MGF Based Analysis of Area under the ROC Curve in Energy Detection

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***Abstract*—The area under the receiver operating characteristic (ROC) curve (AUC), an important performance measure of the energy detector, is derived for Nakagami-m and η-μ fading channels. The analysis is based on the moment generating function (MGF) of the received signal-to-noise ratio (SNR). The derived closed-form expressions do not include special functions, thus reducing computational issues. The analytical framework can also be applied in cases with other fading channels, with diversity reception, or with cooperative spectrum sensing.**

***Index Terms*—Area under ROC curve (AUC), energy detection, moment generating function (MGF).**

I. Introduction

The presence of unknown deterministic signals of an active wireless communications network can be detected by the energy detection. This technique is frequently analyzed in the wireless communications community, especially for the spectrum sensing in cognitive radio networks and the signal detection in ultra-wideband (UWB) communications. In cognitive radio networks, energy detection is a promising spectrum sensing technique because it is a non-coherent technique with low implementation complexity. The conventional energy detector measures the energy associated with the received signal over a specified time period and a bandwidth.

The performance of an energy detector is analyzed by using two key metrics: detection probability (or missed-detection probability) and false alarm probability. Typically, detection capability at a given threshold is illustrated using a receiver operating characteristic (ROC) curve – a plot of the detection probability versus the false alarm probability when the threshold varies from 0 to ∞. In the open literature, the performance of an energy detector, based on ROC curve, has been analyzed for small scale and large scale fading, different diversity reception techniques, and data and decision fusion cooperative spectrum sensing [1]–[4]. However, it is desirable to have a single parameter to measure the overall detection capability. Thus, the area under the ROC curve (AUC) has been introduced as an alternative performance measure [5]. AUC is also used as a metric for fault detection performance of wavelet-based detection techniques [6].

AUC and complementary AUC (CAUC, which is the area under the complementary ROC curve) are proposed in [5] and [7], respectively, and subsequently investigated in [8], as single-valued measures for overall detection capability of an energy detector. In these references, the AUC and CAUC of systems with no diversity reception, with several popular diversity schemes, with channel estimation errors, with fading correlations, and with relay networks are comprehensively analyzed. However, the analytical approaches of [5], [7], [8] may not suffice for some network scenarios (e.g., cooperative spectrum sensing). Moreover, the special functions (e.g., Marcum-Q, confluent hypergeometric, and regularized confluent hypergeometric functions) in the detection probability and the AUC expressions lead to high computational complexity. To circumvent these drawbacks, based on moment generating function (MGF) of the received signal-to-noise ratio (SNR), this letter proposes an alternative analytical approach to evaluate the AUC by reformulating it as a complex integral, which can be evaluated by calculating the residues [9]. The residue calculations are simple, with no special functions involved, and are readily available in all modern mathematical softwares (e.g., Mathematica and MAPLE).

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II. System Model

A. Channel and Signal Models

Depending on the unknown signal status (absent or present), the received signal status at the receiver can be described by using a binary hypothesis: $\mathcal{H}_0$ (signal absent) and $\mathcal{H}_1$ (signal present). The additive white Gaussian noise (AWGN) at the receiver is assumed to be a circularly symmetric complex Gaussian random variable with mean zero and one-sided power spectral density N_0. The conventional energy detection, which has an ideal bandpass filter with bandwidth W and an integrator within time interval T, is considered. For an unknown deterministic signal, the probabilities of false alarm, $P_f(\lambda)$, and detection, $P_d(\gamma,\lambda)$, can be evaluated as [1]

$$P_f(\lambda)=\frac{\Gamma(u,\frac{\lambda}{2})}{\Gamma(u)} \quad \text{and} \quad P_d(\gamma,\lambda)=Q_u(\sqrt{2\gamma},\sqrt{\lambda}), \qquad (1)$$

where $u=WT$, λ is a predefined threshold value, γ is SNR given as $\gamma=E_s|h|^2/N_0$, E_s is the power budget at the transmitter, h is wireless channel gain, $Q_u(\cdot,\cdot)$ is the uth order generalized Marcum-Q function, $\Gamma(\cdot)$ is the gamma function, and $\Gamma(\cdot,\cdot)$ is the upper incomplete gamma function.

B. Performance Measurement

We next consider the AUC, which measures overall detection capability of the energy detector. Different from [5], [7], [8], this measure is derived by using an alternative approach as follows.

Applying the threshold averaging method, the instantaneous AUC, $A(\gamma)$, can be given as $A(\gamma)=-\int_0^\infty P_d(\gamma,\lambda)\frac{\partial P_f(\lambda)}{\partial\lambda}d\lambda$.

