Computing the Outage Probability in Mobile Radio Networks Using the Sampling Theorem

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***Abstract*— This letter provides a novel, efficient numerical solution to evaluate the moment generating function of a Suzuki probability density function using the sampling theorem. Applications include outage probability calculations for mobile radio networks in Rayleigh fading and shadowing.**

***Index Terms*—Approximation methods, cochannel interference, fading channels, land mobile radio, sampling methods.**

I. Introduction

THE commercial proliferation of mobile and cellular communication services has generated much research into mobile radio systems and their performance. The mobile radio channel is characterized by random impairments such as fading and shadowing. Consequently, the effect of combined lognormal shadowing and fading has been investigated by many researchers [1]–[6]. Consider computing the image function defined as

$$\phi_\alpha(s) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{e^{-x^2}}{1 + s e^{\alpha x}}\, dx \tag{1}$$

where $s,\ \alpha > 0$. The Laplace transform of a Suzuki probability density function (pdf) is a special case with $\alpha = \sqrt{2}\sigma/4.34$, where σ is the standard deviation of shadowing in decibels. The range of interest may be $3 < \sigma \leq 12$ and $0 < s \leq 10^3$. This image function has extensive applications in evaluating the outage performance of multi-user mobile radio networks [2], [3]. Since no analytical formula exists, an n-point Gauss–Hermite quadrature (GHQ) formula can be used to compute $\phi_\alpha(s)$ [2]. While this approach can be accurate, it is quite difficult to estimate the remainder term, which can only be bounded using a $2n$th-order derivative [7, p. 890].

Our main result is the following series:

$$\phi_\alpha(s) = \frac{h}{\sqrt{\pi}} \sum_{n=-\infty}^{\infty} \frac{e^{-(nh - \ln s/\alpha)^2}}{1 + e^{nh\alpha}} + E_c \tag{2}$$

where h is a small parameter controlling the correction term, which has the form

$$E_c = \frac{2h^2\alpha}{\sqrt{\pi}} \sum_{n=-\infty}^{\infty} (-1)^n n e^{-(nh - \ln s/\alpha)^2} \cdot \sum_{k=1,k\,\mathrm{odd}}^{\infty} \frac{e^{-k\pi^2/(\alpha h)}}{(n\alpha h)^2 + (k\pi)^2}. \tag{3}$$

The value of h should not be too large or too small. It is found that an h value between 0.2 and 0.4 is sufficient for this application. To derive (2), we use a method due to Rybicki [8]. The basis of this method is to express e^{-x^2} appearing in (1) as a sampling series representation and to select a sampling phase to minimize the correction term.

Consider a typical evaluation of $\phi_\alpha(s)$. The advantages of the series solution are as follows.

- The series can be truncated to a few terms.
- The truncation points can be determined in advance for given s, σ, and the required accuracy.
- The correction term is negligible and can be ignored.

II. Derivation

According to the celebrated Shannon sampling theorem, a function $g(x)$ can be reconstructed exactly from its samples $g(x_0 + nh), n = 0, \pm 1, \pm 2, \cdots$, provided the Fourier transform (FT) $G(\omega) = 0$ for $|\omega| > \pi/h$. While $g(x) = e^{-x^2}$ is not strictly bandlimited, its FT $G(\omega) = \sqrt{\pi} e^{-\omega^2/4}$ decays rapidly. Thus, we can write

$$e^{-x^2} = \sum_{n=-\infty}^{\infty} e^{-x_n^2} \operatorname{sinc} \frac{\pi}{h}(x - x_n) + \epsilon(x) \tag{4}$$

where $h > 0$, $x_n = x_0 + nh$, and $\operatorname{sinc} x = \sin x/x$. Here x_0 is a sampling phase[1] allowing an arbitrary shift of the sampling grid. It can be shown that $|\epsilon(x)| < e^{-(\pi/2h)^2}$ [8]. Thus, if $h = 0.1$, $|\epsilon(x)| < 10^{-107}$!

Substituting (4) in (1) and neglecting $\epsilon(x)$ contribution, we obtain

$$\phi_\alpha(s) = \frac{1}{\sqrt{\pi}} \sum_{n=-\infty}^{\infty} q_n e^{-x_n^2} \tag{5}$$

where $f(x) = (1 + s e^{\alpha x})^{-1} = (1 + e^{\beta + \alpha x})^{-1}$, $\beta = \ln s$, and

$$q_n \triangleq \int_{-\infty}^{\infty} f(x) \operatorname{sinc} \frac{\pi}{h}(x - x_n)\, dx. \tag{6}$$

From the Appendix, we have

$$q_n = h f(x_n) + \mathrm{Real}\left\{ 2h \sum_{k=1,k\,\mathrm{odd}} \frac{e^{-k\pi^2/(\alpha h) - i(x_n + \beta/\alpha)\pi/h}}{\alpha x_n + \beta - ik\pi} \right\} \tag{7}$$

where $i = \sqrt{-1}$. The set of functions $\operatorname{sinc}(\pi(x - x_n)/h$, for $n = 0, \pm 1, \pm 2, \cdots$, forms an orthogonal basis for any function

[1] The FT of $g(x - x_0)$ is $G(\omega)e^{-j\omega x_0}$, which means if $g(x)$ is bandlimited, so is $g(x - x_0)$.

