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Derivation of Craig's formula for Gaussian probability function

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A recent work provides two proofs for Craig's formula, which is an integral formula for the Gaussian probability density function. It also highlights the fact the original proof of the Craig formula is somewhat unclear. The authors of the work provide two proofs, one based on the Stieltjes transform of a Gaussian pulse and another based on the moment generation function (MGF) of a unit Gaussian random variable (GRV). Another integral formula for $Q(x)$ based on the characteristic function (CHF) of a unit GRV is also provided. This formula can be applied to the analysis of coherent predetection equal-gain combining diversity receivers.

Introduction: A recent Letter [1] by Lever furnishes two interesting but long proofs for the so-called Craig formula:

$$Q(x) = \frac{1}{\pi}\int_0^{\pi/2} \exp\left(-\frac{x^2}{2\sin^2\theta}\right) d\theta \qquad (1)$$

for $x > 0$. Lever further comments that 'there is little indication of the origin of this result'. The purpose of this Letter is therefore two-fold: (i) to show that an equivalent integral form has been known for a long time, and (ii) to offer a simple proof based on the definition of $Q(x)$ itself. Apart from the proofs in [1], there are at least three ways of deriving eqn. 1. First, the finite integral with trigonometric argument for the Gaussian Q-function can be obtained in a straightforward fashion from a standard mathematical integral (see [2] and eqn. 2 below) based on the Stieltjes transform. Secondly, the proof due to Craig has evolved largely through geometric relations (i.e. from the study of the symbol error probability for MPSK [3 – 5] as a special case when $M = 2$). It should be highlighted that even though the finite integral expression for MPSK can be traced back to Weinstein [3], and a few years later in [4], only Craig [5] pointed out the 'new' definite integral form for the $Q(x)$. Thirdly, we offer a new proof based on the MGF of a GRV. Finally, we also derive another integral formula for $Q(x)$ using the CHF of a unit GRV.

Integral representations for $Q(x)$:
(i) *Stieltjes transform method:* From [6], we find that

$$\operatorname{erfc} x = \frac{2}{\pi}\int_0^{\infty} \frac{e^{-x^2(t^2+1)}}{t^2+1} dt \quad x > 0 \qquad (2)$$

Using the substitution $t = \cot\theta$ and $Q(x) = 0.5\operatorname{erfc}(x/\sqrt{2})$, we immediately obtain Craig's formula. So we can see that eqns. 1 and 2 are equivalent forms. Incidentally, see [7] for the use of eqn. 2 to analyse the performance of trellis coding in various fading channels.

Curiously enough, the proof of eqn. 2 itself is not mentioned anywhere in [6]. Hence, we next give a direct proof for eqn. 2. The single-sided Stieltjes transform of a signal $x(t)$ is defined as [8]:

$$X^s(z) = \int_0^{\infty} \frac{x(t)}{z+t} dt$$

We can modify the integration range to be $-\infty$ to ∞. Therefore, the modified Stieltjes transform of a Gaussian pulse e^{-t^2} is defined to be

$$s(z) = \frac{j}{\pi}\int_{-\infty}^{\infty} \frac{e^{-t^2}}{z-t} dt \qquad (3)$$

where $j = \sqrt{-1}$. By substituting $t - z = y$, this can be rewritten as

$$s(z) = \frac{1}{j\pi}\int_{-\infty-z}^{\infty-z} \frac{e^{-(y^2+2zy+z^2)}}{y} dy \qquad (4a)$$

$$= \frac{e^{-z^2}}{j\pi}\int_{-\infty+jc}^{\infty+jc} \frac{e^{-(y^2+2zy)}}{y} dy \qquad (4b)$$

where the constant c is negative if $\operatorname{Im}(z) > 0$ or is positive if $\operatorname{Im}(z) < 0$. Note that we can fix the limits of the integral from $\pm\infty - z$ to $\pm\infty + jc$, because there are no singularities in the integrand, except at $y = 0$ (so the Cauchy theorem implies that the value of the integral in eqn. 4*b* is only changed when c changes from positive to negative, or vice versa). Multiplying both sides of eqn. 4*b* by e^{z^2} and differentiating over z, we find

$$\frac{\partial\left[s(z)e^{z^2}\right]}{\partial z} = \frac{-2}{j\pi}\int_{-\infty+jc}^{\infty+jc} e^{-(y^2+2zy)} dy \qquad (5a)$$

$$= \frac{2je^{z^2}}{\sqrt{\pi}} \qquad (5b)$$

The error function is defined as [6]

$$\operatorname{erfc}(-jz) = \frac{2}{\sqrt{\pi}}\int_{-jz}^{\infty} e^{-t^2} dt \qquad (6)$$