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Using the derivative of $P_f(\lambda)$ with respect to (w.r.t.) λ and an alternative representation of $P_d(\gamma,\lambda)$ [4, eq. (4)], $A(\gamma)$ can be written as

$$\begin{aligned}A(\gamma) &= \int_0^\infty \frac{e^{-\frac{\lambda}{2}}}{j2\pi}\oint_\Omega \frac{e^{(\frac{1}{z}-1)\gamma+\frac{\lambda}{2}z}}{z^u(1-z)}dz\frac{\lambda^{u-1}e^{-\frac{\lambda}{2}}}{2^u\Gamma(u)}d\lambda \\ &= \frac{1}{j2\pi}\oint_\Omega \frac{e^{(\frac{1}{z}-1)\gamma}}{z^u(1-z)}\int_0^\infty \frac{\lambda^{u-1}e^{-(1-\frac{z}{2})\lambda}}{2^u\Gamma(u)}d\lambda dz \\ &= \frac{1}{j2\pi}\oint_\Omega \frac{e^{(\frac{1}{z}-1)\gamma}}{z^u(1-z)(2-z)^u}dz,\end{aligned} \quad (2)$$

where Ω is a circular contour of radius $r \in [0,1)$, and the second equality of (2) results after changing integration orders.

III. Average AUC

The average AUC under different fading channels is given in this section. With direct integration over the channel SNR distribution $f_\gamma(x)$, the average AUC, $\bar{A}=\int_0^\infty A(x)f_\gamma(x)dx$, can be written using (2) as

$$\bar{A} = \frac{1}{j2\pi}\oint_\Omega g(z)dz, \quad (3)$$

where

$$g(z) = \frac{\mathcal{M}_\gamma\left(1-\frac{1}{z}\right)}{z^u(1-z)(2-z)^u},$$

$\mathcal{M}_\gamma(s)=\mathbb{E}(e^{-s\gamma})$ is MGF of the received SNR γ, and $\mathbb{E}(\cdot)$ means expectation. The Residue Theorem in complex analysis is one of the effective techniques to evaluate the contour integral in (3). If $g(z)=\sum_{i=-\infty}^{\infty}a_i(z-z_0)^i$, the integration of $g(z)$ in a closed contour Ω encircling z_0 is given by $\oint_\Omega g(z)dz = j2\pi a_{-1}$ where a_{-1} is the complex residue. If the contour encloses multiple poles, then the general result is

$$\oint_\Omega g(z)dz = j2\pi \sum_{a_i\in\mathcal{A}} \mathrm{Res}(g;a_i), \quad (4)$$

where $\mathcal{A}$ is the set of poles contained inside the contour, and $\mathrm{Res}(g;a_i)$ denotes residue of function $g(z)$ at $z=a_i$ [9]. Readers are referred to our paper [4], in which a detailed discussion on calculation of residues is given.

A. Average AUC over Nakagami-m and η-μ Fading Channels

We consider Nakagami-m and η-μ (Format I) fading models, as they are popularly used to model the non-line of sight small-scale fading of a wireless channel. Note that Rayleigh fading model is a special case of the Nakagami-m model.

1) Nakagami-m: The MGF of Nakagami-m fading model is $1/\left(1+\frac{\bar{\gamma}}{m}s\right)^m$, where $\bar{\gamma}$ is the average SNR and m is the fading parameter. The average AUC over Nakagami-m fading, $\bar{A}_{\mathrm{Nak}}$, is derived based on (3) and (4), as

$$\bar{A}_{\mathrm{Nak}} = \begin{cases}\frac{1}{\left(1+\frac{\bar{\gamma}}{m}\right)^m}\left[\mathrm{Res}\left(g;\frac{\bar{\gamma}}{m+\bar{\gamma}}\right)+\mathrm{Res}\left(g;0\right)\right] : & u>m \\ \frac{1}{\left(1+\frac{\bar{\gamma}}{m}\right)^m}\mathrm{Res}\left(g;\frac{\bar{\gamma}}{m+\bar{\gamma}}\right) : & u\le m.\end{cases} \quad (5)$$

$\mathrm{Res}\left(g;\frac{\bar{\gamma}}{m+\bar{\gamma}}\right) = \frac{1}{(m-1)!}D^{m-1}\left(\frac{1}{z^{u-m}(1-z)(2-z)^u}\right)|_{z=\frac{\bar{\gamma}}{m+\bar{\gamma}}}$,
$\mathrm{Res}(g;0) = \frac{1}{(u-m-1)!}D^{u-m-1}\left(\frac{1}{\left(z-\frac{\bar{\gamma}}{m+\bar{\gamma}}\right)^m(1-z)(2-z)^u}\right)|_{z=0}$,
and $D^n(f(z))$ denotes the nth derivative of $f(z)$ w.r.t. z. When $m=1$, $\bar{A}_{\mathrm{Nak}}$ is the average AUC over Rayleigh fading.

