

where h is the sampling period, $\mathrm{sinc}(t) \triangleq \sin(\pi t)/(\pi t)$, and [5]

$$|\epsilon(t)| \leq \frac{2}{\pi} \int_{\pi/h}^\infty |F(\omega)|\, d\omega. \tag{3}$$

This bound shows that if $f(t)$ is bandlimited, i.e., its Fourier transform (FT) $F(\omega) = 0$ for $|\omega| > \pi/h$, then $\epsilon(t)$ vanishes, resulting the Shannon sampling theorem. There are two conditions for (2) to be really useful for numerical purposes. First, $f(nh)$ must decrease rapidly as $|n|$ increases. Second, $|\epsilon(t)|$ must be very small. These conditions translate to the requirements that both $f(t)$ and $F(\omega)$ simultaneously be localized in time- and frequency–domains. These are mutually exclusive conditions—a function and its FT cannot both decay too fast. In fact, of all functions, the Gaussian $f(t) = e^{-t^2}$ is the most rapidly decaying in both t and ω [6]. Its FT $F(\omega) = \sqrt{\pi}e^{-\omega^2/4}$ is Gaussian too. Therefore, e^{-t^2} can be reconstructed from its samples with arbitrary accuracy. Using (2), we find

$$e^{-t^2} = \sum_{n=-\infty}^{\infty} e^{-n^2h^2}\, \mathrm{sinc}\left(\frac{t-nh}{h}\right) + \epsilon(t). \tag{4}$$

Applying (3), we find

$$|\epsilon(t)| \leq 2\mathrm{erfc}\left(\frac{\pi}{2h}\right). \tag{5}$$

Thus, if $h = 0.1$, $|\epsilon(t)| < 10^{-108}$! This idea to expand e^{-t^2} was suggested by Rybicki [6], who then derived a series for

0090–6778/00$10.00 © 2000 IEEE

$\operatorname{erfc}(z)$ where z is complex. However, his series only holds for $\operatorname{Imag}(z) > 0$ and hence cannot be used to compute $\operatorname{erfc}(x)$ for real x.

B. Error Function

We now use the above representation to derive a series for the error function. Using [3, eq. (7.4.11)], we can write, for $x > 0$

$$\operatorname{erfc}(x) = \frac{xe^{-x^2}}{\pi} \int_{-\infty}^{\infty} \frac{e^{-t^2}}{t^2 + x^2}\, dt. \tag{6}$$

We can show that

$$\int_{-\infty}^{\infty} \operatorname{sinc}\left(\frac{t - nh}{h}\right) \frac{1}{t^2 + x^2}\, dt$$
$$= \frac{h}{2x} \int_{-\pi/h}^{\pi/h} e^{j\omega nh - x|\omega|}\, d\omega \tag{7a}$$
$$= \frac{h}{n^2h^2 + x^2} - \frac{he^{-\pi x/h}(-1)^n}{(n^2h^2 + x^2)} \tag{7b}$$

where n is an integer ($0, \pm 1, \pm 2, \cdots$) and $j = \sqrt{-1}$. Equation (7a) follows from the use of Parseval's theorem to transform the left-hand integral into the frequency-domain. The integration limits are $\pm\pi/h$ because the function $\operatorname{sinc}(t/h)$ corresponds to an ideal low-pass filter of bandwidth $2\pi/h$ in the frequency-domain. Combining (4), (6), and (7b), we obtain

$$\operatorname{erfc}(x) = \frac{hxe^{-x^2}}{\pi} \sum_{n=-\infty}^{n=\infty} \frac{e^{-n^2h^2}}{n^2h^2 + x^2} + \epsilon_1(x) + \epsilon_2(x) \tag{8}$$

where the error term due to the second right-hand term in (7b) is

$$\epsilon_2(x) = \frac{-he^{-x^2 - \pi x/h}}{\pi x} + \frac{2hxe^{-x^2 - \pi x/h}}{\pi} \cdot \sum_{n=1}^{n=\infty} \frac{e^{-n^2h^2}(-1)^{n+1}}{n^2h^2 + x^2} \tag{9}$$