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bandlimited to $|\omega| < \pi/h$. So if $f(x)$ is such a function, then $q_n = hf(x_n)$. However, since $f(x)$ above is not strictly bandlimited, the second term in the right side of (7) is a correction term. Since x_0 is arbitrary, we choose $x_0 = -\beta/\alpha$; therefore, $x_n + \beta/\alpha = nh$ and (2) and (3) follow.

A. Series Truncation

The series truncation points are explicitly determined so that the total truncation error is less than ϵ. The basic idea is to get a suitable upper bound to E_t (see below) and choose the truncation point so that the upper bound is less than $\epsilon/2$. This is done for both positive and negative indexes. For ease of computation, the summation index n can be centered around the point $nh = \ln s/\alpha$. Define $\hat{n}$ to be the integer nearest to $\ln s/(\alpha h)$ and $\vartheta = \hat{n}h - \ln s/(\alpha)$. Let the summation (2) be limited to $\hat{n} - M \leq n \leq \hat{n} + N$, where $M, N \geq 0$. Part of the truncation error is then

$$E_t = \frac{h}{\sqrt{\pi}} \sum_{n=\hat{n}+N+1}^{\infty} \frac{e^{-(nh-\ln s/\alpha)^2}}{1+e^{\alpha nh}} \leq \frac{1}{\sqrt{\pi}} \int_{x=Nh}^{\infty} \frac{e^{-(x+\vartheta)^2}}{1+e^{(x+\hat{n}h)\alpha}}\, dx. \tag{8}$$

Since $1 + e^{(x+\hat{n}h)\alpha} > e^{(x+\hat{n}h)\alpha}$ and $\operatorname{erfc} x \leq e^{-x^2}$ for $x > 0$, we obtain

$$N \geq \frac{1}{2h}\left(\sqrt{(\alpha+2\vartheta)^2 - 4\ln\left(\epsilon e^{\vartheta^2+\alpha\hat{n}h}\right)} - (\alpha+2\vartheta)\right) \tag{9}$$

where $\epsilon/2$ is the required accuracy. Similarly, considering the terms for $-\infty \leq n < \hat{n} - M$, we can show that

$$M \geq \frac{1}{h}\left(\vartheta + \operatorname{erfc}^{-1}\epsilon\right). \tag{10}$$

Choosing N and M accordingly ensures the total truncation error be less than ϵ. Note also that the series in (2) can be readily used even without explicitly determining N and M at the outset. The summation can terminate at a certain index $n = N$ when the next term is below a small fraction of the partial sum of the N terms.

B. Correction Term E_c

The largest contribution to E_c occurs when $nh = \ln s/\alpha$. From (3), we thus have approximately

$$|E_c| \approx \frac{2h|\ln s|}{\sqrt{\pi}} \sum_{k=1, k\,\mathrm{odd}}^{\infty} \frac{e^{-k\pi^2/(\alpha h)}}{(\ln s)^2 + (k\pi)^2}. \tag{11}$$

A numerical example can demonstrate how relatively small the correction term is. For $h = 0.2$ and $s = 100$, as α varies from one to four (σ from 3–12 dB), $|E_c|$ varies from 10^{-24} to 10^{-7}. Note that $\phi_\alpha(s)$ in this case varies from 0.0124 to 0.0809. Thus, the correction term can easily be ignored. For small h, ignoring $k > 1$ terms and numerical constants in (11), we find

$$|E_c| \sim \begin{cases} O(he^{-\pi^2/(\alpha h)}/\ln s), & \text{if } s \to \infty \\ O(he^{-\pi^2/(\alpha h)}\ln s), & \text{if } s \to 1. \end{cases} \tag{12}$$

TABLE I
COMPARISON OF TRUNCATED (2) AND 20-POINT GHQ

s	ρ	%error	$h=0.2$		$h=0.4$	
			%error	K	%error	K
10	4	2×10^{-9}	1×10^{-10}	47	6×10^{-10}	25
10	8	5×10^{-5}	7×10^{-11}	44	1×10^{-5}	23
10	12	1×10^{-2}	8×10^{-9}	42	3×10^{-3}	22
100	4	5×10^{-9}	1×10^{-9}	46	4×10^{-9}	24
100	8	2×10^{-4}	3×10^{-8}	43	7×10^{-6}	23
100	12	3×10^{-2}	5×10^{-9}	41	3×10^{-3}	21

The above is just a qualitative statement to highlight the parameters that govern the error term. For instance, it shows that E_c decreases very rapidly with decreasing h.