2) η-μ (Format I): The MGF of η-μ (Format I) fading model is $K^\mu/[(s+c_1)^\mu(s+c_2)^\mu]$, where $K=4\mu^2h/\bar{\gamma}^2$, $c_1=2(h-H)\mu/\bar{\gamma}$, $c_2=2(h+H)\mu/\bar{\gamma}$, μ represents the number of multipath clusters ($\mu>0$), and the respective H and h are defined as $H=(\eta^{-1}-\eta)/4$ and $h=(2+\eta^{-1}+\eta)/4$ where η (>0) is the power ratio of the in-phase component to the quadrature component [10].

The average AUC over η-μ fading, $\bar{A}_{\eta\mu}$, is derived based on (3) and (4), as (6) at the top of the next page, where
$\mathrm{Res}\left(g;\frac{1}{1+c_1}\right) = \frac{D^{\mu-1}\left(\frac{1}{(z-\frac{1}{1+c_2})^\mu z^{u-2\mu}(1-z)(2-z)^u}\right)|_{z=\frac{1}{1+c_1}}}{(\mu-1)!}$,
$\mathrm{Res}\left(g;\frac{1}{1+c_2}\right) = \frac{D^{\mu-1}\left(\frac{1}{(z-\frac{1}{1+c_1})^\mu z^{u-2\mu}(1-z)(2-z)^u}\right)|_{z=\frac{1}{1+c_2}}}{(\mu-1)!}$,
$\mathrm{Res}(g;0) = \frac{D^{u-2\mu-1}\left(\frac{1}{(z-\frac{1}{1+c_1})^\mu(z-\frac{1}{1+c_2})^\mu(1-z)(2-z)^u}\right)|_{z=0}}{(u-2\mu-1)!}$, and μ (the number of multipath clusters) is an integer value.

The AUC expressions (5) and (6) are closed-form. These expressions can be exactly evaluated by symbolic mathematical softwares (e.g., Mathematica and MAPLE). They do not include any special functions (such as confluent hypergeometric and Marcum-Q functions appearing in [5], [7], [8]).

B. AUC in other Network Scenarios

If the MGF of γ is in a simple rational form (e.g., in Nakagami-m and η-μ fading), the proposed analytical approach based on residue evaluation is effective. Unfortunately, MGFs of some fading models (e.g., K or K_G model) or some network scenarios (e.g., cooperative spectrum sensing) do not have a rational form. For those scenarios, we make the following suggestions: 1) [11] proposes a mixture gamma (MG) model for the distribution of the SNR, which can accurately approximate existing fading channels. The MGF of the MG model is in a simple rational form; 2) for cooperative spectrum sensing in cognitive radio, we have derived in [4] the rational form MGFs; and 3) in general, the Taylor series and the Padé approximation of the MGF can generate rational forms. In summary, if the exact MGF is not in a suitable rational form, an accurate rational approximation can be derived, which can be used in (4). Thus, the proposed analytical approach based on the MGF furnishes a unified framework.

IV. Numerical and Simulation Results

This section provides numerical and semi-analytical Monte-Carlo simulation results. For numerical results, equations (5) and (6) are used for Nakagami-m and η-μ channels, respectively. For semi-analytical Monte-Carlo simulations, 10^6 channel realizations are generated, each of which is used by eq. (9) in [5] to calculate the instantaneous AUC, and the average AUC is calculated over the 10^6 instantaneous AUCs.

The average CAUC (CAUC=1-AUC) versus average SNR curves are plotted, with $u=2$. Fig. 1 shows the average CAUC for Nakagami-m fading channel, which demonstrates the effect of fading parameter m on overall detection capability. When m increases, the average CAUC decreases, which means the overall detection capability increases. Moreover, the detection performance increases with order m as m increases from 1, 2, 3, to 5. Thus, the detection diversity order is m.

$$\bar{A}_{\eta\mu} = \begin{cases} \frac{K^{\mu}}{(1+c_1)^{\mu}(1+c_2)^{\mu}} \left[\text{Res}\left(g; \frac{1}{1+c_1}\right) + \text{Res}\left(g; \frac{1}{1+c_2}\right) + \text{Res}\left(g; 0\right) \right] : & u > 2\mu \\ \frac{K^{\mu}}{(1+c_1)^{\mu}(1+c_2)^{\mu}} \left[\text{Res}\left(g; \frac{1}{1+c_1}\right) + \text{Res}\left(g; \frac{1}{1+c_2}\right) \right] : & u \leq 2\mu \end{cases} \tag{6}$$