Therefore, the derivative over z is given as

$$\frac{\partial[\operatorname{erfc}(-jz)]}{\partial z} = \frac{2je^{z^2}}{\sqrt{\pi}} \qquad (7)$$

Comparing eqn. 5*b* and eqn. 7, we may write

$$s(z)e^{z^2} = \operatorname{erfc}(-jz) + C \qquad (8)$$

where C is a constant independent of z. It can be easily verified numerically that $C = 0$ if $\operatorname{Im}(z) > 0$. Hence from eqns. 3 and 8, we have

$$e^{-z^2}\operatorname{erfc}(-jz) = \frac{j}{\pi}\int_{-\infty}^{\infty} \frac{e^{-t^2}}{z-t} dt \quad \operatorname{Im}(z) > 0 \qquad (9)$$

Note that eqn. 9 is given in [6] without any proof at all. Now substituting $z = jx$(real x) in eqn. 9 and after some minor simplifications,

$$\operatorname{erfc}(x) = \frac{2x}{\pi}\int_0^{\infty} \frac{e^{-(t^2+x^2)}}{x^2+t^2} dt \quad x > 0 \qquad (10)$$

which yields eqn. 1 after the substitution $t = x\cot\theta$, or equivalently eqn. 10 yields eqn. 2 if t is replaced by tx in eqn. 10. Note that in this derivation, we have used a more general definition for the error function. In the communications literature, $\operatorname{erfc}(x)$ is always defined for real x. This restriction is not necessary and the error function can be defined for complex arguments. In the sum, this derivation rests on the fact that the Stieltjes transform of a Gaussian pulse e^{-t^2} is $e^{-z^2}\operatorname{erfc}(-jz)$.

(ii) *MGF method:* From its definition, $Q(x)$ is the tail probability that a zero-mean, unit-variance GRV X exceeds x. The MGF of such an RV is $\phi(s) = E[e^{-sX}] = e^{s^2}/2$. Therefore, $Q(x)$ can be written as a standard Laplace inversion integral. So using [9], we find

$$Q(x) = \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} \frac{e^{s^2/2+sx}}{-s} ds \quad -\infty < c < 0 \qquad (11)$$

The vertical contour of integration can be placed anywhere in the regularity domain of $\phi(s)$, which in this case is the entire left-half plane. Therefore, for $x > 0$, we choose $c = -x$. Using the variable substitution $s = -x + jt$, we find

$$Q(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{-\frac{1}{2}(t^2+x^2)}}{x-jt} dt \qquad (12)$$

which is equivalent to

$$Q(x) = \frac{x}{\pi}\int_0^{\infty} \frac{e^{-\frac{1}{2}(t^2+x^2)}}{x^2+t^2} dt \qquad (13)$$

because the imaginary part of the integrand in eqn. 12 is an odd function (hence, the integration from $-\infty$ to ∞ equals zero) and the real part of the integrand is an even function (hence, the integration from $-\infty$ to ∞ equals two times the integral from 0 to ∞). Using the substitution $t = x\cot\theta$ in eqn. 13, we obtain the Craig formula (eqn. 1) at once. The above derivation is much simpler than those due to Lever.

(iii) *CHF method:* From its definition, $Q(x)$ is the tail probability that a zero-mean, unit-variance GRV X exceeds x. The CHF of such an RV is $\phi(t) = E[e^{jtX}] = e^{-t^2/2}$. The Gil-Pelaez inversion theorem [10] expresses the complementary distribution function of an RV in terms of its CHF. Hence we immediately obtain

$$Q(x) = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \frac{1}{t}\,\mathrm{Im}\left[\phi(t)e^{-jtx}\right] dt \qquad (14a)$$

$$= \frac{1}{2} - \frac{1}{\pi}\int_0^\infty \frac{1}{t}e^{-t^2/2}\sin(tx)\, dt \qquad (14b)$$