III. EXAMPLES

Since Hermitian integration is often used to evaluate $\phi_\alpha(s)$, we compare the truncated series (2) and a 20-point GHQ rule for (1). Following Section II-A, the number of terms computed in (2) is $K = M + N + 1$. The relative error of the estimates is given by

$$\%\text{error} = \left|\frac{\phi_\alpha(s) - \hat{\phi_\alpha}(s)}{\phi_\alpha(s)}\right| \times 100 \tag{13}$$

where the reference $\phi_\alpha(s)$ is computed using MATLAB's *quad8* function with an absolute tolerance of 10^{-15} and $\hat{\phi_\alpha}(s)$ denotes the truncated series or the 20-point GHQ rule.

The third column in Table I shows the relative error for the GHQ rule. For $h = 0.2$, the truncated series is more accurate, but needs about twice as many terms. For $h = 0.4$, the two approximations have roughly the same accuracy and computational complexity. The relative error for the GHQ rule increases rapidly for large σ, the degree of shadowing. This increase can be explained as follows. The accuracy of an n-point GHQ rule depends on a $2n$-point derivative [7, p. 890]. We can show that this derivative decreases with n only if $\sigma < 4.3$ [9, Appendix 1], and for large σ, the bound on the remainder term is not sufficiently tight to guarantee a negligible error. Therefore, the truncated series can provide far better accuracy for heavy shadowing cases.

For an application example, we consider the following base-to-mobile link. We write [2, eq. (12)] as

$$P_{\text{out}}(n) = 1 - \frac{1}{\sqrt{\pi}} \sum_{l=1}^{m} w_l \prod_{k=1}^{n} \phi_\alpha\left(\frac{zr_o^\beta}{r_k^\beta e^{-\alpha x_l}}\right) \tag{14}$$

where x_l, w_l are the GHQ nodes and weights, z is the protection ratio, n is the number of active interfering base stations (BS's) which are located on a circle of radius D (the normalized reuse distance) centered around the primary BS, $r_0 \leq 1$ is the distance from the primary BS to the mobile, r_k, $1 \leq k \leq n$, are the distances from the mobile to the interferers, and β is the propagation-loss index. For the following numerical results, we use a 20-point GHQ (i.e., $m = 20$) rule in (14). Note that the above formula uses GHQ only once, whereas [2, eq. (12)] uses repeated GHQ's for a multidimensional integral. When using (2), we set $h = 0.2$. We have up to six interferers, so $n \leq 6$, and choose $\beta = 4$.

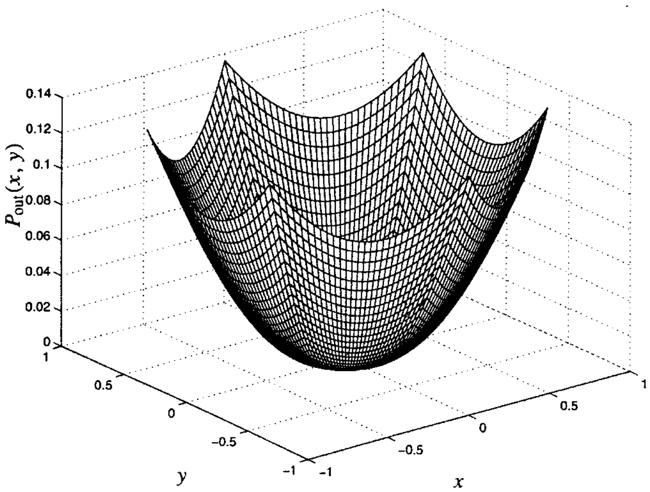

Fig. 1. The interferers are 120 ° apart on the circle with radius D.

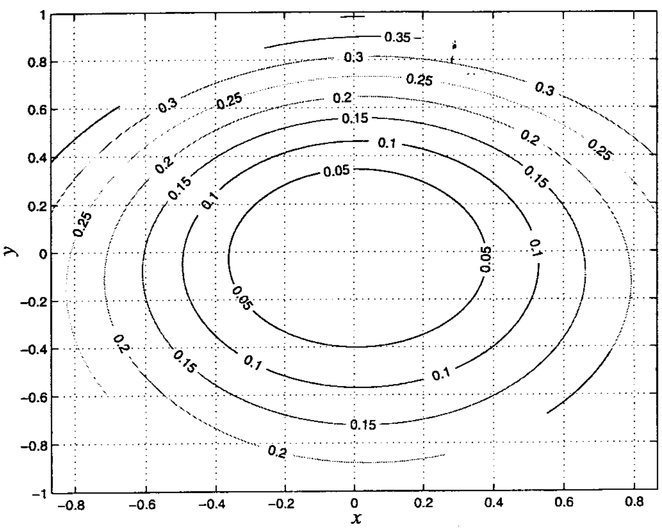

Fig. 2. Outage contours as the mobile traverses the cell. BS is located at (0, 0). $n = 2$, $D = 4$, and $\sigma = 10$ dB.