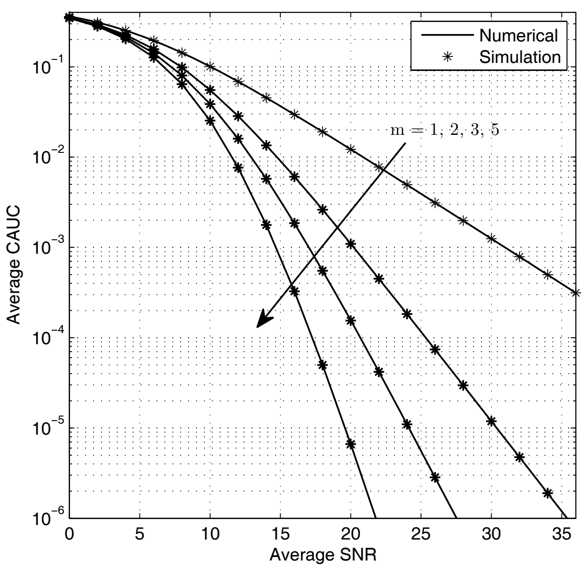

Fig. 1. Average CAUC for Nakagami-m fading channel.

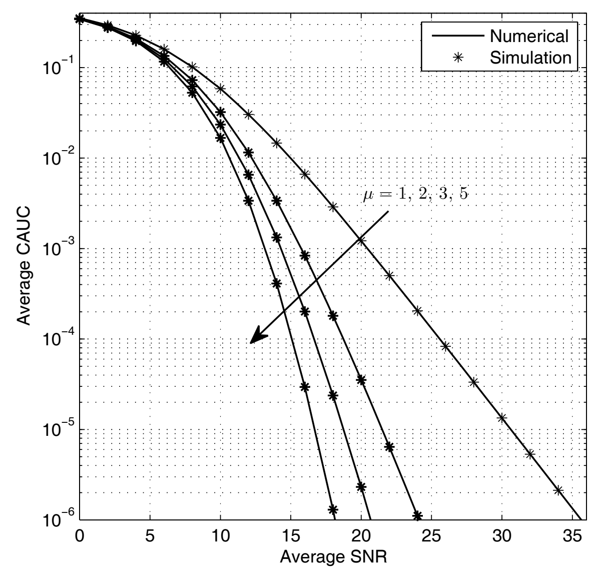

Fig. 2. Average CAUC for η-μ fading channel (Format I), $\eta = 0.5$.

Since a general proof for detection diversity order may need more space, we give a numerical example. When $m = 2$ and $u = 3$, from (5), $\bar{A}_{\text{Nak}}$ can be evaluated as

$$\begin{aligned} \bar{A}_{\text{Nak}} &= \frac{4}{(2+\bar{\gamma})^2} \left[\frac{1/8}{\left(\frac{\bar{\gamma}}{2+\bar{\gamma}}\right)^2} - \frac{1}{\frac{2}{2+\bar{\gamma}}\left(\frac{\bar{\gamma}}{2+\bar{\gamma}}\right)^2\left(\frac{4+\bar{\gamma}}{2+\bar{\gamma}}\right)^3} \right. \\ &\left. + \frac{3}{\frac{2}{2+\bar{\gamma}}\left(\frac{\bar{\gamma}}{2+\bar{\gamma}}\right)\left(\frac{4+\bar{\gamma}}{2+\bar{\gamma}}\right)^4} + \frac{1}{\left(\frac{2}{2+\bar{\gamma}}\right)^2\left(\frac{\bar{\gamma}}{2+\bar{\gamma}}\right)\left(\frac{4+\bar{\gamma}}{2+\bar{\gamma}}\right)^3} \right] \\ &= 1 - 15.5\bar{\gamma}^{-2} + \mathcal{O}(\bar{\gamma}^{-3}). \end{aligned} \tag{7}$$

Thus, the detection diversity order is 2. Fig. 2 shows the average CAUC for η-μ fading model (Format I), which demonstrates the effect of the number (μ) of multipath clusters on overall detection capability. When $\mu = 1, 2, 3$, and 5, the average CAUC decreases in the order of $2, 4, 6$, and 10, respectively. This means overall detection capability increases with order 2μ, and the detection diversity order is 2μ.

Similar to [1-3], a small value of u $(= 2)$ is considered in above results. A larger u will increase the computational complexity, since our approach needs to compute a derivative of order u. It takes less than 10 minutes for a 3.4 GHz computer processor to compute $u = 3500$ case. For values of u up to 10,000, a more powerful workstation may be needed.

V. Conclusion

Closed-form expressions for the AUC of energy detector over Nakagami-m and η-μ fading channels are derived. This analysis uses the MGF of the received SNR at the energy detector output, and subsequently applies the residue theorem. The analytical results do not include special functions, thus reducing computational issues, and are amenable to rapid computer evaluation by using mathematical softwares. This analysis can be extended to other fading models and different network scenarios, e.g., cooperative spectrum sensing.

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