and using (5) in (6), the error term due to $\epsilon(t)$ in (4) is bounded as

$$|\epsilon_1(x)| \le 2e^{-x^2} \operatorname{erfc}\left(\frac{\pi}{2h}\right) \tag{10}$$

since $\int_{-\infty}^{\infty} 1/(t^2 + x^2)\, dt = \pi/x$. Term-by-term integration used to arrive at the series (8) is justified because (8) is absolutely convergent for any positive x except for $x = 0$. If, say, $h = 0.2$, the magnitude of $\epsilon_1(x)$ vanishes rapidly. For example, for $x = 2, 4, 8$, the bound is in the order of 10^{-30}, 10^{-35}, 10^{-56}, respectively. These values are upper bounds and the actual error is even smaller! Taking the magnitude of the series in (9), we find

$$\begin{aligned} |\epsilon_2(x)| &\le \frac{he^{-x^2 - \pi x/h}}{\pi x} + \frac{2hxe^{-x^2 - \pi x/h}}{\pi} \sum_{n=1}^{n=\infty} e^{-n^2h^2} \\ &\le \frac{he^{-x^2 - \pi x/h}}{\pi x} + \frac{2xe^{-x^2 - \pi x/h}}{\pi} \int_{t=0}^{\infty} e^{-t^2}\, dt \\ &\le \frac{e^{-x^2 - \pi x/h}}{\pi x}(h + \sqrt{\pi}). \end{aligned} \tag{11}$$

The series $\epsilon_2(x)$ is treated as an error term in (8) because the ratio $|\epsilon_2(x)|$ to the first series in the right-hand side of (8) is less than $e^{-\pi x/h}/x^2$. So $\epsilon_2(x)$ is a negligible error term as $x \to \infty$.

For convenience, we rewrite (8) as

$$\operatorname{erfc}(x) = \frac{hxe^{-x^2}}{\pi} \left(\frac{1}{x^2} + 2\sum_{n=1}^{n=N} \frac{e^{-n^2h^2}}{n^2h^2 + x^2}\right) + \epsilon_a(x) \tag{12}$$

where N is the series truncation point and the total approximation error $\epsilon_a(x)$ arises from the following three factors: 1) the series truncation error in (8); 2) the sampling series error for non-bandlimited functions (4); and 3) the second right-hand term in (7b). Using (10) and (11), the absolute approximation error is therefore bounded as

$$|\epsilon_a(x)| \le \underbrace{\frac{2hxe^{-x^2}}{\pi} \left(\sum_{n=N+1}^{n=\infty} \frac{e^{-n^2h^2}}{n^2h^2 + x^2}\right)}_{T_e} + \underbrace{2e^{-x^2} \operatorname{erfc}\left(\frac{\pi}{2h}\right) + \frac{e^{-x^2 - \pi x/h}}{\pi x}(\sqrt{\pi} + h)}_{S_e} \tag{13}$$

where T_e and S_e are the series truncation and undersampling errors, respectively. Therefore, the approximation error depends on the three parameters: x, h, and N. Moreover, for fixed h and N, the error bound *decreases* with increasing x. This suggests that once suitable values for h and N are chosen for, say, $x = x_0$, those values can be used for all $x \ge x_0$.

III. Discussion

We next compare the series (12) with two other methods for computing the error function.

A. Fourier Series Expansion

Beaulieu [2] derives a series solution for $Q(x)$ which can be modified to give the series

$$\operatorname{erfc}(x) = 1 - \frac{4}{\pi} \sum_{\substack{n=1 \\ n \text{ odd}}}^{2N-1} \frac{e^{-n^2h^2} \sin(2nhx)}{n} + \epsilon_a(x) \tag{14}$$

where the parameter T in [2] has been replaced by h for the ease of comparison ($h = \sqrt{2}\pi/T$) and $n = 2N - 1$ is the series truncation point. Using another one of Beaulieu's results [7, eq. (29)], we write