Conclusions: Two proofs for the Craig formula have been derived, one based on the Stieltjes transform of a Gaussian pulse and the other based on the MGF of a unit GRV. We also provide another integral formula for $Q(x)$ based on the CHF of a unit GRV. The Craig formula and its equivalent (eqn. 2) are suitable for the fading averaging required in maximal ratio combining (MRC) type problems (i.e. averaging of erfc($\sqrt{\gamma}$)) Because they express the error function as an integral of form $\int e^{-x^2 f(t)}g(t)dt$, erfc($\sqrt{\gamma}$) is transformed to an integral of form $\int e^{-\gamma f(t)}g(t)dt$. Hence the average of erfc($\sqrt{\gamma}$) directly relates to the MGF of γ. However, the integral representation given in eqn. 14*b* is not suitable for this case. On the other hand, it is suitable for performing the average over the distribution of the fading amplitude, encountered in the analysis of coherent predetection equal-gain combining (EGC) diversity receivers. In this case, we have the CHF of $x = \sqrt{\gamma}$ [11]. So the average error rate is directly obtained by averaging eqn. 14*a* over x, which expresses the average error rate in terms of the CHF. Hence, we can anticipate that the integral representations emanating from the MGF of the GRV are suitable for MRC type problems, while those arising from the CHF of the GRV are suitable for EGC type problems.

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DS/CDMA closed-loop power control with adaptive algorithm

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Closed-loop power control in the reverse link of a DS/CDMA system is analysed. The transmitter/receiver based on IS-95 and the radio propagation channel under mobile communication environments are modelled and a new power control algorithm is proposed. This algorithm utilises the burst error characteristics of the channel and can be directly applied to the current IS-95 system. The new power control algorithm, with respect to the SIR (signal to interference ratio), increases the service quality, and finally, enhances the system capacity.

Introduction: Direct sequence code division multiple access (DS/CDMA) is a strong candidate for future personal communications systems because of its robustness to multipath interference and varying channel conditions [1, 2]. A difficult problem in applying DS/CDMA to cellular mobile radio, however, is the near-far problem. The capacity of a DS/CDMA cellular system is limited by the total interference generated by other users [3]. Thus, the power control technique which equalises the average signal power of all users received at the base station is a major design criterion in DS/CDMA systems.

A number of power control algorithms have been proposed to minimise the effects of fading, shadowing and near-far effects [4, 5]. These algorithms need several bits to form a power control command (PCC) for updating a transmitter's (mobile) power. The IS-95 DS/CDMA system's base station, however, uses a traffic channel to deliver PCC bits to the mobile [6]. These power control algorithms decrease the capacity of the forward link traffic channel. In this Letter, we propose an adaptive closed-loop power control algorithm which does not decrease the capacity of the forward link and has better performance than that of the conventional algorithm.

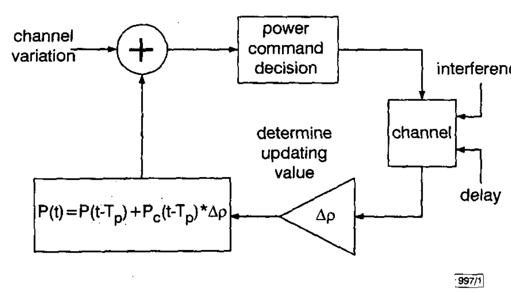

Fig. 1 *Closed-loop power control model*

Adaptive algorithm for closed loop power control: The power control function of the IS-95 reverse link consists of two main parts: open-loop power control (OLPC) and closed-loop power control (CLPC). In OLPC, which is an average power control technique, attempts are made to compensate for slowly varying shadowing and for path loss. CLPC, or feedback power control, is used to mitigate the effects of rapid fading [4]. In this Letter, we only treat CLPC and do not consider path loss.

Fig. 1 shows the CLPC model. The transmitted signal power of the mobile, $P(t)$, is updated by Δp every T_p (1.25ms in IS-95) seconds. As shown in Fig. 1, the reverse link receiver estimates the SIR of the received signal every T_p seconds. If the SIR exceeds a threshold, a power-down power control bit '−1' is sent. Otherwise, a power-up power control bit '1' is transmitted to the mobile via the forward link.

For conventional CLPC, Δp is fixed (typically at 1dB). In this Letter, we propose CLPC using an adaptive algorithm, the main characteristic of which is that Δp is not fixed. This algorithm