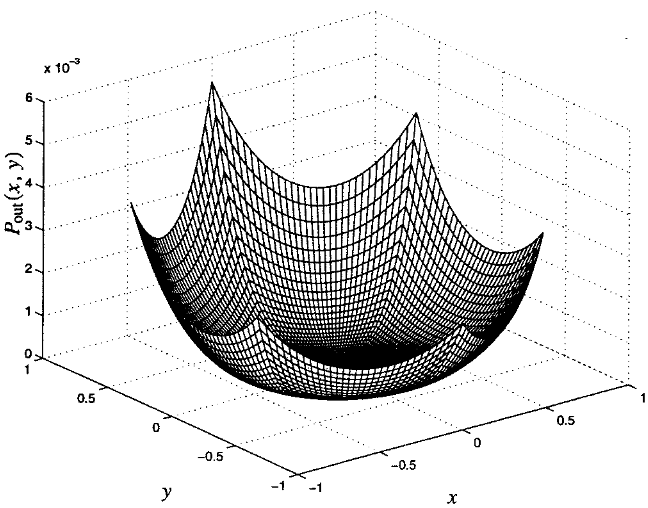

Fig. 3. Outage contours as the mobile traverses the cell. BS is located at (0, 0). $n = 1$, $D = 10$, and $\sigma = 5$ dB.

Figs. 1–3 provide some numerical results. A symmetrical case with three equispaced (angular) interfering BS's is shown in Fig. 1. The outage peaks occur when the mobile is located at one of the vertices of its hexagonal cell. Outage contours as a function of the mobile's location are shown in Fig. 2. The two interfering BS's are at roughly 90 ° and 150 ° (measured counter clockwise from the x axis). An asymmetrical situation is depicted in Fig. 3 for a single interfering BS.

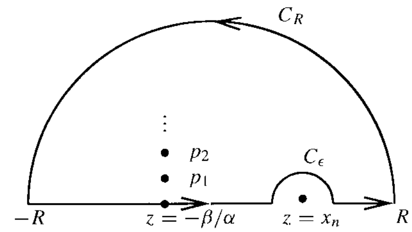

Fig. 4. Outage as the mobile traverses the cell. BS is located at (0, 0). $n = 3$, $D = 8$, and $\sigma = 10$ dB.

IV. Conclusion

A novel series solution has been derived for the Laplace transform of a Suzuki pdf. Rigorous error bounds and truncation bounds have been presented. Programming this formula is particularly simple and can be done in only a few lines of MATLAB code. The series solution provides a valuable tool for computing the outage curves for some practical situations.

Appendix

We use standard contour integration techniques [10]. To evaluate q_n (6) define

$$\psi(z) = f(z)\frac{e^{i(z-x_n)\pi/h}}{(z-x_n)\pi/h} \tag{A.1}$$

where $f(z)$ is as in (6). Consider the integration of $\psi(z)$ over the contour depicted in Fig. 4. As $R \to \infty$, the function $\psi(z)$ is analytic at every point of the contour and its interior except at $z = p_k$, where it has a simple pole. The poles of $f(z)$ are the roots of $1 + e^{\beta+\alpha z} = 0$. Since $e^{ik\pi} = -1$ for $k = 1, 3, 5, \cdots$, the relevant poles are given by $\beta + \alpha z = k\pi$. Therefore, $p_k = -\beta/\alpha + ik\pi/\alpha$ for $k = 1, 3, 5, \cdots$.

Using Cauchy's theorem, we have

$$\left(\int_{-R}^{x_n-\epsilon} + \int_{x_n+\epsilon}^{R}\right)\psi(z)\,dz + \int_{C_\epsilon}\psi(z)\,dz + \int_{C_R}\psi(z)\,dz = 2\pi i\sum_k r_k \tag{A.2}$$

where r_k is the residue of $\psi(z)$ at $z = p_k$. Let $z = x_n + \epsilon e^{i\theta}$. As θ varies from π to zero, we can show that

$$\lim_{\epsilon\to 0}\int_{C_\epsilon}\psi(z)\,dz = -ihf(x_n). \tag{A.3}$$

For $z = Re^{i\theta}$, $0 \le \theta < \pi$, i.e., on C_R, $|\psi(z)| \le e^{-R\sin\theta}/(R-|x_n|)$. So the third integral in (A.2) vanishes for large R. Taking the limit as $R \to \infty$, calculating the simple residues at $z = p_k$, and identifying the imaginary part of the first term in (A.2) as q_n, we obtain (7).

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