$$|\epsilon_a(x)| < \underbrace{\left|\frac{4}{\pi} \sum_{\substack{n=2N+1 \\ n \text{ odd}}}^{\infty} \frac{e^{-n^2h^2} \sin(2nhx)}{n}\right|}_{T_e} + \underbrace{\operatorname{erfc}\left(\frac{\pi}{2h} - x\right)}_{S_e} \tag{15}$$

using our h notation. T_e and S_e are the series truncation and undersampling errors, respectively. This bound illustrates the difficulty with (14) in computing the error function for large arguments. For fixed h and N, as x increases T_e in (13) rapidly decreases due to the multiplicative factor e^{-x^2}, while T_e in (15) remains more or less constant ($\sin(2nhx)$ terms are simply oscillating terms, which may do little to reduce the truncation error). Again, for fixed h, as x increases, S_e in (13) rapidly decreases while S_e in (15) *increases*. Overall, for fixed h and N, as x increases, the above bound *increases* in contrast to (13). Therefore, as x increases, h should decrease to keep the total error small. The series (14) then converges much slower (see Table II). However, unlike (13), the error bound (15) decreases as $x \to 0$. The series (14) is in fact a special instance of the more general series derived by Beaulieu for the distribution of a sum of random variables (rv's) [8], which is exact only for bounded rv's. So to derive the series (14), a Gaussian rv is truncated to a finite interval. If $\operatorname{erfc}(x)$ is computed for small x, this truncation does not limit the accuracy, but for large x, it does.

B. Asymptotic Series

The following asymptotic expansion has widely been used in various numerical libraries (e.g., GNU libc, Matlab) to compute the error function for large x:

$$\operatorname{erfc}(x) = \frac{e^{-x^2}}{\sqrt{\pi}x}\left[1+\sum_{n=1}^{N-1}(-1)^n\frac{1\cdot 3\cdots(2n-1)}{(2x^2)^n}+R_N(x)\right] \tag{16}$$

where from [3, eq. (7.1.24)], the relative error term is bounded as

$$|R_N(x)| \le \frac{1\cdot 3\cdots(2N-1)}{(2x^2)^N}. \tag{17}$$

It is clear that this series diverges for all $x > 0$ (because, for fixed x, $R_N(x) \to \infty$ as $N \to \infty$). However, for a given x, a good approximation of $\operatorname{erfc}(x)$ can be obtained by taking a fixed number of terms in the sum [see (20)]. However, for fixed x, taking more and more terms of the series does not improve the accuracy, since the series diverges. Note that $R_N(x)$ is less in absolute value than the first neglected term.

What is the value of N to minimize $R_N(x)$ for a given x? Using [3, eq. (6.1.12)] in the above, we find

$$|R_N(x)| \le \frac{1}{\sqrt{\pi}}\frac{\Gamma\left(N+\frac{1}{2}\right)}{(x^2)^N} \tag{18}$$

where $\Gamma(z)$ is the Gamma function. Taking the logarithm of this and differentiating over N, the optimum N given by

$$\psi\left(N+\frac{1}{2}\right) - \ln(x^2) = 0 \tag{19}$$

where $\psi(z)$ is the Digamma function [3, eq. (6.3.1)]. Using the asymptotic expansion [3, eq. (6.3.18)], the optimum number of terms is given by

$$N \approx x^2 - 0.5. \tag{20}$$

Combining (18) and (20) and using Stirling's approximation for $\Gamma(z)$ [3, eq. (6.1.37)], we can bound the relative error [see (22)] as

$$|R_N(x)| \le \sqrt{2}e^{-x^2}. \tag{21}$$

TABLE I
USE OF (12) TO COMPUTE $\operatorname{erfc}(x)$. PARAMETERS h AND N TO ACHIEVE A RELATIVE ERROR LESS THAN 1×10^{-10} FOR ALL $x \ge x_0$

x_0	h	N	erfc (x_0)	*RE* (22)	$\lvert\epsilon_a(x_0)\rvert$	bound (13)
1	0.24	19	1.57299207050(1)	5(11)	8(11)	4(7)
2	0.43	10	4.67773498105(3)	7(11)	3(13)	2(8)
3	0.54	8	2.20904969985(5)	5(11)	1(15)	1(8)
4	0.6	7	1.54172579002(8)	5(11)	8(19)	5(11)
5	0.6	7	1.53745979442(12)	2(11)	3(23)	6(15)
8	0.6	7	1.12242971729(29)	5(11)	5(40)	7(32)
10	0.6	7	2.088487583762(45)	5(11)	1(55)	2(47)

TABLE II
USE OF (14) TO COMPUTE $\operatorname{erfc}(x)$. PARAMETERS h AND N TO ACHIEVE A RELATIVE ERROR LESS THAN 1×10^{-10} FOR ALL $x \ge x_0$

x_0	h	N	erfc (x_0)	*RE* (22)	$\lvert\epsilon_a(x_0)\rvert$	bound (15)
1	0.2598	9	1.57299207050(1)	1(12)	2(13)	2(12)
2	0.2116	12	4.67773498105(3)	9(12)	4(14)	5(14)
3	0.1806	15	2.209049699858(5)	7(11)	2(15)	2(15)
4	0.1511	20	1.541725790028(8)	2(11)	3(19)	6(19)
5	0.1326	25	1.53745979442(12)	2(11)	3(23)	8(22)
8	0.0896	51	1.12242971729(29)	2(13)	2(42)	3(41)

Here one cannot do much to control the accuracy because it is a function of x itself. Whereas in (12), the accuracy can be increased by finer sampling (decreased h).

C. Numerical Results

In the following, the relative error is defined as

$$\text{RE} = \frac{|\epsilon_a(x)|}{\operatorname{erfc}(x)} \tag{22}$$

where the numerator is defined in (12) or (14). The reference error function is from the standard Maple implementation. In the following tables, $a(n)$ denotes $a \times 10^{-n}$.

For a given x, there exists an optimum h value for the use of (12). If h is too small, then T_e in (13) tends to be large. If h is too large, then S_e in (13) tends to increase. However, it is unnecessary to perform a fine search for this optimum. In Table I, using Maple with 200-digit precision, we have empirically determined suitable h and N values so that the relative error is less than 10^{-10}. The series is highly accurate and gets even more accurate as x increases. Also, a relative error of less than 10^{-10} is a rather stringent requirement (not even necessary). For instance, using (12) with$N = 5$ and $h = 0.6$, the error function can be evaluated with a relative error less than 2×10^{-6} for any $x \ge 2$.

Table II shows numerical results for the use of (14). Unlike the trend exhibited in Table I, the number of terms required to achieve a specified relative error increases with x. This trend occurs because h decreases for increasing x, making it increasingly difficult to compute $\operatorname{erfc}(x)$ using (14) for large x. As well, small h values make (14) much more susceptible to round-off errors. For instance, some entries in Table II cannot be obtained using double precision arithmetic (provided by Matlab) but require arbitrary precision arithmetic (provided by Maple). The series (12) does not suffer from these drawbacks.

TABLE III
EQUATION (12) VERSUS (16)

x	N	$\|R_N(x)\|$	$T(x)$
3	9	2(4)	1.0
4	16	2(7)	1.3
5	25	2(11)	1.8
6	36	3(16)	2.5
7	49	7(22)	3.3
8	64	2(28)	4.3
10	100	5(44)	6.5

TABLE IV
EQUATION (12) VERSUS (16)

x	N	$\|R_N(x)\|$	$T(x)$
3	9	2(4)	N/A
4	16	2(7)	N/A
5	25	2(11)	N/A
6	14	2(12)	1.2
7	11	2(12)	1.0
8	9	4(12)	1.0
10	8	8(13)	0.9

From Table I and (21), it is clear that both the new series (12) and the asymptotic series (16) can provide extremely accurate estimates of $\mathrm{erfc}(x)$ for large x. So Tables III and IV show timing comparisons between (12) and (16), which were performed on a Sun Ultra SPARC 10 machine by repeatedly computing $\mathrm{erfc}(x)$ for 10^6 times. For each x, $T(x)$ is the ratio of the time taken by (16) to that by (12). The bound $|R_N(x)|$ is computed using (21) or (17). For the use of (12), $h = 0.5$ and $N = 10$. This combination guarantees a relative error less than 4×10^{-12} for (12) for all $x \geq 3$. The exponential factors $e^{-n^2h^2}$ in (12) are precomputed and stored in an array. This array implementation improves its speed dramatically. Table III shows the case where N is the optimum number of terms given by (20). Not surprisingly, $T(x)$ increases with x. In Table IV, (16) is terminated when N is the smallest integer to guarantee $|R_N(x)| \leq 4 \times 10^{-12}$. For $x \leq 5$, even with optimum N, (16) does not achieve this accuracy level (hence marked N/A). However, for $x > 5$, both (12) and (16) take roughly the same amount of time to execute for the same accuracy level.

Some rational approximations to $\mathrm{erfc}(x)$ can also be found in [3]. However, they guarantee only an upper bound on the absolute error. This means for large x, the relative error can be fairly large. For instance, the rational approximation [3, eq. (7.1.25)] guarantees $|\epsilon_a(x)| \leq 2.5 \times 10^{-5}$ for $0 \leq x < \infty$. However, the relative error is 3×10^{-3}, 4×10^{-2}, and $1 \times^{-1}$ at $x = 2,\ 5$, and 10, respectively. Also, the routine [4, p. 221] based on Chebysev fitting, computes $\mathrm{erfc}(x)$ for all x with relative error less than 1.2×10^{-7}.

IV. CONCLUSION

An infinite series representation for the error function has been developed. It becomes more accurate and efficient as x increases. The total approximation error has been bounded to show its decay with increasing x. In comparison to (14), this series solution is more accurate and requires less terms as x increases. However, the series (14) is useful where averaging of the error function is required over the distribution of x (average of $\sin(ax)$ is simply given by the imaginary part of the characteristic function of x). Such applications include intersymbol and cochannel interference problems [7]. The new asymptotic expansion of $\mathrm{erfc}(x)$ can be useful for numerical computations in cases where standard library functions are not sufficiently accurate. Unlike the asymptotic series, the accuracy of the new series can be improved by decreasing the sampling interval. Finally, the paper considers three infinite series representations for the error function. Mathematically, both (12) and (14) are absolutely convergent series for any $x > 0$. Whereas (16) is divergent for any $x > 0$. Numerically, (14) is most useful as $x \to 0$, while (12) and (16) are most useful as $x \to \infty$.

REFERENCES

[1] P. O. Börjesson and C. E. Sundberg, "Simple approximations of the error function $Q(x)$ for communications applications," *IEEE Trans. Commun.*, vol. COM-27, pp. 639–643, Mar. 1979.

[2] N. C. Beaulieu, "A simple series for personal computer computation of the error function $Q(.)$," *IEEE Trans. Commun.*, vol. 37, pp. 989–991, Sept. 1989.

[3] M. Abramovitz and I. A. Stegun, *Handbook of Mathematical Functions*. New York: Dover, 1972.

[4] W. H. Press, S. A. Teukolsky, W. T. Vetterling, and B. P. Flannery, *Numerical Recipes in C—The Art of Scientific Computing*, 2nd ed. Cambridge, U.K.: Cambridge Univ. Press, 1992.

[5] J. L. Brown, Jr, "On the error in reconstructing a nonbandlimited function by means of the bandpass sampling theorem," *J. Math. Anal. Appl.*, vol. 18, pp. 75–84, 1967.

[6] G. B. Rybicki, "Dawson's integral and the sampling theorem," *Comput. Phys.*, vol. 3, pp. 85–87, Mar. 1989.

[7] N. C. Beaulieu, "The evaluation of error probabilities for intersymbol and cochannel interference," *IEEE Trans. Commun.*, vol. 39, pp. 1740–1749, Dec. 1991.

[8] ——, "An infinite series for the computation of the complementary probability distribution function of a sum of independent random variables and its application to the sum of Rayleigh random variables," *IEEE Trans. Commun.*, vol. 38, pp. 1463–1474, Sept. 1990.